Debts from last week

① "Needle in the haystack" argument:

Fix a tester V. Pick $\vec{x}_0 \in \mathbb{F}^m$ uniformly at random.

Let $p \in \mathbb{F}_{\le d}[x_1, \dots, x_m]$, $f_{\vec{x}_0} : \mathbb{F}^m \to \mathbb{F}$

$$f_{\vec{x}_0}(\vec{x}) = \begin{cases} p(\vec{x}) & \vec{x} \neq \vec{x}_0 \\ p(\vec{x}) + 1 & \vec{x} = \vec{x}_0 \end{cases}$$

- $f_{\vec{x}_0}(\vec{x}) \notin \mathbb{F}_{\le d}[x_1, \dots, x_m]$
- $$\underset{\vec{x}_0, r}{P}\Big[V^{\overset{\text{the proof associated with } p}{f_{\vec{x}_0}, \pi}} \underset{\text{on random } r}{\text{accepts}}\Big] \geq \underset{\vec{x}_0, r}{P}\Big[V^{p,\pi} \underset{\text{on random. } r}{\text{accepts}}\Big] - \frac{\overset{\text{\# of queries performed by V}}{q}}{|\mathbb{F}^m|} \geq 1 - \frac{q}{|\mathbb{F}^m|}$$

$\Longrightarrow$ There exists $\vec{x}_0 \in \mathbb{F}^m$, such that

$$\exists \pi \ \underset{r}{P}\Big[V^{f_{\vec{x}_0}, \pi} \underset{\text{on random } r}{\text{accepts}}\Big] \geq 1 - \frac{q}{|\mathbb{F}^m|} > \frac{1}{2}$$

Hence, there is no tester V making $q < |\mathbb{F}^m|/2$ queries to func. and proof sat. the following:

Comp $p \in \mathbb{F}_{\le d}[x_1, -, x_m] \Longrightarrow \exists \pi \ \underset{r}{P}\Big[V^{p,\pi} \underset{\text{on random } r}{\text{accepts}}\Big] = 1$

Sound $f \notin \mathbb{F}_{\le d}[x_1 \_\_\_, x_m] \Longrightarrow \forall \pi \ \underset{r}{P}\Big[V^{f,\pi} \underset{\text{on random } r}{\text{accepts}}\Big] \leq \frac{1}{2}$

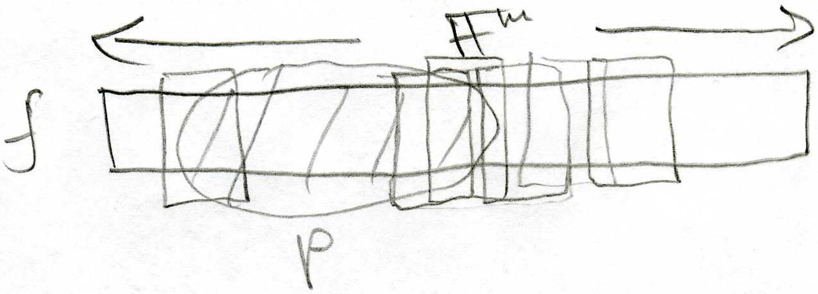
② LDT Theorem $\Longrightarrow$ low degree tester

LDT Theorem For any $f : \mathbb{F}^m \to \mathbb{F}$,

$$\left| \mathrm{agr}_{\leq d}(f) - \mathop{\mathbb{E}}_{s \in S_2^m} \left[\mathrm{agr}_{\leq d}(f_{|s}) \right] \right| \leq m^{O(1)} \cdot \left(\frac{d}{|\mathbb{F}|} \right)^{\Omega(1)}$$

$\max\limits_{p \in \mathbb{F}_{\leq d}[x_1 \ldots x_m]} \Pr_{\vec{x}} [f(\vec{x}) = p(\vec{x})]$

all planes in $\mathbb{F}^m$



Reminder Parameters:

$d = m \cdot (h-1)$
$= \mathrm{polylog}\, n$

$h^m = n$

$h = \log n$

$m = \frac{\log n}{\log \log n}$

$|\mathbb{F}|$ chosen so that $m^{O(1)} \cdot \left(\frac{d}{|\mathbb{F}|} \right)^{\Omega(1)}$ in the above theorem is small.

