Multiplicity one theorem for $(\mathrm{GL}_{n+1}(\mathbb{R}), \mathrm{GL}_n(\mathbb{R}))$

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Abstract. Let F be either $\mathbb{R}$ or $\mathbb{C}$. Consider the standard embedding $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F)$ and the action of $\mathrm{GL}_n(F)$ on $\mathrm{GL}_{n+1}(F)$ by conjugation.

In this paper we show that any $\mathrm{GL}_n(F)$-invariant distribution on $\mathrm{GL}_{n+1}(F)$ is invariant with respect to transposition.

We show that this implies that for any irreducible admissible smooth Fréchet representations π of $\mathrm{GL}_{n+1}(F)$ and τ of $\mathrm{GL}_n(F)$,

$$\dim \mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi, \tau) \leq 1.$$

For p-adic fields those results were proven in [AGRS].

Contents

1. Introduction

Let F be an archimedean local field, i.e. $F = \mathbb{R}$ or $F = \mathbb{C}$. Consider the standard imbedding $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F)$. We consider the action of $\mathrm{GL}_n(F)$ on $\mathrm{GL}_{n+1}(F)$ by conjugation. In this paper we prove the following theorem:

Theorem A. *Any* $\mathrm{GL}_n(F)$ *- invariant distribution on* $\mathrm{GL}_{n+1}(F)$ *is invariant with respect to transposition.*

It has the following corollary in representation theory.

Theorem B. *Let* π *be an irreducible admissible smooth Fréchet representation of* $\mathrm{GL}_{n+1}(F)$ *and* τ *be an irreducible admissible smooth Fréchet representation of* $\mathrm{GL}_n(F)$. *Then*

$$\dim \mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi, \tau) \leq 1. \tag{1}$$

We deduce Theorem B from Theorem A using an argument due to Gelfand and Kazhdan adapted to the archimedean case in [AGS].

Property (1) is sometimes called *strong Gelfand property* of the pair $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$. It is equivalent to the fact that the pair $(\mathrm{GL}_{n+1}(F) \times \mathrm{GL}_n(F), \Delta \mathrm{GL}_n(F))$ is a *Gelfand pair.*

Remark C. *Using the tools developed here, combined with* [AGRS], *one can easily show that Theorem A implies an analogous theorem for the unitary groups.*

Remark. *After the completion of this work we found out that Chen-Bo Zhu and Sun Binyong have obtained the same results simultaneously, independently and in a different way, see* [SZ].

They also proved an analogous theorem for the orthogonal groups.

1.1. Some related results

For non-archimedean local fields of characteristic zero Theorems A and B were proven in [AGRS]. The proof in [AGRS] does not work in the archimedean case because of the presence of transversal derivatives. For this reason we need to use a new ingredient - the theory of D-modules and in particular the Integrability Theorem (see Theorem 2.3.6 below).

We hope that this method will be very useful in the future. It already has been used in subsequent works [AS08, Say09, Aiz08].

The proof given here cannot be literally repeated to get a new proof in the non-Archimedean case since the theory of D-modules is not available there. However one can develop a non-Archimedean analog of the tools that we gain from the theory of D-modules and obtain a proof that works uniformly in both cases. This is done in the subsequent work [Aiz08].

In [AGS], a special case of Theorem B was proven for all local fields; namely the case when τ is one-dimensional.

Theorem A easily implies the following corollary.

Corollary D. *Let* $P_n \subset \mathrm{GL}_n$ *be the subgroup consisting of all matrices whose last row is* $(0, ..., 0, 1)$. *Let* GL_n *act on itself by conjugation. Then every* $P_n(F)$ *- invariant distribution on* $\mathrm{GL}_n(F)$ *is* $\mathrm{GL}_n(F)$ *- invariant.*

This theorem has been proven in [Bar] for eigendistributions with respect to the center of $U_{\mathbb{C}}(\mathrm{gl}_n)$. In [Bar] it is also shown that this implies Kirillov's conjecture.

1.2. Structure of the proof

We will now briefly sketch the main ingredients of our proof of Theorem A.

First we show that we can switch to the following problem. The group $\mathrm{GL}_n(F)$ acts on a certain linear space X_n and σ is an involution of X_n. We have to prove that every $\mathrm{GL}_n(F)$-invariant distribution on X_n is also σ-invariant. We do that by induction on n. Using the Harish-Chandra descent method we show that the induction hypothesis implies that this holds for distributions on the complement to a certain small closed subset $S \subset X_n$. We call this set *the singular set*. This is done in section 3.

Next we assume the contrary: there exists a non-zero $\mathrm{GL}_n(F)$-invariant distribution ξ on X which is anti-invariant with respect to σ.

We use the notion of singular support of a distribution from the theory of D-modules. Let $T \subset T^*X$ denote the singular support of ξ. Using Fourier transform and the fact any such distribution is supported in S we obtain that T is contained in $\check{S}$ where $\check{S}$ is a certain small subset in T^*X. This is done in section 4.

Then we use a deep result from the theory of D-modules which states that the singular support of a distribution is a coisotropic variety in the cotangent bundle.

This enables us to show, using a complicated but purely geometric argument, that the support of ξ is contained in a much smaller subset of S. This is done in section 5.

Finally it remains to prove that any $\mathrm{GL}_n(F)$-invariant distribution that is supported on this subset together with its Fourier transform is zero. This is proven in subsection 4.1 using Homogeneity Theorem (Theorem 2.2.13) which in turn uses Weil representation.

1.3. Content of the paper

In section 2 we give the necessary preliminaries for the paper.

In subsection 2.1 we fix the general notation that we will use.

In subsection 2.2 we discuss invariant distributions and introduce some tools to work with them. The most advanced are

- The Homogeneity theorem and Fourier transform.
- The Harish-Chandra descent method.

In subsection 2.3 we discuss the notion of singular support of a distribution. The most important for us property of this singular support is being coisotropic. This fact is a crucial tool of this paper.

In subsection 2.4 we introduce notation that we will use in our proof.

In section 3 we use the Harish-Chandra descent method.

In subsection 3.1 we linearize the problem to a problem on the linear space $X = \mathrm{sl}(V) \times V \times V^*$, where $V = F^n$.

In subsection 3.2 we perform the Harish-Chandra descent on the $\mathrm{sl}(V)$-coordinate and $V \times V^*$ coordinate separately and then use automorphisms ν_λ of X to descend further to the singular set S.

In section 4 we reduce Theorem A to the following geometric statement: any coisotropic subvariety of $\check{S}$ is contained in a certain set $\check{C}_{X \times X}$. The reduction is done using the fact that the singular support of a distribution has to be coisotropic, and the following proposition: any $\mathrm{GL}(V)$-invariant distribution on X such that it and its Fourier transform are supported on $\mathrm{sl}(V) \times (V \times 0 \cup 0 \times V^*)$ is zero.

In subsection 4.1 we prove this proposition using Homogeneity theorem.

In section 5 we prove the geometric statement. Technically this is the most complicated part of the paper. However we would like to note that it is purely algebro-geometric statement that involves no analysis.

In subsection 5.1 we give preliminaries on coisotropic subvarieties. In particular, we give a geometric partial analog of Frobenius reciprocity for coisotropic subvarieties (Corollaries 5.1.7 and 5.1.8).

In subsection 5.2 we stratify the set $\check{S}$ and use an inductive argument on the strata. This reduces the geometric statement to a proposition on one stratum that we call *the Key Proposition*.

In subsection 5.3 we analyze a stratum of $\check{S}$ and then use the geometric analog of Frobenius reciprocity to reduce the Key Proposition to a lemma on $V \times V^* \times V \times V^*$ that we call *the Key Lemma*.

In subsection 5.4 we prove the Key Lemma.

In Appendix A we prove that Theorem A implies Theorem B using an archimedean analog of Gelfand-Kazhdan technique.

In Appendix B we give more details on the facts concerning the theory of D-modules listed in subsection 2.3.

1.4. Acknowledgements

2. Preliminaries

2.1. General notation

- In this paper all the algebraic varieties are defined over F.
- For an algebraic variety X we denote by $X(F)$ the topological space or smooth manifold of F points of X.
- We consider linear spaces as algebraic varieties and treat them in the same way.
- For an algebraic variety X defined over $\mathbb{R}$ we denote by $X_{\mathbb{C}}$ the natural algebraic variety defined over $\mathbb{R}$ such that $X_{\mathbb{C}}(\mathbb{R}) = X(\mathbb{C})$. Note that over $\mathbb{C}$, $X_{\mathbb{C}}$ is isomorphic to $X \times X$.
- For a group G acting on a set X and a point $x \in X$ we denote by Gx or by $G(x)$ the orbit of x and by G_x the stabilizer of x.
- An action of a Lie algebra $\mathfrak{g}$ on a (smooth, algebraic, etc) manifold M is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on M. Note that an action of a (Lie, algebraic, etc) group on M defines an action of its Lie algebra on M.
- For a Lie algebra $\mathfrak{g}$ acting on M, an element $\alpha \in \mathfrak{g}$ and a point $x \in M$ we denote by $\alpha(x) \in T_x M$ the value at point x of the vector field corresponding to α. We denote by $\mathfrak{g}x \subset T_x M$ or by $\mathfrak{g}(x)$ the image of the map $\alpha \mapsto \alpha(x)$ and by $\mathfrak{g}_x \subset \mathfrak{g}$ its kernel.
- For manifolds $L \subset M$ we denote by $N_L^M := (T_M|_L)/T_L$ the normal bundle to L in M.
- Denote by $CN_L^M := (N_L^M)^*$ the conormal bundle.

