Multiplicity One Theorems and Invariant Distributions

Thesis submitted for the degree
"Doctor of Philosophy"
by

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Submitted to the Scientific Council of the
Weizmann Institute of Science
Rehovot, Israel

Prepared at the Mathematics Department under the supervision of
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May 2009

To the memory of my grandfathers
Naum Gourevitch and David Kagan

This work was carried out under the supervision of

Professor Joseph Bernstein and
Professor Stephen Gelbart

Acknowledgements

First of all I would like to thank my advisors **Joseph Bernstein** and **Stephen Gelbart** for guiding me through this project.

I cordially thank **Joseph Bernstein** for teaching me most of the mathematics I know and for showing me his approach to mathematics and to research in general.

I would like to thank my parents, **Lena and Ilya** for teaching me their approach to life and to work and for their support in my carrier and in all decisions that I made in my life.

I am pleased to thank my wife **Anna** for her advise, help and support in life in general and in mathematics in particular. I also cordially thank our children **Mark** and **Galina** for their patience to me while I was busy working on this thesis.

I am glad to thank by best friend **Avraham Aizenbud** with whom large portion of this work was done in collaboration. We have been studding and working together since 1997. His friendship always supported me both in the darkest and in the happiest times in my life.

I would like to thank my "scientific elder brother" **Eitan Sayag** for his encouragement and guiding in the last 13 years, which is almost half of my life. I am very glad that my old wish to collaborate with Eitan was fulfilled during my PhD studies.

I cordially thank **Steve Rallis** and **Gérard Schiffmann** for releasing their ideas on the subject in their preprint [RS07]. This allowed us to write our preprint [AG07]. My first paper on this subject, [AGRS07], was a synthesis of these two preprints. I also thank **David Soudry** for showing us the preprint [RS07] just at the time we needed it.

I would like to thank **David Kazhdan** who drew my interest to this subject by suggesting us to prove Theorem C in the Archimedean case.

During the years of my PhD studies I have benefitted from talks with many people. I cannot list all of them, but I would like to thank **Moshe Baruch**, **Vladimir Berkovich**, **Patrick Delorme**, **Gerrit van Dijk**, **Yuval Flicker**, **Maria Gorelik**, **Nadia Gurevich**, **Herve Jacquet**, **David Kazhdan**, **Bernhard Kroetz**, **Erez Lapid**, **Omer Offen**, **Andrey Reznikov**, **Siddhartha Sahi**, **Yiannis Sakellaridis**, **Eugenii Shustin**, **David Soudry**, **Ilya Tyomkin**, **Yakov Varshavsky**, **Nolan Wallach**, **Wee Teck Gan** and **Oksana Yakimova** for fruitful discussions.

I would also like to thank **Semyon Alesker**, **Erez Lapid** and **Binyong Sun** for useful remarks.

I wish to thank **Vladimir Berkovich**, **Maria Gorelik**, **Dmitry Novikov** and **Sergei Yakovenko** for their encouragement, guidance and support.

Finally, I would like to thank the secretary of the pure math department, **Gizel Maymon**, and the previous secretary **Terry Debesh** for defending me from all kinds of burocracy.

During the work on this project I twice visited the Hausdorff Center of Mathematics and Max Planck Institute fur Mathematik in Bonn. An important part of this work was done during those visits. I wish to thank both institutions for inspiring environment and working conditions.

During the work on this thesis I was supported in part by a BSF grant, a GIF grant, and an ISF center of excellency grant.

Summary

Let F be a local field of characteristic zero, e.g. $F = \mathbb{R}$ or $F = \mathbb{Q}_p$.

Consider the standard imbedding $\mathrm{GL}(n, F) \subset \mathrm{GL}(n+1, F)$. Let $\mathrm{GL}(n, F)$ act on $\mathrm{GL}(n+1, F)$ by conjugation.

The main goal of this thesis was to prove the following theorem.