$|\mathbb{F}| = \mathrm{polylog}\, n$.

Plane vs. Point tester

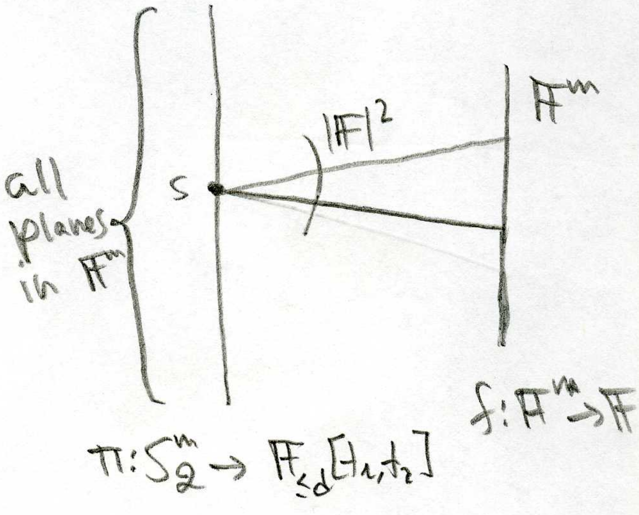
"uniformly at random"

1. Pick u.a.r $s \in S_2^m$, $\vec{x} \in s$.

2. Check $\pi(s)(t_1,t_2) \overset{?}{=} f(\vec{x})$

   where $\vec{x} = s(t_1,t_2)$

   canonical representation of s as a func. $s: \mathbb{F}^2 \to \mathbb{F}^m$



Claim Assuming LDT Theorem,

Comp $p \in \mathbb{F}_{\le d}[x_1,\dots,x_m] \Rightarrow \exists \pi \; \underset{s,\vec{x}\in s}{P}\left(\text{Plane-vs-Point accepts given access to } p,\pi\right) = 1.$

Sound $agr(f, \mathbb{F}_{\le d}[x_1,\dots,x_m]) \le 0.8 \Rightarrow \forall \pi$

$$\underset{s,\vec{x}\in s}{P}\left(\begin{array}{l}\text{Plane-vs-Point accepts}\\ \text{given access to } f,\pi\end{array}\right) \le 0.9$$

Note LDT Thm gives an almost-tight result for all the spectrum.

Pf Comp follows for $\pi(s) \doteq f|_s$.

Sound: assuming $agr(f, \mathbb{F}_{\le d}[x_1,\dots,x_m]) \le 0.8$. Fix some π.

$$\underset{s,x\in s}{P}\left(\text{Plane-vs-Point}^{f,\pi}\right) \le \underset{s\in S_2^m}{\mathbb{E}} agr_{\le d}(f|_s) \le agr_{\le d}(f) + m^{O(1)}\left(\frac{d}{|\mathbb{F}|}\right)^{\Omega(1)}$$

$$\le 0.9$$

choosing closest to $f|_s$ is best possible π

for app. $|\mathbb{F}|$

∎

Today Will show:

$$\mathrm{agr}_{\le d}(f) \ge \mathop{\mathbb{E}}_{s\in S_2^m}\left[\mathrm{agr}_{\le d}(f_{|s})\right] - m^{O(1)}\cdot\left(\frac{d}{|\mathbb{F}|}\right)^{-\Omega(1)}$$