- For a point $y \in L$ we denote by $N_{L,y}^M$ the normal space to L in M at the point y and by $CN_{L,y}^M$ the conormal space.

2.2. Invariant distributions

2.2.1. Distributions on smooth manifolds.

Notation 2.2.1. *Let X be a smooth manifold. Denote by $C_c^\infty(X)$ the space of test functions on X, that is smooth compactly supported functions, with the standard topology, i.e. the topology of inductive limit of Fréchet spaces.*

Denote $\mathcal{D}(X) := C_c^\infty(X)^*$ to be the dual space to $C_c^\infty(X)$.

For any vector bundle E over X we denote by $C_c^\infty(X, E)$ the space of smooth compactly supported sections of E and by $\mathcal{D}(X, E)$ its dual space. Also, for any finite dimensional real vector space V we denote by $C_c^\infty(X, V)$ the space of smooth compactly supported sections of the trivial bundle with fiber V and by $\mathcal{D}(X, V)$ its dual space.

2.2.2. Schwartz distributions on Nash manifolds.

Our proof of Theorem A widely uses Fourier transform which cannot be applied to general distributions. For this we require a theory of Schwartz functions and distributions as developed in [AG1].

This theory is developed for Nash manifolds. Nash manifolds are smooth semi-algebraic manifolds but in the present work only smooth real algebraic manifolds are considered. Therefore the reader can safely replace the word *Nash* by *smooth real algebraic.*

Schwartz functions are functions that decay, together with all their derivatives, faster than any polynomial. On $\mathbb{R}^n$ it is the usual notion of Schwartz function. For precise definitions of those notions we refer the reader to [AG1]. We will use the following notations.

Notation 2.2.2. *Let X be a Nash manifold. Denote by $\mathcal{S}(X)$ the Fréchet space of Schwartz functions on X.*

Denote by $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ the space of Schwartz distributions on X.

For any Nash vector bundle E over X we denote by $\mathcal{S}(X, E)$ the space of Schwartz sections of E and by $\mathcal{S}^*(X, E)$ its dual space.

Notation 2.2.3. *Let X be a smooth manifold and let $Z \subset X$ be a closed subset. We denote $\mathcal{S}_X^*(Z) := \{\xi \in \mathcal{S}^*(X) | \mathrm{Supp}(\xi) \subset Z\}$.*

For a locally closed subset $Y \subset X$ we denote $\mathcal{S}_X^*(Y) := \mathcal{S}_{X \setminus (\overline{Y} \setminus Y)}^*(Y)$. In the same way, for any bundle E on X we define $\mathcal{S}_X^*(Y, E)$.

Remark 2.2.4. *Schwartz distributions have the following two advantages over general distributions:*

(i) For a Nash manifold X and an open Nash submanifold $U \subset X$, we have the following exact sequence

$$0 \to \mathcal{S}_X^*(X \setminus U) \to \mathcal{S}^*(X) \to \mathcal{S}^*(U) \to 0.$$

(ii) Fourier transform defines an isomorphism $\mathcal{F} : \mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}^*(\mathbb{R}^n)$.

The following theorem allows us to switch between general distributions and Schwartz distributions.

Theorem 2.2.5. *Let a reductive group G act on a smooth affine variety X. Let V be a finite dimensional continuous representation of $G(F)$ over $\mathbb{R}$. Suppose that $\mathcal{S}^*(X(F), V)^{G(F)} = 0$. Then $\mathcal{D}(X(F), V)^{G(F)} = 0$.*

For proof see [AG2], Theorem 4.0.2.

2.2.3. Basic tools.

We present here some basic tools on equivariant distributions that we will use in this paper.

Proposition 2.2.6. *Let a Nash group G act on a Nash manifold X. Let $Z \subset X$ be a closed subset.*

Let $Z = \bigcup_{i=0}^{l} Z_i$ be a Nash G-invariant stratification of Z. Let χ be a character of G. Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq l$ we have $\mathcal{S}^*(Z_i, Sym^k(CN_{Z_i}^X))^{G,\chi} = 0$. Then $\mathcal{S}_X^*(Z)^{G,\chi} = 0$.

For proof see section B.2 in [AGS].

Proposition 2.2.7. *Let G_i be Nash groups acting on Nash manifolds X_i for $i = 1 \dots n$. Let $E_i \to X_i$ be G_i-equivariant Nash vector bundles.*
(i) Suppose that $\mathcal{S}^*(X_j, E_j)^{G_j} = 0$ for some $1 \leq j \leq n$. Then

$$\mathcal{S}^*(\prod_{i=1}^{n} X_i, \boxtimes E_i)^{\prod G_i} = 0,$$

where $\boxtimes$ denotes the external product of vector bundles.
(ii) Let $H_i < G_i$ be Nash subgroups. Suppose that $\mathcal{S}^*(X_i, E_i)^{H_i} = \mathcal{S}^*(X_i, E_i)^{G_i}$ for all i. Then

$$\mathcal{S}^*(\prod X_i, \boxtimes E_i)^{\prod H_i} = \mathcal{S}^*(\prod X_i, \boxtimes E_i)^{\prod G_i},$$

The proof is trivial and the same as the proof of Proposition 3.1.5 in [AGS].

Theorem 2.2.8 (Frobenius reciprocity). *Let a unimodular Nash group G act transitively on a Nash manifold Z. Let $\varphi : X \to Z$ be a G-equivariant Nash map. Let $z \in Z$. Suppose that its stabilizer G_z is unimodular. Note that this implies that there exists a G-invariant measure on Z. Fix such a measure. Let X_z be the fiber of z. Let χ be a character of G. Then $\mathcal{S}^*(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z)^{G_z,\chi}$.*

Moreover, for any G-equivariant bundle E on X, the space $\mathcal{S}^*(X, E)^{G,\chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z, E|_{X_z})^{G_z,\chi}$.

For proof see [AG2], Theorem 2.5.7.

2.2.4. Fourier transform.

From now till the end of the paper we fix an additive character κ of F given by $\kappa(x) := e^{2\pi i \operatorname{Re}(x)}$.

Notation 2.2.9. *Let V be a vector space over F. Let B be a non-degenerate bilinear form on V. Then B defines Fourier transform with respect to the self-dual Haar measure on V. We denote it by $\mathcal{F}_B : \mathcal{S}^*(V) \to \mathcal{S}^*(V)$.*

For any Nash manifold M we also denote by $\mathcal{F}_B : \mathcal{S}^*(M \times V) \to \mathcal{S}^*(M \times V)$ the fiberwise Fourier transform.

If there is no ambiguity, we will write $\mathcal{F}_V$ instead $\mathcal{F}_B$.

We will use the following trivial observation.

Lemma 2.2.10. *Let V be a finite dimensional vector space over F. Let a Nash group G act linearly on V. Let B be a G-invariant non-degenerate symmetric bilinear form on V. Let M be a Nash manifold with an action of G. Let $\xi \in \mathcal{S}^*(V(F) \times M)$ be a G-invariant distribution. Then $\mathcal{F}_B(\xi)$ is also G-invariant.*

2.2.5. Homogeneity Theorem.

Notation 2.2.11. *Let V be a vector space over F. Consider the homothety action of $F^\times$ on V by $\rho(\lambda)v := \lambda^{-1}v$. It gives rise to an action ρ of $F^\times$ on $\mathcal{S}^*(V)$.*

Also, for any $\lambda \in F$ we denote $|\lambda|_F := |\lambda|^{dim_\mathbb{R} F}$.

Notation 2.2.12. *Let V be a vector space over F. Let B be a non-degenerate symmetric bilinear form on V. We denote*

$$Z(B) := \{x \in V(F) | B(x, x) = 0\}.$$

Theorem 2.2.13 (Homogeneity Theorem). *Let V be a vector space over F. Let B be a non-degenerate symmetric bilinear form on V. Let M be a Nash manifold. Let $L \subset \mathcal{S}^*_{V(F) \times M}(Z(B) \times M)$ be a non-zero subspace such that $\forall \xi \in L$ we have $\mathcal{F}_B(\xi) \in L$ and $B\xi \in L$ (here B is interpreted as a quadratic form).*

Then there exist a non-zero distribution $\xi \in L$ and a unitary character u of $F^\times$ such that either $\rho(\lambda)\xi = |\lambda|_F^{\frac{dimV}{2}} u(\lambda)\xi$ for any $\lambda \in F^\times$ or $\rho(\lambda)\xi = |\lambda|_F^{\frac{dimV}{2}+1} u(\lambda)\xi$ for any $\lambda \in F^\times$.

For proof see [AG2], Theorem 5.1.7.

2.2.6. Harish-Chandra descent.

Definition 2.2.14. *Let an algebraic group G act on an algebraic variety X. We say that an element $x \in X(F)$ is G-**semisimple** if its orbit $G(F)x$ is closed.*

Theorem 2.2.15 (Generalized Harish-Chandra descent). *Let a reductive group G act on smooth affine varieties X and Y. Let χ be a character of $G(F)$. Suppose that for any G-semisimple $x \in X(F)$ we have*

$$\mathcal{S}^*((N^X_{Gx,x} \times Y)(F))^{G(F)_x, \chi} = 0.$$

Then $\mathcal{S}^*(X(F) \times Y(F))^{G(F)_x, \chi} = 0$.

For proof see [AG2], Theorem 3.1.6.