Theorem. *Every* $\mathrm{GL}(n, F)$*-invariant distribution on* $\mathrm{GL}(n+1, F)$ *is invariant with respect to transposition.*

We proved it in [AGRS07] for non-Archimedean F and in [AG08d] for Archimedean F. For Archimedean F it was also proven independently, simultaneously and in a different way in [SZ08]. The proof we present here is a combination of these three approaches and works uniformly for both Archimedean and non-Archimedean fields.

This theorem is important in representation theory, since it implies the following multiplicity one theorem.

Theorem. *Let* π *be an irreducible admissible representation of* $\mathrm{GL}(n+1, F)$ *and* ρ *be an irreducible admissible smooth representation of* $\mathrm{GL}(n, F)$*. Then*

$$\dim \mathrm{Hom}_{\mathrm{GL}(n,F)}(\pi|_{\mathrm{GL}(n,F)}, \rho) \leq 1.$$

Another corollary of the main theorem is the following one.

Theorem. *Let* $GL(n, F)$ *act on itself by conjugation. Let* $P(n, F) < GL(n, F)$ *be the subgroup of matrices whose last row is* $(0, \ldots, 0, 1)$*. Then any* $P(n, F)$*-invariant distribution on* $GL(n, F)$ *is* $GL(n, F)$*-invariant.*

This theorem in turn implies Kirillov's conjecture that states that any irreducible unitary representation $GL(n, F)$ remains irreducible unitary representation when restricted to $P(n, F)$. In this way Kirillov's conjecture was originally proven in [Ber84] (for non-archimedean F) and in [Bar03](for archimedean F).

Analogs of our theorems hold also for the orthogonal groups. Namely, let V be a finite dimensional vector space over F and q be any non-degenerate quadratic form on V. Extend q to $V \oplus F$ in the natural way and consider the embedding $O(V) \hookrightarrow O(V \oplus F)$. Let $O(V)$ act on $O(V \oplus F)$ by conjugation.

Theorem. *Every* $O(V)$*-invariant distribution on* $O(V \oplus F)$ *is invariant with respect to transposition. Also, for any irreducible admissible smooth representations* π *of* $O(V \oplus F)$ *and* ρ *of* $O(V)$ *we have*

$$\dim \mathrm{Hom}_{O(V)}(\pi|_{O(V)}, \rho) \leq 1.$$

For non-archimedean F this theorem proved in [AGRS07] and for archimedean F in [SZ08]. Both proofs use the theorem for $GL(n, F)$.

In our proofs for non-archimedean F we used tools dealing with invariant distributions developed in [BZ76], [Ber84] and [JR96]. For archimedean F we had to develop archimedean analogs of some of these tools (see chapter 1).

However, the archimedean case is much more difficult and we needed another, crucial, tool to solve it. This tool was the theory of D-modules. In particular, we used a theorem on the singular support of a D-module, see section 1.8. To make our proof be uniform for both kinds of local fields we use the paper [Aiz08] in which a non-Archimedean analog of this tool is developed.

Contents

Chapter 0

Introduction

One may divide the task of representation theory into two parts:

- Classify all irreducible representations of a given group G.
- Understand how a given representation of G "decomposes" into irreducible ones.

For a representation π and an irreducible representation τ one can ask with what multiplicity τ occurs in the decomposition of π. It is interesting to know what happens if we consider an irreducible representation of G as a representation of a subgroup $H \subset G$. Consider its "decomposition" to irreducible representations of H. It is said that the pair (G, H) has *multiplicity one* property if all those irreducible representations appear in the decomposition with multiplicity one. Such pairs are also called *strong Gelfand pairs*. If this property holds just for the trivial representation instead of τ then the pair (G, H) is called a *Gelfand pair*.

Those notions were introduced by I.M. Gelfand in the 50s for pairs of compact topological groups. In the 70s those notions were developed by Gelfand and Kazhdan (in [GK75]) in the realm of reductive algebraic groups over non-Archimedean local fields (like the field of p-adic numbers). We recall their definitions and techniques, as well as an adaptation to Archimedean local fields (i.e. $\mathbb{R}$ and $\mathbb{C}$) in chapter 2.