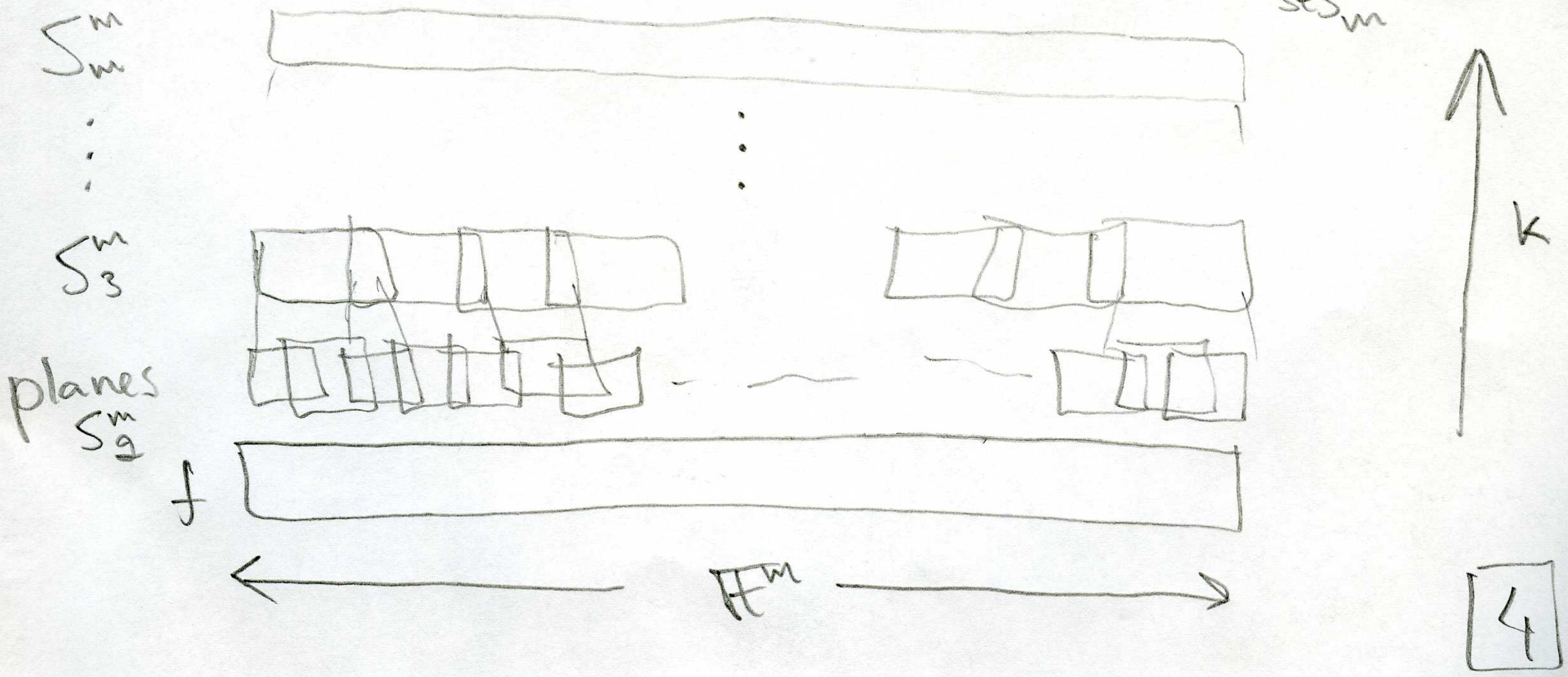
Together with Ex, this will conclude the proof of the LDT Theorem.

(and hence the proof that $NP \subseteq PCP[O(\log n), O(1)]_{\{0,1\}^{\mathrm{poly}\log n}}$).

Proof strategy Will show by induction that for **every** $2\le k\le m$,

$$\mathop{\mathbb{E}}_{s\in S_k^m}\left[\mathrm{agr}_{\le d}(f_{|s})\right] \ge \mathop{\mathbb{E}}_{s\in S_2^m}\left[\mathrm{agr}_{\le d}(f_{|s})\right] - m^{O(1)}\cdot\left(\frac{d}{|\mathbb{F}|}\right)^{-\Omega(1)}$$

In particular, for $k=m$. **Note** $\mathrm{agr}_{\le d}(f)=\mathop{\mathbb{E}}_{s\in S_m^m}\left[\mathrm{agr}_{\le d}(f_{|s})\right]$