2.3. D-modules and singular support

In this paper we will use the algebraic theory of D-modules. We will now summarize the facts that we need and give more details in Appendix B. For a good introduction to the algebraic theory of D-modules we refer the reader to [Ber] and [Bor].

More specifically, we will use the notion of singular support of a distribution. For those who are not familiar with the theory of D-modules, Corollary 2.3.7 and the facts that are listed after it are the only properties of singular support that we use.

In this subsection $F = \mathbb{R}$.

Definition 2.3.1. *Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Consider the D_X-submodule $\mathcal{M}_\xi$ of $\mathcal{S}^*(X(\mathbb{R}))$ generated by ξ. We define the **singular support** of ξ to be the singular support of $\mathcal{M}_\xi$. We denote it by $SS(\xi)$.*

Remark 2.3.2. *A similar, but not equivalent notion is sometimes called in the literature a 'wave front of ξ'.*

Notation 2.3.3. *Let (V, B) be a quadratic space. Let X be a smooth algebraic variety. Consider B as a map $B : V \to V^*$. Identify $T^*(X \times V)$ with $T^*X \times V \times V^*$. We define $F_V : T^*(X \times V) \to T^*(X \times V)$ by $F_V(\alpha, v, \phi) := (\alpha, -B^{-1}\phi, Bv)$.*

Definition 2.3.4. *Let M be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it M-**coisotropic** if one of the following equivalent conditions holds.*
(i) The ideal sheaf of regular functions that vanish on $\overline{Z}$ is closed under Poisson bracket.
(ii) At every smooth point $z \in Z$ we have $T_z Z \supset (T_z Z)^\perp$. Here, $(T_z Z)^\perp$ denotes the orthogonal complement to $(T_z Z)$ in $(T_z M)$ with respect to ω.
(iii) For a generic smooth point $z \in Z$ we have $T_z Z \supset (T_z Z)^\perp$.
If there is no ambiguity, we will call Z a coisotropic variety.

Note that every non-empty M-coisotropic variety is of dimension at least $\frac{1}{2} dim M$.

Notation 2.3.5. *For a smooth algebraic variety X we always consider the standard symplectic form on T^*X. Also, we denote by $p_X : T^*X \to X$ the standard projection.*

The following theorem is crucial in this paper.

Theorem 2.3.6. *[Integrability Theorem] Let X be a smooth algebraic variety. Let $\mathcal{M}$ be a finitely generated D_X-module. Then $SS(\mathcal{M})$ is a T^*X-coisotropic variety.*

This is a special case of Theorem I in [Gab]. For similar versions see also [KKS, Mal].

Corollary 2.3.7. *Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Then $SS(\xi)$ is coisotropic.*

We will also use the following well-known facts from the theory of D-modules. Let X be a smooth algebraic variety.

Fact 2.3.8. *Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Then $\overline{\mathrm{Supp}(\xi)}_{Zar} = p_X(SS(\xi))(\mathbb{R})$, where $\overline{\mathrm{Supp}(\xi)}_{Zar}$ denotes the Zariski closure of $\mathrm{Supp}(\xi)$.*

Fact 2.3.9.
Let an algebraic group G act on X. Let $\mathfrak{g}$ denote the Lie algebra of G. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))^{G(\mathbb{R})}$. Then

$$SS(\xi) \subset \{(x, \phi) \in T^*X \mid \forall \alpha \in \mathfrak{g} \, \phi(\alpha(x)) = 0\}.$$

Fact 2.3.10. *Let (V, B) be a quadratic space. Let $Z \subset X \times V$ be a closed subvariety, invariant with respect to homotheties in V. Suppose that $\mathrm{Supp}(\xi) \subset Z(\mathbb{R})$. Then*

$$SS(\mathcal{F}_V(\xi)) \subset F_V(p_{X \times V}^{-1}(Z)).$$

For proofs of those facts see Appendix B.

2.4. Specific notation

The following notations will be used in the body of the paper.

- Let $V := V_n$ be the standard n-dimensional linear space defined over F.
- Let $\mathrm{sl}(V)$ denote the Lie algebra of operators with zero trace.
- Denote $X := X_n := \mathrm{sl}(V_n) \times V_n \times V_n^*$
- $G := G_n := \mathrm{GL}(V_n)$
- $\mathfrak{g} := \mathfrak{g}_n := \mathrm{Lie}(G_n) = \mathrm{gl}(V_n)$
- $\widetilde{G} := \widetilde{G}_n := G_n \rtimes \{1, \sigma\}$, where the action of the 2-element group $\{1, \sigma\}$ on G is given by the involution $g \mapsto g^{t^{-1}}$.
- We define a character χ of $\widetilde{G}$ by $\chi(G) = \{1\}$ and $\chi(\widetilde{G} - G) = \{-1\}$.
- Let G_n act on G_{n+1}, $\mathfrak{g}_{n+1}$ and on $\mathrm{sl}(V_n)$ by $g(A) := gAg^{-1}$.
- Let G act on $V \times V^*$ by $g(v, \phi) := (gv, (g^*)^{-1}\phi)$. This gives rise to an action of G on X.
- Extend the actions of G to actions of $\widetilde{G}$ by $\sigma(A) := A^t$ and $\sigma(v, \phi) := (\phi^t, v^t)$.
- We consider the standard scalar products on $\mathrm{sl}(V)$ and $V \times V^*$. They give rise to a scalar product on X.
- We identify the cotangent bundle T^*X with $X \times X$ using the above scalar product.
- Let $\mathcal{N} := \mathcal{N}_n \subset \mathrm{sl}(V_n)$ denote the cone of nilpotent operators.
- $C := (V \times 0) \cup (0 \times V^*) \subset V \times V^*$.
- $\check{C} := (V \times 0 \times V^* \times 0) \cup (0 \times V^* \times 0 \times V^*) \subset V \times V^* \times V \times V^*$.
- $\check{C}_{X \times X} := (\mathrm{sl}(V) \times V \times 0 \times \mathrm{sl}(V) \times V \times 0) \cup (\mathrm{sl}(V) \times 0 \times V^* \times \mathrm{sl}(V) \times 0 \times V^*) \subset X \times X$.
- $S := \{(A, v, \phi) \in X_n \mid A^n = 0 \text{ and } \phi(A^i v) = 0 \text{ for any } 0 \leq i \leq n\}$.

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$$\check{S} := \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in X \times X \mid \forall i, j \in \{1, 2\}$$
$$(A_i, v_j, \phi_j) \in S \text{ and } \forall \alpha \in \mathrm{gl}(V), \alpha(A_1, v_1, \phi_1) \bot (A_2, v_2, \phi_2)\}$$

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- Note that

$$\check{S} = \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in X \times X \mid \forall i, j \in \{1, 2\}$$
$$(A_i, v_j, \phi_j) \in S \text{ and } [A_1, A_2] + v_1 \otimes \phi_2 - v_2 \otimes \phi_1 = 0\}.$$

- $\check{S}' := \check{S} - \check{C}_{X \times X}$.
- $\Gamma := \{(v, \phi) \in V \times V^* \mid \phi(V) = 0\}$.
- For any $\lambda \in F$ we define $\nu_\lambda : X \to X$ by $\nu_\lambda(A, v, \phi) := (A + \lambda v \otimes \phi - \lambda \frac{\langle \phi, v \rangle}{n} \mathrm{Id}, v, \phi)$.
- It defines $\check{\nu}_\lambda : X \times X \to X \times X$. It is given by

$$\check{\nu}_\lambda((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) =$$
$$= ((A_1 + \lambda v_1 \otimes \phi_1 - \lambda \frac{\langle \phi_1, v_1 \rangle}{n} \mathrm{Id}, v_1, \phi_1), (A_2, v_2 - \lambda A_2 v_1, \phi_2 - \lambda A_2^* \phi_1)).$$

3. Harish-Chandra descent

3.1. Linearization

In this subsection we reduce Theorem A to the following one

Theorem 3.1.1. $\mathcal{S}^*(X(F))^{\widetilde{G}(F), \chi} = 0$.

    We will divide this reduction to several propositions.

Proposition 3.1.2. *If* $\mathcal{D}(G_{n+1}(F))^{\widetilde{G}_n(F), \chi} = 0$ *then Theorem A holds.*

    The proof is straightforward.

Proposition 3.1.3. *If* $\mathcal{S}^*(G_{n+1}(F))^{\widetilde{G}_n(F), \chi} = 0$ *then* $\mathcal{D}(G_{n+1}(F))^{\widetilde{G}_n(F), \chi} = 0$.

    Follows from Theorem 2.2.5.

Proposition 3.1.4. *If* $\mathcal{S}^*(\mathfrak{g}_{n+1}(F))^{\widetilde{G}_n(F), \chi} = 0$ *then* $\mathcal{S}^*(G_{n+1}(F))^{\widetilde{G}_n(F), \chi} = 0$.