Let F be a non-Archimedean local field of characteristic 0. Consider the standard imbedding $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F)$. We consider the action of $\mathrm{GL}_n(F)$ on $\mathrm{GL}_{n+1}(F)$ by conjugation.

In this thesis we prove

Theorem A. *The pair* $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ *is a strong Gelfand pair.*

Gelfand and Kazhdan also provided criteria for proving both Gelfand properties. The criterion for strong Gelfand property is the following. Suppose that we are given an anti-involution σ of G that preserves H and preserves all distributions on

G that are invariant with respect to cojugation by elements of H. Then (G, H) is a strong Gelfand pair. By this criterion Theorem A follows from the following one:

Theorem B. *Any* $\mathrm{GL}_n(F)$ *- invariant distribution on* $\mathrm{GL}_{n+1}(F)$ *is invariant with respect to transposition.*

For me, this theorem and invariant distributions in general are of great interest on their own. Also, they have other applications in representation theory. For example, Theorem B implies the following one.

Theorem C. *Let* $P_n \subset \mathrm{GL}_n$ *be the subgroup consisting of all matrices whose last row is* $(0, ..., 0, 1)$. *Let* GL_n *act on itself by conjugation. Then every* $P_n(F)$ *- invariant distribution on* $\mathrm{GL}_n(F)$ *is* $\mathrm{GL}_n(F)$ *- invariant.*

Proof. Since $\mathrm{GL}_{n-1} < P_n$, any $P_n(F)$ - invariant distribution ξ on $\mathrm{GL}_n(F)$ is $\mathrm{GL}_{n-1}(F)$ - invariant. Hence by Theorem B ξ is transposition invariant. Since P_n and P_n^t generate GL_n, ξ is $\mathrm{GL}_n(F)$ - invariant. □

This theorem in turn implies Kirillov's conjecture that states that any irreducible unitary representation $GL(n, F)$ remains irreducible unitary representation when restricted to $P_n(F)$. In this way Kirillov's conjecture was originally proven by Bernstein in [Ber84] for non-Archimedean F and by Baruch in [Bar03] for Archimedean F.

0.1 The techniques and the structure of our proof

Over non-Archimedean fields there is a powerful tool for proving theorems on invariant distributions, due to Bernstein, Gelfand, Kazhdan and Zelevinski. It says that if an algebraic group G acts on an algebraic variety X and σ is an involutive automorphism of $X(F)$ that normalizes the action of G and preserves all $G(F)$-orbits then all $G(F)$-invariant distributions on $X(F)$ are σ-invariant. For a precise formulation see Theorem 1.3.5.

This theorem is very powerful since it reduces a statement on invariant distributions to a check of simple geometric conditions. Many theorems on invariant distributions over non-Archimedean fields were proven using this theorem.

However, in more difficult cases, including the ones under consideration in this thesis, σ preserves a majority of $G(F)$-orbits but not all of them. In such cases there actually exist $G(F)$-invariant but not σ-invariant distributions defined on some locally closed subsets of $X(F)$ and one has to prove that they cannot be invariantly continued to $X(F)$. For this one has to use some non-geometric tools, like the ones described in sections 1.7 and 1.8.

One can formulate a geometric condition which is necessary (though not sufficient) in all cases. Namely, in order to preserve $G(F)$ invariant distributions σ must preserve all **closed** $G(F)$-orbits.

In [AG08c] we formulated a sufficient condition that one has to check for all closed orbits. This condition is not geometric - it is a statement on invariant distributions on the normal space to the orbit (see section 1.4). However, the normal spaces to most orbits have dimension smaller than X so one can hope that the problem on invariant distribution on them is simpler. Also, the normal spaces are linear spaces which enables to apply additional techniques, like the Fourier transform. We called this method Generalized Harish-Chandra descent, since in the case of a reductive group acting on itself by conjugation it was developed by Harish-Chandra.