Important Observation "Symmetry"

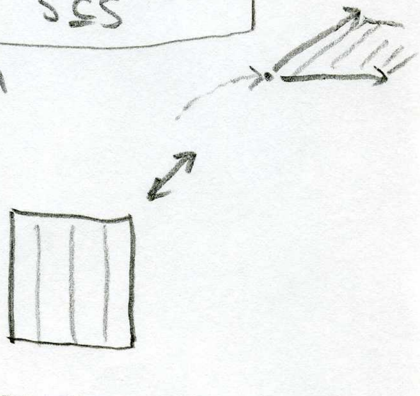
Note for every $s' \in S_k^m$, there's the same # of $s \in S_{k'}^m$, $k' > k$ s.t. $s' \subseteq s$

For every $s \in S_{k+1}^m$,

there's a bijection $s \leftrightarrow \mathbb{F}^{k+1}$

(given by the canonical representation)

$s' \in S_k^m,\ s' \subseteq s \leftrightarrow s' \in S_k^{k+1}$



Hence, suffices to prove for $f : \mathbb{F}^{k+1} \rightarrow \mathbb{F}$,

$$(*)\quad agr_{\le d}(f) \ge \mathop{\mathbb{E}}_{s \in S_k^{k+1}} \left[agr_{\le d}(f_{|s}) \right] - \varepsilon \quad \left[\text{where } \varepsilon = \left(\frac{d}{|\mathbb{F}|}\right)^{\Omega(1} \right]$$

Since

(*)

$$\mathop{\mathbb{E}}_{s \in S_k^m} \left[agr_{\le d}(f_{|s}) \right] \ge \mathop{\mathbb{E}}_{s \in S_k^m} \left[\mathop{\mathbb{E}}_{s' \in S_k^{k+1}} \left[agr_{\le d}(f_{|s'}) \right] - \varepsilon \right]$$

(*)

$$\ge \mathop{\mathbb{E}}_{s \in S_k^m} \left[\mathop{\mathbb{E}}_{s' \in S_k^{k+1}} \left[\mathop{\mathbb{E}}_{s'' \in S_{k-1}^k} \left[agr_{\le d}(f_{|s''}) \right] - \varepsilon \right] - \varepsilon - \varepsilon \right]$$

$\dots$

(*)

$$\ge \mathop{\mathbb{E}}_{s \in S_2^m} \left[agr_{\le d}(f_{|s}) \right] - \varepsilon m$$

Bootstrapping argument

We will only prove :

$$\text{agr}_{\textcircled{\le 2}d}(f) \ge \mathbb{E}_{s\in S_k^{k+1}}\left[\text{agr}_{\le d}(f_{|s})\right]^{\textcircled{2}} - \varepsilon \qquad \varepsilon \le O\left(\sqrt{\tfrac{d}{|\mathbb{F}|}}\right)$$

In $\text{ex.}^{\#3}$, Show this *implies*

$$\text{agr}_{\le d}(f) \ge \mathbb{E}_{s\in S_k^{k+1}}\left[\text{agr}_{\le d}(f_{|s})\right] - \varepsilon' \qquad \varepsilon' \le \left(\frac{d}{|\mathbb{F}|}\right)^{\Omega(1)}$$

Hyperplanes Consistency

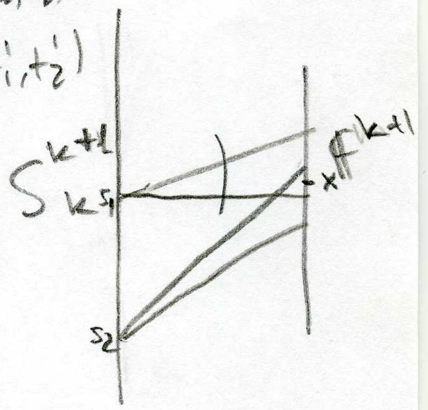
$\pi : S_k^{k+1} \to \mathbb{F}_{\le d}[t_1, \ldots, t_k]$.