Proof. Let $\xi \in \mathcal{S}^*(G_{n+1}(F))^{\widetilde{G}_n(F), \chi}$. We have to prove $\xi = 0$. Assume the contrary. Take $p \in \mathrm{Supp}(\xi)$. Let $t = \det(p)$. Let $f \in \mathcal{S}(F)$ be such that f vanishes in a neighborhood of 0 and $f(t) \neq 0$. Consider the determinant map $\det : G_{n+1}(F) \to F$. Consider $\xi' := (f \circ \det) \cdot \xi$. It is easy to check that $\xi' \in \mathcal{S}^*(G_{n+1}(F))^{\widetilde{G}_n(F), \chi}$ and $p \in \mathrm{Supp}(\xi')$. However, we can extend ξ' by zero to $\xi'' \in \mathcal{S}^*(\mathfrak{g}_{n+1}(F))^{\widetilde{G}_n(F), \chi}$, which is zero by the assumption. Hence ξ' is also zero. Contradiction. $\square$

Proposition 3.1.5. *If $\mathcal{S}^*(X_n(F))^{\widetilde{G}_n(F),\chi} = 0$ then $\mathcal{S}^*(\mathfrak{g}_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$.*

Proof. The $\widetilde{G}_n(F)$-space $\mathrm{gl}_{n+1}(F)$ is isomorphic to $X_n(F) \times F \times F$ with trivial action on $F \times F$. This isomorphism is given by

$$\begin{pmatrix} A_{n\times n} & v_{n\times 1} \\ \phi_{1\times n} & \lambda \end{pmatrix} \mapsto ((A - \frac{\mathrm{Tr}\,A}{n}\mathrm{Id}, v, \phi), \lambda, \mathrm{Tr}\,A).$$

$\square$

3.2. Harish-Chandra descent

Now we start to prove Theorem 3.1.1. The proof is by induction on n. Till the end of the paper we will assume that Theorem 3.1.1 holds for all $k < n$ for both archimedean local fields.

The theorem obviously holds for $n = 0$. Thus from now on we assume $n \geq 1$. The goal of this subsection is to prove the following theorem.

Proposition 3.2.1. $\mathcal{S}^*(X(F) - S(F))^{\widetilde{G}(F),\chi} = 0$.

In fact, one can prove this theorem directly using Theorem 2.2.15. However, this will require long computations. Thus, we will divide the proof to several steps and use some tricks to avoid part of those computations.

Proposition 3.2.2. $\mathcal{S}^*(X(F) - (\mathcal{N} \times V \times V^*)(F))^{\widetilde{G}(F),\chi} = 0$.

Proof. By Theorem 2.2.15 it is enough to prove that for any semi-simple $A \in \mathrm{sl}(V)$ we have

$$\mathcal{S}^*((N_{GA,A}^{\mathrm{sl}(V)} \times (V \times V^*))(F))^{\widetilde{G}(F)_A,\chi} = 0.$$

Now note that $\widetilde{G}(F)_A \cong \prod \widetilde{G}_{n_i}(F_i)$ where $n_i < n$ and F_i are some field extensions of F. Note also that

$$(N_{GA,A}^{\mathrm{sl}(V)} \times V \times V^*)(F) \cong \mathrm{sl}(V)_A \times (V \times V^*)(F) \cong \prod X_{n_i}(F_i) \times \mathcal{Z}(\mathrm{sl}(V)_A)(F),$$

where $\mathcal{Z}(\mathrm{sl}(V)_A)$ is the center of $\mathrm{sl}(V)_A$. Clearly, $\widetilde{G}_A$ acts trivially on $\mathcal{Z}(\mathrm{sl}(V)_A)$.

Now by Proposition 2.2.7 the induction hypothesis implies that

$$\mathcal{S}^*(\prod X_{n_i}(F_i) \times \mathcal{Z}(\mathrm{sl}(V)_A)(F))^{\prod \widetilde{G}_{n_i}(F_i),\chi} = 0.$$

$\square$

In the same way we obtain the following proposition.

Proposition 3.2.3. $\mathcal{S}^*(X(F) - (\mathrm{sl}(V) \times \Gamma)(F))^{\widetilde{G}(F),\chi} = 0$.

Corollary 3.2.4. $\mathcal{S}^*(X(F) - (\mathcal{N} \times \Gamma)(F))^{\widetilde{G}(F),\chi} = 0$.

Lemma 3.2.5. *Let $A \in \mathrm{sl}(V)$, $v \in V$ and $\phi \in V^*$. Suppose $A + \lambda v \otimes \phi$ is nilpotent for all $\lambda \in F$. Then $\phi(A^i v) = 0$ for any $i \geq 0$.*

Proof. Since $A + \lambda v \otimes \phi$ is nilpotent, we have $tr(A + \lambda v \otimes \phi)^k = 0$ for any $k \geq 0$ and $\lambda \in F$. By induction on i this implies that $\phi(A^i v) = 0$. $\square$

Proof of Theorem 3.2.1. By the previous lemma, $\bigcap_{\lambda \in F} \nu_\lambda(\mathcal{N} \times \Gamma) \subset S$. Hence $\bigcup_{\lambda \in F} \nu_\lambda(X - \mathcal{N} \times \Gamma) \supset X - S$.

By Corollary 3.2.4 $\mathcal{S}^*(X(F) - (\mathcal{N} \times \Gamma)(F))^{\widetilde{G}(F), \chi} = 0$. Note that ν_λ commutes with the action of $\widetilde{G}(F)$. Thus $\mathcal{S}^*(\nu_\lambda(X(F) - (\mathcal{N} \times \Gamma)(F)))^{\widetilde{G}(F), \chi} = 0$ and hence $\mathcal{S}^*(X(F) - S(F))^{\widetilde{G}(F), \chi} = 0$. $\qquad \square$

4. Reduction to the geometric statement

In this section coisotropic variety means $X \times X$-coisotropic variety.

The goal of this section is to reduce Theorem 3.1.1 to the following statement, which is purely geometric and involves no distributions.

Theorem 4.0.1 (geometric statement). *For any coisotropic subvariety of $T \subset \check{S}$ we have $T \subset \check{C}_{X \times X}$.*

Till the end of this section we will assume the geometric statement.

Proposition 4.0.2. *Let $\xi \in \mathcal{S}^*(X(F))^{\widetilde{G}(F), \chi} = 0$. Then $\mathrm{Supp}(\xi) \subset (\mathrm{sl}(V) \times C)(F)$.*

Proof for the case $F = \mathbb{R}$.
Step 1. $SS(\xi) \subset \check{S}$.
We know that

$$\mathrm{Supp}(\xi), \mathrm{Supp}(\mathcal{F}_{\mathrm{sl}(V)}^{-1} \xi), \mathrm{Supp}(\mathcal{F}_{V \times V^*}^{-1}(\xi)), \mathrm{Supp}(\mathcal{F}_X^{-1}(\xi)) \subset S(F).$$

By Fact 2.3.10 this implies that

$$SS(\xi) \subset (S \times X) \cap F_{\mathrm{sl}(V)}(S \times X) \cap F_{V \times V^*}(S \times X) \cap F_X(S \times X).$$

On the other hand, ξ is $G(F)$-invariant and hence by Fact 2.3.9

$$SS(\xi) \subset \{((x_1, x_2) \in X \times X \mid \forall g \in \mathfrak{g}, g(x_1) \perp x_2\}.$$

Thus $SS(\xi) \subset \check{S}$.
Step 2. $SS(\xi) \subset \check{C}_{X \times X}$.
By Corollary 2.3.6, $SS(\xi)$ is $X \times X$-coisotropic and hence by the geometric statement $SS(\xi) \subset \check{C}_{X \times X}$.
Step 3. $\mathrm{Supp}(\xi) \subset (\mathrm{sl}(V) \times C)(F)$.
Follows from the previous step by Fact 2.3.8. $\qquad \square$

The case $F = \mathbb{C}$ is proven in the same way using the following corollary of the geometric statement.

Proposition 4.0.3. *Any $(X \times X)_{\mathbb{C}}$-coisotropic subvariety of $\check{S}_{\mathbb{C}}$ is contained in $(\check{C}_{X \times X})_{\mathbb{C}}$.*

Now it is left to prove the following proposition.

Proposition 4.0.4. *Let $\xi \in \mathcal{S}^*(X(F))^{\widetilde{G}(F),\chi}$ be such that*

$$\mathrm{Supp}(\xi), \mathrm{Supp}(\mathcal{F}_{V \times V^*}(\xi)) \subset (\mathrm{sl}(V) \times C)(F).$$

Then $\xi = 0$.

4.1. Proof of proposition 4.0.4

Proposition 4.0.4 follows from the following lemma.

Lemma 4.1.1. *Let $F^\times$ act on $V \times V^*$ by $\lambda(v,\phi) := (\lambda v, \frac{\phi}{\lambda})$. Let $\xi \in \mathcal{S}^*((V \times V^*)(F))^{F^\times}$ be such that*

$$\mathrm{Supp}(\xi), \mathrm{Supp}(\mathcal{F}_{V \times V^*}(\xi)) \subset C(F).$$

Then $\xi = 0$.

By Homogeneity Theorem (Theorem 2.2.13) it is enough to prove the following lemma.

Lemma 4.1.2. *Let μ be a character of $F^\times$ given by $|\cdot|_F^n u$ or $|\cdot|_F^{n+1} u$ where u is some unitary character. Let $F^\times \times F^\times$ act on $V \times V^*$ by $(x,y)(v,\phi) = (\frac{y}{x} v, \frac{1}{xy} \phi)$. Then $\mathcal{S}^*_{(V \times V^*)(F)}(C(F))^{F^\times \times F^\times, \mu \times 1} = 0$.*

By Proposition 2.2.6 this lemma follows from the following one.

Lemma 4.1.3. *For any $k \geq 0$ we have*
(i) $\mathcal{S}^*(((V-0) \times 0)(F), Sym^k(CN_{(V-0)\times 0}^{V \times V^*}(F)))^{F^\times \times F^\times, \mu \times 1} = 0$.
(ii) $\mathcal{S}^*((0 \times (V^*-0))(F), Sym^k(CN_{0 \times (V^*-0)}^{V \times V^*}(F)))^{F^\times \times F^\times, \mu \times 1} = 0$.
(iii) $\mathcal{S}^*(0, Sym^k(CN_0^{V \times V^*}(F)))^{F^\times \times F^\times, \mu \times 1} = 0$.