In the case of GL_n acting on GL_{n+1} by conjugation, for all closed orbits except one, the problem we get on the normal space is a product of problems of the same kind, possibly for other local fields but with smaller n. Hence when proving the main theorem by induction we may assume those problems to be solved, and there is only one problematic orbit. This orbit consists of one element - the identity. The normal space to it is the Lie algebra $\mathfrak{gl}_{n+1}(F)$. Using the fact that the problem is solved near other closed orbits, we show that any distribution ξ which is GL_n-invariant and transposition anti-invariant must be supported in the nilpotent cone. Since the Fourier transform of ξ has the same invariance properties, it also must be supported in the nilpotent cone.

Now, when we have restrictions on the support of both ξ and its Fourier transform we can apply two kinds of "uncertainty principles", that we develop in sections 1.7 and 1.8. We apply them and show that such ξ must be zero. This is the most complicated part of the proof.

0.2 Remarks

Theorem B is not proven yet for local fields of positive characteristic (like $\mathbb{F}_q(t)$). The main difficulty is that the Harish - Chandra descent technique does not work in this case. We managed to prove a partial analog of Theorem B, see chapter 3.

Some chapters of this thesis are written uniformly for all local fields. In other chapters or sections we assume that the field has characteristic 0 and then we specify that in the beginning of the chapter or section.

The first chapter of the thesis is dedicated to tools for working with invariant distributions. Some of these tools existed before. Many tools existed only in the non-Archimedean case and our contribution was to develop their Archimedean analogs. However, we developed the Generalized Harish-Chandra descent tool for all local fields, and the tool of singular support (see section 1.8) was adapted to this kind of problems first in the Archimedean case (in [AG08d]) and then continued to the non-Archimedean case by Aizenbud in [Aiz08].

Theorems A and B were first proven for the non-Archimedean case in [AGRS07]. For the Archimedean case it was proven simultaneously, independently and in different ways in [AG08d] and [SZ08]. The proof we present here is a synthesis of those three proofs. It works uniformly in both cases thanks to [Aiz08].

The main tools of the proof in [AGRS07] are the Homogeneity Theorem (see section 1.7) and certain non-linear automorphisms ν_λ (see subsection 4.3.2). They are used in the current proof for convenience, but are not essential any more. The reason that they became less important for us is that we have used a very powerful tool - the singular support (see section 1.8). In the Archimedean case this tool uses a deep result from the theory of D-modules. However, the Homogeneity Theorem and the automorphisms ν_λ are still used in an essential way in the proofs of multiplicity one theorem for orthogonal groups. I hope that they will continue to be used in further problems of this kind.

0.3 Applications

Remark D. *Using the tools developed here, one can show that Theorem B implies an analogous theorem for the unitary groups. Theorem B will be used during the Harish-Chandra descent and the main part of the proof will be similar to the one we describe in chapter 4, since U_n is a form of GL_n.*

Analogs of Theorems A and B hold also for the orthogonal groups. Namely, let V be a finite dimensional vector space over a local field F of characteristic zero and q be any non-degenerate quadratic form on V . Extend q to $V \oplus F$ in the natural way and consider the embedding $O(V) \hookrightarrow O(V \oplus F)$. Let $O(V)$ act on $O(V \oplus F)$ by conjugation.

Theorem E. *Any $O(V)$ - invariant distribution on $O(V \oplus F)$ is invariant with respect to transposition.*

Corollary F. *$(O(V \oplus F), O(V))$ is a strong Gelfand pair.*

This theorem was conjectured by Bernstein and Rallis in the 1980s. It was proven in [AGRS07] for non-Achimedean F and in [SZ08] for Archimedean F. Both proofs use Theorem B.

These proofs use the same techniques as the proof we present here, except the singular support, which is not used in [AGRS07] and only a partial analog of it is used in [SZ08].

Theorem B and its analog for unitary groups are used in [AGRS07] and [SZ08] during Harish-Chandra descent since the centralizer of a semisimple orthogonal operator is a product of orthogonal, unitary and general linear groups over some finite field extensions.