Claim For any $f: \mathbb{F}^{k+1} \to \mathbb{F}$,

$\vec{x} = s_1(t_1,t_2)$

$\vec{x} = s_2(t_1',t_2')$

$$\underset{\substack{s_1,s_2\in S_k^{k+1}\\ \vec{x}\in s_1,s_2}}{P}\left(\pi(s_1)(t_1,t_2) = f(\vec{x}) = \pi(s_2)(t_1',t_2')\right) \ge$$



$$\underset{s,\vec{x}\in s}{P}\left(\pi(s)(t_1,t_2) = f(\vec{x})\right)^{\textcircled{2}}$$

$s(t_1,t_2) = \vec{x}$

Note possibly $s_1 = s_2$;
counting triplets
$(s_1, s_2, \vec{x})$ s.t
$\vec{x} \in s_1, s_2$

Technique Counting + Convexity

$$|\mathbb{F}^m| \cdot |\{s\in S_k^{k+1} \mid x\in s\}|^2$$

some x

$\forall x$ this is the same

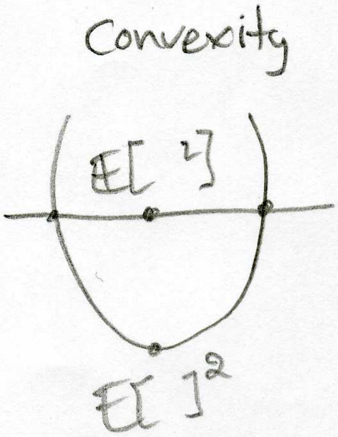
Proof Abbrev. $I_{s,x}$ = indicator for $\pi(s)(t_1,t_2) = f(\vec{x})$ where $\vec{x} = s(t_1,t_2)$

Want to show:

$$\underset{\substack{s_1,s_2\\ x\in s_1,s_2}}{\mathbb{E}}\left[I_{s_1,x}\cdot I_{s_2,x}\right] \geq \left(\underset{s,\, x\in s}{\mathbb{E}}\left[I_{s,x}\right]\right)^2$$

$$\rotatebox{90}{=}$$

$$\underset{x}{\mathbb{E}}\left[\left(\underset{s\ni x}{\mathbb{E}}\left[I_{s,x}\right]\right)^2\right]$$



Inequality follows from Jensen.

$\square$

Claim For every s_1, s_2 that intersect on a $(k-1)$-dim affine sub.,
$\underset{\vec{x}\in s_1,s_2}{\mathbb{E}}\left[I_{s_1,\vec{x}}\cdot I_{s_2,\vec{x}}\right] > \frac{d}{|\mathbb{F}|} \Longrightarrow \pi(s_1)$ & $\pi(s_2)$ agree on intersec.

Proof Schwartz-Zippel.

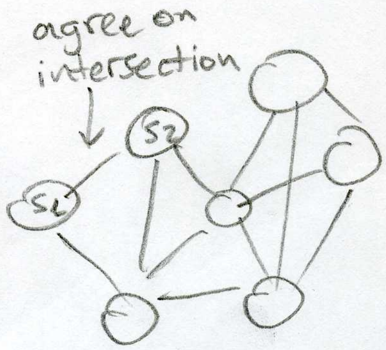
$\square$

Corr $\underset{\substack{s_1,s_2\\ x\in s_1,s_2}}{P}$ [$\pi(s_1)$&$\pi(s_2)$ agree on intersection + $\pi(s_1)$&$\pi(s_2)$ agree with f on $\vec{x}$] $\geq \underset{s\in S_k^{k+1}}{\mathbb{E}}\left[agr_{\leq d}(f_{|s})\right]^2 - \frac{d}{|\mathbb{F}|}$

Hyperplanes Graph

Vertices = hyperplanes S_k^{k+1}

Edges = (s_1, s_2) s.t. s_1, s_2 agree on their intersection (possibly $s_1 \cap s_2 = \emptyset$).



Note I Graph is dense by Corr.

II This is not true for dimensions < k.