Proof.
(i) Cover $V-0$ by standard affine open sets $V_i := \{x_i \neq 0\}$. It is enough to show that $\mathcal{S}^*((V_i \times 0)(F), Sym^k(CN_{(V_i \times 0)(F)}^{V \times V^*}(F)))^{F^\times \times F^\times, \mu \times 1} = 0$.

Note that V_i is isomorphic as an $F^\times \times F^\times$-manifold to $F^{n-1} \times F^\times$ with the action given by $(x,y)(v,\alpha) = (v, \frac{y}{x}\alpha)$. Note also that the bundle $Sym^k(CN_{(V_i \times 0)(F)}^{V \times V^*}(F))$ is a constant bundle with fiber $Sym^k(V)$.

Hence by Proposition 2.2.7 it is enough to show that $\mathcal{S}^*(F^\times, Sym^k(V))^{F^\times \times F^\times, \mu \times 1} = 0$. Let $H := (F^\times \times F^\times)_1 = \{(t,t) \in F^\times \times F^\times\}$. Now by Frobenius reciprocity (Theorem 2.2.8) it is enough to show that $(Sym^k(V^*(F)) \otimes_{\mathbb{R}} \mathbb{C})^{H, \mu \times 1|_H} = 0$. This is clear since (t,t) acts on $(Sym^k(V^*(F))$ by multiplication by t^{-2k}.
(ii) is proven in the same way.
(iii) is equivalent to the statement $((Sym^k(V \times V^*)(F)) \otimes_{\mathbb{R}} \mathbb{C})^{F^\times \times F^\times, \mu \times 1} = 0$. This is clear since $(t,1)$ acts on $Sym^k(V \times V^*)(F)$ by multiplication by t^{-k}. $\qquad \square$

5. Proof of the geometric statement

5.1. Preliminaries on coisotropic subvarieties

Proposition 5.1.1. *Let M be a smooth algebraic variety with a symplectic form on it. Let $R \subset M$ be an algebraic subvariety. Then there exists a maximal M-coisotropic subvariety of R i.e. an M-coisotropic subvariety $T \subset M$ that includes all M-coisotropic subvarieties of R.*

Proof. Let T' be the union of all smooth M-coisotropic subvarieties of R. Let T be the Zariski closure of T' in R. Clearly, T includes all M-coisotropic subvarieties of R. Let U denote the set of regular points of T. Clearly $U \cap T'$ is dense in U. On the other hand, for any $x \in U \cap T'$, the tangent space to T at x is coisotropic. Hence T is coisotropic. $\square$

Remark 5.1.2. *Suppose M is affine. Then T can be computed explicitly in the following way. Let I be the ideal of regular functions that vanish on $\overline{R}$. We can iteratively close it with respect to Poisson brackets and taking radical. Since $\mathcal{O}(M)$ is Noetherian, this process will stabilize. Let J denote the obtained closure and $Z(J)$ denote the zero set of J. Then $T = Z(J) \cap R$.*

The following lemma is trivial.

Lemma 5.1.3. *Let M be a smooth algebraic variety and ω be a symplectic form on it. Let a group G act on M preserving ω. Let S be a G-invariant subvariety. Then the maximal M-coisotropic subvariety of S is also G-invariant.*

Definition 5.1.4. *Let Y be a smooth algebraic variety. Let $Z \subset Y$ be a smooth subvariety and $R \subset T^*Y$ be any subvariety. We define* **the restriction $R|_Z \subset T^*Z$ of R to Z** *in the following way. Let $R' = p_Y^{-1}(Z) \cap R$. Let $q : p_Y^{-1}(Z) \to T^*Z$ be the projection. We define $R|_Z := q(R')$.*

Lemma 5.1.5. *Let Y be a smooth algebraic variety. Let $Z \subset Y$ be a smooth subvariety and $R \subset T^*Y$ be a coisotropic subvariety. Assume that any smooth point $z \in p_Y^{-1}(Z) \cap R$ is also a smooth point of R and we have $T_z(p_Y^{-1}(Z) \cap R) = T_z(p_Y^{-1}(Z)) \cap T_z R$.*
Then $R|_Z$ is T^*Z coisotropic.

In the proof we will use the following straightforward lemma.

Lemma 5.1.6. *Let W be a linear space. Let $L \subset W$ be a linear subspace and $R \subset W \oplus W^*$ be a coisotropic subspace. Then $R|_L$ is $L \oplus L^*$ coisotropic.*

Proof of lemma 5.1.5. Without loss of generality we assume that R is irreducible. Let $R' = p_Y^{-1}(Z) \cap R$. Without loss of generality we assume that R' is irreducible. Let R'' be the set of smooth points of R'. Let $q : p_Y^{-1}(Z) \to T^*Z$ be the projection. Let R''' be the set of smooth points in $q(R'')$. Clearly R''' is dense in $R|_Z$. Hence it is enough to prove that for any $x \in R'''$ the space $T_x(R|_Z)$ is coisotropic. Let $y \in R''$ s.t. $x = q(y)$. Denote $W := T_{p_Y(y)}Y$. Let

$Q := T_y R \subset W \oplus W^*$. Let $L := T_{p_Y(y)} Z$. By the assumption $T_x(R|_Z) \supset Q|_L$. By the lemma, $Q|_L$ is coisotropic and hence $T_x(R|_Z)$ is also coisotropic. $\quad\square$

Corollary 5.1.7. *Let Y be a smooth algebraic variety. Let an algebraic group H act on Y. Let $q : Y \to B$ be an H-equivariant morphism. Let $O \subset B$ be an orbit. Consider the natural action of G on T^*Y and let $R \subset T^*Y$ be an H-invariant subvariety. Suppose that $p_Y(R) \subset q^{-1}(O)$. Let $x \in O$. Denote $Y_x := q^{-1}(x)$. Then*
  – if R is T^*Y-coisotropic then $R|_{Y_x}$ is $T^*(Y_x)$-coisotropic.

Corollary 5.1.8. *In the notation of the previous corollary, if $R|_{Y_x}$ has no (non-empty) $T^*(Y_x)$-coisotropic subvarieties then R has no (non-empty) $T^*(Y)$-coisotropic subvarieties.*

Note that the converse statement does not hold in general.

5.2. Reduction to the Key Proposition

In this subsection coisotropic variety means $X \times X$-coisotropic variety.
We will use the following notation.

Notation 5.2.1.
(i) For any nilpotent operator $A \in \mathrm{sl}(V)$ we denote

$$Q_A := \{(v, \phi) \in V \times V^* \mid v \otimes \phi \in [A, \mathfrak{g}]\} = \{(v, \phi) \in V \times V^* \mid (v \otimes \phi) \perp \mathfrak{g}_A\}.$$

(ii) Denote by T the maximal coisotropic subvariety of $\check{S}'$.
(iii) For any two nilpotent orbits $O_1, O_2 \subset N$ denote

$$U(O_1, O_2) := \{(A_1, v_1, \phi_1, A_2, v_2, \phi_2) \in X \times X \mid \forall i, j \in \{1, 2\}$$
$$A_i \in O_i, (v_j, \phi_j) \in Q_{A_i}, [A_1, A_2] + v_1 \otimes \phi_2 - v_2 \otimes \phi_1 = 0, (v_1, \phi_1, v_2, \phi_2) \notin \check{C}\}.$$

The geometric statement is equivalent to the following theorem

Theorem 5.2.2. $T = \emptyset$.

The goal of this subsection is to reduce the geometric statement to the following Key Proposition.

Proposition 5.2.3 (Key Proposition). *For any two nilpotent orbits O_1, O_2 there are no (non-empty) coisotropic subvarieties in $U(O_1, O_2)$.*

The reduction will be in the spirit of the beginning of section 3 in [AGRS].

Notation 5.2.4. *Let*

$$\mathcal{N}^i = \{(A_1, A_2) \in \mathcal{N} \times \mathcal{N} \mid \dim G(A_1) + \dim G(A_2) \le i\}.$$

$$\widehat{\mathcal{N}^i} := \{(A_1, v_1, \phi_1, A_2, v_2, \phi_2) \in \check{S}' \mid (A_1, A_2) \in \mathcal{N}^i\}.$$

We will prove by descending induction that $T \subset \widehat{\mathcal{N}^i}$. From now on we fix i, suppose that this holds for i and prove that holds for $i-1$. Let $\mathfrak{S}$ denote the subgroup of automorphisms of $X \times X$ generated by $\check{\nu}_\lambda$, $F_{\mathrm{sl}(V)}$ and $F_{V \times V^*}$.

Denote $\widetilde{\mathcal{N}^i} := \bigcap_{\nu \in \mathfrak{S}} \nu(\widehat{\mathcal{N}^i})$. We know that $T \subset \widehat{\mathcal{N}^i}$, and hence $T \subset \widetilde{\mathcal{N}^i}$. Let $U^i := \widetilde{\mathcal{N}^i} - \widehat{\mathcal{N}^{i-1}}$. It is enough to show that U^i does not have (non-empty) coisotropic subvarieties.

Notation 5.2.5. *Let O_1, O_2 be nilpotent orbits such that $\dim O_1 + \dim O_2 = i$. Denote $U'(O_1, O_2) := \{(A_1, v_1, \phi_1, A_2, v_2, \phi_2) \in U^i | A_1 \in O_1,\ A_2 \in O_2\}$.*

Since the sets $U'(O_1, O_2)$ form an open cover of U^i, it is enough to show that each $U'(O_1, O_2)$ does not have (non-empty) coisotropic subvarieties. This fact clearly follows from the Key Proposition using the following easy lemma.

Lemma 5.2.6. $U'(O_1, O_2) \subset U(O_1, O_2)$.

5.3. Reduction to the Key Lemma

We will use the following notation

Notation 5.3.1.

$$R_A := \{(v_1, \phi_1, v_2, \phi_2) \in Q_A \times Q_A - \check{C} \,|$$
$$\exists B \in [A, \mathfrak{g}] \cap \mathcal{N} \text{ such that } [A, B] + v_1 \otimes \phi_2 - v_2 \otimes \phi_1 = 0\}.$$

The goal of this subsection is to reduce the Key Proposition to the following Key Lemma.

Lemma 5.3.2 (Key Lemma). R_A *does not have (non-empty) $V \times V^* \times V \times V^*$-coisotropic subvarieties.*

Notation 5.3.3. *Denote*

$$U''(O_1, O_2) := \{(A_1, v_1, \phi_1, A_2, v_2, \phi_2) \in U(O_1, O_2) | \mathfrak{g}_{A_1} \perp \mathfrak{g}_{A_2}\}.$$

Lemma 5.3.4. *Any $X \times X$-coisotropic subvariety of $U(O_1, O_2)$ lies in $U''(O_1, O_2)$.*

Proof. Denote $M := O_1 \times V \times V^* \times O_2 \times V \times V^*$. Note that $U(O_1, O_2) \subset M$. Note that any coisotropic subvariety of M is contained in $M' := \{(A_1, v_1, \phi_1, A_2, v_2, \phi_2) \in M \,|\, \mathfrak{g}_{A_1} \perp \mathfrak{g}_{A_2}\}$. Hence any coisotropic subvariety of $U(O_1, O_2)$ is contained in $U(O_1, O_2) \cap M'$. $\square$

The following straightforward lemma together with Corollary 5.1.8 finish the reduction.

Lemma 5.3.5. $U''(O_1, O_2)|_{A \times V \times V^*} \subset R_A$.

5.4. Proof of the Key Lemma

We will first give a short description of the proof for the case when A is one Jordan block. Then we will present the proof in the general case.

During the whole subsection coisotropic variety means $V \times V^* \times V \times V^*$-coisotropic variety.

5.4.1. Proof in the case when A is one Jordan block.
In this case $Q_A = \bigcup_{i=0}^n (Ker A^i) \times (Ker(A^*)^{n-i})$. Hence

$$Q_A \times Q_A = \bigcup_{i,j=0}^n (Ker A^i) \times (Ker(A^*)^{n-i}) \times (Ker A^j) \times (Ker(A^*)^{n-j}).$$

Denote $L_{ij} := (Ker A^i) \times (Ker(A^*)^{n-i}) \times (Ker A^j) \times (Ker(A^*)^{n-j})$.

It is easy to see that any coisotropic subvariety of $Q_A \times Q_A$ is contained in $\bigcup_{i=0}^n L_{ii}$. Hence it is enough to show that for any i, $\dim R_A \cap L_{ii} < 2n$. For $i = 0, n$ it is clear since $R_A \cap L_{ii}$ is empty. So we will assume $0 < i < n$.

Let $f \in \mathcal{O}(L_{ii})$ be the polynomial defined by $f(v_1, \phi_1, v_2, \phi_2) := (v_1)_i(\phi_2)_{i+1} - (v_2)_i(\phi_1)_{i+1}$, where $(\cdot)_i$ means the i-th coordinate. It is enough to show that $f(R_A \cap L_{ii}) = \{0\}$.

Let $(v_1, \phi_1, v_2, \phi_2) \in L_{ii}$. Let $M := v_1 \otimes \phi_2 - v_2 \otimes \phi_1$. Clearly, M is of the form

$$M = \begin{pmatrix} 0_{i \times i} & * \\ 0_{(n-i) \times i} & 0_{(n-i) \times (n-i)} \end{pmatrix}.$$

Note also that $M_{i,i+1} = f(v_1, \phi_1, v_2, \phi_2)$.

It is easy to see that any B satisfying $[A, B] = M$ is upper triangular. On the other hand, we know that there exists a nilpotent B satisfying $[A, B] = M$. Hence this B is upper nilpotent, which implies $M_{i,i+1} = 0$ and hence $f(v_1, \phi_1, v_2, \phi_2) = 0$.

5.4.2. Notation on filtrations.

Notation 5.4.1.
(i) Let L be a vector space with a gradation $G^i L$. It defines a filtration $G^{\geq i} L$ by $G^{\geq i} L := \bigoplus_{j \geq i} G^j L$.
(ii) Let L be a vector space with a descending filtration $F^{\geq i}$. We define $F^{>i} L := \bigcup_{j>i} F^{\geq j} L$.

Notation 5.4.2. *Let L and M be vector spaces with descending filtrations $F^{\geq i} L$ and $F^{\geq i} M$.*
Define filtrations $F^{\geq i}(L \otimes M) := \sum_{k+l=i} F^{\geq k} L \otimes F^{\geq l} M$ and $F^{\geq i}(L^*) := (F^{>-i} L)^\perp$.
Similarly for gradations $G^i L$ and $G^i M$ we define gradations $G^i(L \oplus M) := \bigoplus_{k+l=i} G^k L \otimes G^l M$ and $G^i(L^*) := (\bigoplus_{j \neq -i} G^j L)^\perp$.

We fix a standard basis $\{E, H, F\}$ of sl_2.

Notation 5.4.3. *Let L be a representation of sl_2. We define*
- *A gradation $W^\alpha(L) := Ker(H - \alpha Id)$ and*
- *An ascending filtration $K_i(L) := Ker(E^i)$.*

Note that if L is an irreducible representation then $K_i(L) = W^{\geq dim L + 1 - 2i}(L)$.

5.4.3. Proof of the Key Lemma.

We will cover R_A by linear spaces and show that every one of them does not include coisotropic subvarieties of R_A.

Fix a morphism of Lie algebras $\psi : \mathrm{sl}_2 \to \mathrm{sl}(V)$ such that $\psi(E) = A$. Decompose V to irreducible representations of sl_2: $V = \bigoplus_{i=1}^{k} V_i$ such that $dim V_i \geq \dim V_{i+1}$.

Notation 5.4.4. *Denote $D_i := \dim V_i$. Let D denote the multiindex $D :=$ $(D_1, ..., D_k)$.*

For any multiindex $I = (I_1, ..., I_k)$ such that $0 \leq I_l \leq D_l$, $I \neq 0$ and $I \neq D$ we define

- *$L_I := W^{\geq D_1 + 1 - 2I_1}(V_1) \oplus ... \oplus W^{\geq D_k + 1 - 2I_k}(V_k) = K_{I_1}(V_1) \oplus ... \oplus K_{I_k}(V_k)$*
- *$L'_I := W^{\geq D_1 + 1 - 2I_1}(V_1^*) \oplus ... \oplus W^{\geq D_k + 1 - 2I_k}(V_k^*) = K_{I_1}(V_1^*) \oplus ... \oplus K_{I_k}(V_k^*)$*
- *$L_{IJ} := L_I \times L'_{D-I} \times L_J \times L'_{D-J}$*

The following two lemmas are straightforward

Lemma 5.4.5.
$$R_A \subset \bigcup_{I,J} L_{IJ}$$

Lemma 5.4.6. *L_{IJ} is not coisotropic if $I \neq J$.*

Hence it is enough to prove the following proposition.

Proposition 5.4.7. $\dim L_{II} \cap R_A < 2n$.

From here on we fix I and suppose that the proposition does not hold for this I. Our aim now is to get a contradiction. Note that if Proposition 5.4.7 holds for I then it holds for $D - I$. Hence without loss of generality we can (and will) assume $I_k < D_k$.

Lemma 5.4.8. *For any $m < l$ we have $D_m - D_l \geq I_m - I_l \geq 0$.*

Before we prove this lemma we introduce some notation.
We fix a Jordan basis of A in each V_i.

Notation 5.4.9. *For any $v \in V, \phi \in V^*, X \in V \otimes V^*$ we define v^l to be the l-th component of v with respect to the decomposition $V = \oplus V_l$ and v_α^l to be its α coordinate.*

Similarly we define $\phi^l, \phi_\alpha^l, X^{lm}, X_{\alpha,\beta}^{lm}$

Proof of lemma 5.4.8. It is enough to prove that for any l, m we have $I_l + (D_m - I_m) \leq max(D_l, D_m)$. Assume that the contrary holds for some l, m. It is enough to show that in this case $\dim(Q_A \cap (L_I \times L'_{D-I})) < n$. Consider the function $g \in \mathcal{O}(L_I \times L'_{D-I})$ defined by $g(v, \phi) = \phi_{I_m+1}^m \cdot v_{I_l}^l$. It is enough to show that $g(Q_A \cap (L_I \times L_{D-I})) = \{0\}$.

Let $B \in V_m \otimes V_l^*$ be defined by $B_{\alpha,\beta} = \delta_{\alpha-\beta, I_m - I_l + 1}$. Consider B as an element of $\mathfrak{g}$. Note that $B \in \mathfrak{g}_A$ and $\langle B, v \otimes \phi \rangle = g(v, \phi)$ for any $(v, \phi) \in L_I \times L'_{D-I}$. Hence $g(Q_A \cap (L_I \times L_{D-I})) = \{0\}$. $\square$

Corollary 5.4.10.
(i) If $I_m = 0$ then $I_l = 0$ for any $l > m$.
(ii) If $I_m = D_m$ then $I_l = D_l$ for any $l > m$.

Corollary 5.4.11. $D_1 > I_1 > 0$.

Notation 5.4.12. *Let k' be the maximal index such that $D_{k'} > I_{k'} > 0$.*

Notation 5.4.13. *Define $f_l \in \mathcal{O}(V \times V^* \times V \times V^*)$ by*

$$f_l(v_1, \phi_1, v_2, \phi_2) := (v_1)^l_{I_l} (\phi_2)^l_{I_l+1} - (v_2)^l_{I_l} (\phi_1)^l_{I_l+1}.$$

Define also $f := \sum_{l=1}^{k'} \frac{D_l - I_l}{D_l} f_l$.

Now it is enough to prove the following proposition.

Proposition 5.4.14.
$$f(R_A \cap L_{II}) = \{0\}.$$

We will need several notations and straightforward lemmas.

Lemma 5.4.15. *For any $\alpha \leq |D_m - D_l|$ we have $W^{\geq \alpha}(V_l \otimes V_m^*) = \{X \in V_l \otimes V_m^* | E(X) \in W^{\geq \alpha+2}(V_l \otimes V_m^*)\}$.*

Definition 5.4.16. *Define gradation W_I^i on V_l by $W_I^i(V_l) = W^{i+(D_l+l-2I_l)}(V_l)$. It gives rise to gradations W_I^i on $V_l^*, V_m \otimes V_l^*, V, V^*$.*

Lemma 5.4.17.
(i) If i is odd then $W_I^i = 0$.
(ii) $W_I^{\geq 0}(V) = L_I$.
(iii) $W_I^{\geq 2}(V^*) = L'_{D-I}$.

Definition 5.4.18. *Let $\mathcal{A}$ be the algebra $W_I^{\geq 0}(V \otimes V^*)$ and $\mathcal{I}$ be its ideal $W_I^{>0}(V \otimes V^*) = W_I^{\geq 2}(V \otimes V^*)$. Clearly $\mathcal{A}/\mathcal{I} \cong \prod End(W_I^i(V))$. This gives rise to a homomorphism $\varepsilon : \mathcal{A} \to End(W_I^0(V))$.*

Lemma 5.4.19.
(i) $\mathcal{A} = \bigoplus_{1 \leq l,m \leq k} W^{\geq D_l - D_m - 2(I_l - I_m)}(V_l \otimes V_m^*)$.
(ii) $\mathcal{I} := \bigoplus_{1 \leq l,m \leq k} W^{\geq D_l - D_m - 2(I_l - I_m)+2}(V_l \otimes V_m^*)$.
(iii) $dim(W_I^0(V)) = k'$
(iv) Consider the basis on $W_I^0(V)$ corresponding to the one on V and identify $End(W_I^0(V))$ with $gl(k')$. Then

$$\varepsilon(X)_{lm} := X^{lm}_{I_l, I_m}.$$

Corollary 5.4.20. $\mathcal{A} = \{X \in End(V) | [A, X] \in \mathcal{I}\}$.

Proof. Follows from the previous lemma using Lemma 5.4.15. $\square$

Proof of Proposition 5.4.14. Let $(v_1, \phi_1, v_2, \phi_2) \in L_{II} \cap R_A$. Let $M := v_1 \otimes \phi_2 - v_2 \otimes \phi_1$. We know that there exists a nilpotent matrix $B \in [A, \mathrm{gl}(V)]$ such that $[A, B] = M$. By Corollary 5.4.20 $B \in \mathcal{A}$. Denote $\Delta := \varepsilon(B)$. Fix $1 \leq l \leq k'$. Denote $a_l := M_{I_l, I_l+1}^{ll}$. Note that $[A_l, B^{ll}] = M^{ll}$. Hence $B_{ll}^{11} = ... = B_{ll}^{I_l, I_l} = \Delta_{ll} = B_{ll}^{I_l+1, I_l+1} - a_l = ... = B_{ll}^{D_l, D_l} - a_l$. Since $B \in [A, End(V)]$ we have $tr(B_{ll}) = 0$. Thus $\Delta_{ll} = \frac{D_l - I_l}{D_l} a_l$.

Since B is nilpotent Δ is nilpotent. Hence $tr(\Delta) = 0$ and thus $\sum_{l=1}^{k'} \frac{D_l - I_l}{D_l} a_l = 0$ which means $f(v_1, \phi_1, v_2, \phi_2) = 0$. $\qquad\square$

Appendix A. Theorem A implies Theorem B

This appendix is analogous to section 1 in [AGRS]. There, the classical theory of Gelfand and Kazhdan (see [GK]) is used. Here we use an archimedean analog of this theory which is described in [AGS], section 2. We work in the notations of [AGS]. In particular, what we call a smooth Fréchet representation is sometimes called in the literature smooth Fréchet representation of moderate growth (see e.g. [Wal]).

We will also use the theory of nuclear Fréchet spaces. For a good brief survey on this theory we refer the reader to [CHM], Appendix A.

Notation A.0.1.
(i) For a smooth Fréchet representation π of a real reductive group we denote by $\widetilde{\pi}$ the smooth dual of π.
(ii) For a representation π of $\mathrm{GL}_n(F)$ we let $\widehat{\pi}$ be the representation of $\mathrm{GL}_n(F)$ defined by $\widehat{\pi} = \pi \circ \theta$, where θ is the (Cartan) involution $\theta(g) = g^{-1^t}$.

We will use the following theorem.

Theorem A.0.2 (Casselman - Wallach globalization). *Let G be a real reductive group. There is a canonical equivalence of categories between the category of admissible smooth Fréchet representations of G and the category of admissible $(\mathfrak{g}, K)$- modules.*

See e.g. [Wal], chapter 11.
We will also use the embedding theorem of Casselman.

Theorem A.0.3. *Any irreducible $(\mathfrak{g}, K)$-module can be imbedded into a $(\mathfrak{g}, K)$-module of principal series.*

Those two theorems have the following corollary.

Corollary A.0.4. *The underlying topological vector space of any admissible smooth Fréchet representation is a nuclear Fréchet space.*

Definition A.0.5. *Let G and H be real reductive groups. Let (π, E) and (τ, W) be admissible smooth Fréchet representations of G and H respectively. We define $\pi \otimes \tau$ to be the natural representation of $G \times H$ on the space $E \widehat{\otimes} W$.*

Proposition A.0.6. *Let G and H be real reductive groups. Let π and τ be irreducible admissible Harish-Chandra modules of G and H respectively. Then $\pi \otimes \tau$ is irreducible Harish-Chandra module of $G \times H$.*

This proposition is well known. For the benefit of the reader we include its proof in subsection A.1. An analogous proposition in the non-Archimedean case appears in [BZ, subsection 2.16], and the proof we give here is along the same lines.

Corollary A.0.7. *Let G and H be real reductive groups. Let π and τ be irreducible admissible smooth Fréchet representations of G and H respectively. Then $\pi \otimes \tau$ is an irreducible representation of $G \times H$.*

Lemma A.0.8. *Let G and H be real reductive groups. Let (π, E) and (τ, W) be admissible smooth Fréchet representations of G and H respectively. Then $Hom_{\mathbb{C}}(\pi, \tau)$ is canonically embedded to $Hom_{\mathbb{C}}(\pi \otimes \widetilde{\tau}, \mathbb{C})$.*

Proof. For a nuclear Fréchet space V we denote by V' its dual space equipped with the strong topology. Let $\widetilde{W}$ denote the underlying space of $\widetilde{\tau}$. By the theory of nuclear Fréchet spaces, we know $Hom_{\mathbb{C}}(E, W) \cong E' \widehat{\otimes} W$ and $Hom_{\mathbb{C}}(E \widehat{\otimes} \widetilde{W}, \mathbb{C}) \cong E' \widehat{\otimes} \widetilde{W}'$. The lemma follows now from the fact that W is canonically embedded to $\widetilde{W}'$. $\square$

We will use the following two archimedean analogs of theorems of Gelfand and Kazhdan.

Theorem A.0.9. *Let π be an irreducible admissible representation of $\mathrm{GL}_n(F)$. Then $\widehat{\pi}$ is isomorphic to $\widetilde{\pi}$.*

For proof see [AGS], Theorem 2.4.4.

Theorem A.0.10. *Let $H \subset G$ be real reductive groups and let σ be an involutive anti-automorphism of G and assume that $\sigma(H) = H$. Suppose $\sigma(\xi) = \xi$ for all H-bi-invariant Schwartz distributions ξ on G. Let π be an irreducible admissible smooth Fréchet representation of G. Then*

$$\dim \mathrm{Hom}_H(\pi, \mathbb{C}) \cdot \dim \mathrm{Hom}_H(\widetilde{\pi}, \mathbb{C}) \leq 1.$$

For proof see [AGS], Theorem 2.3.2.

Corollary A.0.11. *Let $H \subset G$ be real reductive groups and let σ be an involutive anti-automorphism of G such that $\sigma(H) = H$. Suppose $\sigma(\xi) = \xi$ for all Schwartz distributions ξ on G which are invariant with respect to conjugation by H.*

Let π be an irreducible admissible smooth Fréchet representation of G and τ be an irreducible admissible smooth Fréchet representation of H. Then

$$dim \, \mathrm{Hom}_H(\pi, \tau) \cdot \dim \mathrm{Hom}_H(\widetilde{\pi}, \widetilde{\tau}) \leq 1.$$

Proof. Define $\sigma' : G \times H \to G \times H$ by $\sigma'(g,h) := (\sigma(g), \sigma(h))$. Let $\Delta H <$ $G \times H$ be the diagonal. Consider the projection $G \times H \to H$. By Frobenius reciprocity (Theorem 2.2.8), the assumption implies that any ΔH-bi-invariant distribution on $G \times H$ is invariant with respect to σ'.

Hence by the previous theorem, for any irreducible admissible smooth Fréchet representation π' of $G \times H$ we have $\dim \mathrm{Hom}_{\Delta H}(\pi', \mathbb{C}) \cdot \dim \mathrm{Hom}_{\Delta H}(\widetilde{\pi'}, \mathbb{C}) \leq 1$.

Taking $\pi' := \pi \otimes \widetilde{\tau}$ we obtain the required inequality. $\qquad\square$

Corollary A.0.12. *Theorem A implies Theorem B.*

Proof. By Theorem A.0.9, $\dim \mathrm{Hom}_H(\widetilde{\pi}, \widetilde{\tau}) = \dim \mathrm{Hom}_H(\widehat{\pi}, \widehat{\tau}) = \dim \mathrm{Hom}_H(\pi, \tau)$.
$\qquad\square$

A.1. Proof of proposition A.0.6

Notation A.1.1. *Let G be a reductive group, $\mathfrak{g}$ be its Lie algebra and K be its maximal compact subgroup. Let π be an admissible $(\mathfrak{g}, K)$-module.*
Let ρ be an irreducible representation of K.
(i) We denote by $e_\rho : \pi \to \pi$ the projection to the K-type ρ.
(ii) We denote by G_ρ^π the subalgebra of $End(e_\rho(\pi))$ generated by the actions of K and $e_\rho U(\mathfrak{g}) e_\rho$.

The following lemma is well-known

Lemma A.1.2. *Let π be an irreducible admissible $(\mathfrak{g}, K)$-module. Let ρ be an irreducible representation of K. Suppose that $e_\rho(\pi) \neq 0$. Then $e_\rho(\pi)$ is an irreducible representation of G_ρ^π.*

We will also use Bernside theorem.

Theorem A.1.3. *Let V be a finite dimensional complex vector space. Let $A \subset End(V)$ be a subalgebra such that V does not have any non-trivial A-invariant subspaces. Then $A = End(V)$.*

Now we are ready to prove proposition A.0.6.

Proof of proposition A.0.6. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H. Let K and L be maximal compact subgroups of G and H. Let $\omega \subset \pi \otimes \tau$ be a nonzero $(\mathfrak{g} \times \mathfrak{h}, K \times L)$-submodule. Then ω intersects non-trivially some $K \times L$ type. Denote this type by $\rho \otimes \sigma$. By Lemma A.1.2, $e_\rho(\pi)$ is an irreducible representation of G_ρ^π and $e_\sigma(\tau)$ is an irreducible representation of H_σ^τ. Hence by Bernside theorem $G_\rho^\pi = End(e_\rho(\pi))$ and $H_\sigma^\tau = End(e_\sigma(\tau))$. Hence $(G \times H)_{\rho \otimes \sigma}^{\pi \otimes \tau} = End(e_\rho(\pi) \otimes e_\sigma(\tau))$. Thus $\omega \cap e_{\rho \otimes \sigma}(\pi \otimes \tau) = e_{\rho \otimes \sigma}(\pi \otimes \tau)$.

This means that ω contains an element of the form $v \otimes w$, which implies that $\omega = \pi \otimes \tau$. $\qquad\square$

Appendix B. D-modules

In this appendix X denotes a smooth affine variety defined over $\mathbb{R}$. All the statements of this section extend automatically to general smooth algebraic varieties defined over $\mathbb{R}$. In this paper we use only the case when X is an affine space.

Definition B.0.1. *Let $D(X)$ denote the algebra of polynomial differential operators on X. We consider the filtration $F^{\leq i}D(X)$ on $D(X)$ given by the order of differential operator.*

Definition B.0.2. *We denote by $\operatorname{Gr} D(X)$ the associated graded algebra of $D(X)$.*

Define the symbol map $\sigma : D(X) \to \operatorname{Gr} D(X)$ in the following way. Let $d \in D(X)$. Let i be the minimal index such that $d \in F^{\leq i}$. We define $\sigma(d)$ to be the image of d in $(F^{\leq i}D(X))/(F^{\leq i-1}D(X))$

Proposition B.0.3. $\operatorname{Gr} D(X) \cong \mathcal{O}(T^*X)$.

For proof see e.g. [Bor].

Notation B.0.4. *Let (V, B) be a quadratic space.*
(i) We define a morphism of algebras $\Phi_V^D : D(X \times V) \to D(X \times V)$ in the following way.

Consider B as a map $B : V \to V^*$. For any $f \in V^*$ we set $\Phi_V^D(f) := \partial_{B^{-1}(f)}$. For any $v \in V$ we set $\Phi_V^D(\partial_v) := -B(v)$ and for any $d \in D(X)$ we set $\Phi_V^D(d) := d$.
(ii) It defines a morphism of algebras $\Phi_V^O : \mathcal{O}(T^*X) \to \mathcal{O}(T^*X)$.

The following lemma is straightforward.

Lemma B.0.5. *Let f be a homogeneous polynomial. Consider it as a differential operator. Then $\sigma(\Phi_V^D(f)) = \Phi_V^O(\sigma(f))$.*

The D-modules we use in the paper are right D-modules. The difference between right and left D-modules is not essential (see e.g. section VI.3 in [Bor]). We will use the notion of good filtration on a D-module, see e.g. section II.4 in [Bor]. Let us now remind the definition of singular support of a module and a distribution.

Notation B.0.6. *Let M be a $D(X)$-module. Let $\alpha \in M$ be an element. Then we denote by $Ann_{D(X)}$ the annihilator of α.*

Definition B.0.7. *Let M be a $D(X)$-module. Choose a good filtration on M. Consider grM as a module over $\operatorname{Gr} D(X) \cong \mathcal{O}(T^*X)$. We define*

$$SS(M) := \operatorname{Supp}(\operatorname{Gr} M) \subset T^*X.$$

This does not depend on the choice of the good filtration on M (see e.g. [Bor], section II.4).

For a distribution $\xi \in S^*(X(\mathbb{R}))$ we define $SS(\xi)$ to be the singular support of the module of distributions generated by ξ.

The following proposition is trivial.

Proposition B.0.8. *Let $I < D(X)$ be a right ideal. Consider the induced filtrations on I and $D(X)/I$. Then $\mathrm{Gr}(D(X)/I) \cong \mathrm{Gr}(D(X))/\mathrm{Gr}(I)$.*

Corollary B.0.9. *Let $\xi \in S^*(X)$. Then $SS(\xi)$ is the zero set of $\mathrm{Gr}(Ann_{D(X)}\xi)$.*

Corollary B.0.10. *Let $I < \mathcal{O}(T^*X)$ be the ideal generated by $\{\sigma(d) \,|\, d \in Ann_{D(X)}(\xi)\}$. Then $SS(\xi)$ is the zero set of I.*

Corollary B.0.11. *Fact 2.3.9 holds.*

Lemma B.0.12. *Let $\xi \in S^*(X)$. Let $Z \subset X$ be a closed subvariety such that $\mathrm{Supp}(\xi) \subset Z(\mathbb{R})$. Let $f \in \mathcal{O}(X)$ be a polynomial that vanishes on Z. Then there exists $k \in \mathbb{N}$ such that $f^k \xi = 0$.*

Proof.

    Step 1. Proof for the case when X is affine space and f is a coordinate function.

This follows from the proof of Corollary 5.5.4 in [AG1].

    Step 2. Proof for the general case.

Embed X into an affine space A^N such that f will be a coordinate function and consider ξ as distribution on A^N supported in X. By Step 1, $f^k \xi = 0$ for some k. $\qquad\qquad\square$

Corollary B.0.13. *Fact 2.3.8 holds.*

Proposition B.0.14. *Fact 2.3.10 holds. Namely:*

    Let (V, B) be a quadratic space. Let $Z \subset X \times V$ be a closed subvariety, invariant with respect to homotheties in V. Suppose that $\mathrm{Supp}(\xi) \subset Z(\mathbb{R})$. Then $SS(\mathcal{F}_V(\xi)) \subset F_V(p_{X \times V}^{-1}(Z))$.

Proof. Let $f \in \mathcal{O}(X \times V)$ be homogeneous with respect to homotheties in V. Suppose that f vanishes on Z. Then $\Phi_V^D(f^k) \in Ann_{D(X)}(\mathcal{F}_V(\xi))$. Therefore $\sigma(\Phi_V^D(f^k))$ vanishes on $SS(\mathcal{F}_V(\xi))$. On the other hand, $\sigma(\Phi_V^D(f^k)) = \Phi_V^O(\sigma(f^k)) = (\Phi_V^O(\sigma(f)))^k$. Hence $SS(\mathcal{F}_V(\xi))$ is included in the zero set of $\Phi_V^O(\sigma(f))$. Intersecting over all such f we obtain the required inclusion. $\qquad\square$

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