Remark. *One can ask whether analogous theorems hold for special linear, unitary and orthogonal groups.*

The pairs (SL_{n+1}, SL_n) *and* (SU_{n+1}, SU_n) *are not strong Gelfand pairs already for* $n = 1$ *since* SL_1 *and* SU_1 *are trivial and neither* SL_2 *nor* SU_2 *are commutative.*

It is not known whether $(SO(V \oplus F), SO(V))$ *is a strong Gelfand pair. Of course, not every* $SO(V)$ *orbit is preserved by transposition already for* $\dim V = 1$ *(since* $SO(V)$ *is trivial in this case). On the other hand one can use another involution,* σ_2*, obtained by composition of transposition and conjugation by an element of* $O(V) - SO(V)$*. However, one can show that there exists a closed* SO_4*-orbit in* SO_5 *that is not transposition invariant and another closed* SO_4*-orbit that is not* σ_2*-invariant.*

In [Sun09] Theorem B is used to prove (at least in the non-Archimedean case) a theorem on distributions on the semidirect product of a symplectic group with its Heisenberg group, invariant with respect to conjugation by the symplectic group. That theorem in turn implies a certain multiplicity one theorem involving symplectic and metaplectic groups, which in turn is used in [GGP08] to show uniqueness of Fourier-Jacobi models.

All multiplicity one theorems mentioned above can be used to prove splitting of periods of automorphic forms.

Assuming multiplicity at most one, a more difficult question is to find when it is one. Some partial results are known.

For the orthogonal group (in fact the special orthogonal group) this question has been studied by B. Gross and D. Prasad ([GP92, Pra93]) who formulated a precise conjecture. For an up to date account see [GR06, GGP08, Wald09].

Multiplicity one theorems have important applications to the relative trace formula, to automorphic descent, to local and global liftings of automorphic representations, and to determinations of L-functions. In particular, multiplicity at most one is used as a hypothesis in the work [GPSR97] in the study of automorphic L-functions on classical groups.

0.4 Structure of the thesis

In chapter 1 we develop tools to work with invariant distributions.

In chapter 2 we give the definitions of the notions of Gelfand pair and strong Gelfand pair. We also formulate the Gelfand-Kazhdan criteria for these properties and prove them in the Archimedean case. In particular, we show that Theorem B implies Theorem A.

In chapter 3 we prove Gelfand property for the pairs $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ and $(\mathrm{O}_{n+1}(F), \mathrm{O}_n(F))$. For fields of characteristic zero the results of this chapter are much weaker then the main results. I decided to include this chapter for two reasons. First, the proofs in this chapter work over any local field, while the analogs of the main results for fields of positive characteristic are not known. Second, the proofs in this chapter are shorter and clearly show how to use some of the tools.

In chapter 4 we prove the main result, Theorem B.

Chapter 1

Tools to work with invariant distributions

This chapter is based on [BZ76, Ber84, Bar03, AGS08, AG08c, AG08d, Aiz08]. We will give more precise bibliographical note in each section.

1.1 Definitions

1.1.1 Conventions

- Henceforth we fix a local field F. All the algebraic varieties and algebraic groups that we will consider will be defined over F. In some sections and chapters we will assume that the characteristic of F is zero, and we will say so in the beginning of the section or chapter. For simplicity we always assume that the characteristic of F is different from 2.

- For a group G acting on a set X we denote by X^G the set of fixed points of X. Also, for an element $x \in X$ we denote by G_x the stabilizer of x.

- By a reductive group we mean a (non-necessarily connected) algebraic reductive group.

- We consider an algebraic variety X defined over F as an algebraic variety over $\overline{F}$ together with action of the Galois group $Gal(\overline{F}/F)$. On X we only consider the Zariski topology. On $X(F)$ we only consider the analytic (Hausdorff) topology. We treat finite-dimensional linear spaces defined over F as algebraic varieties.

- The tangent space of a manifold (algebraic, analytic, etc.) X at x will be denoted by T_xX.

- Usually we will use the letters X, Y, Z, Δ to denote algebraic varieties and the letters G, H to denote reductive groups. We will usually use the letters $V, W, U, K, M, N, C, O, S, T$ to denote analytic spaces (such as F-points of algebraic varieties) and the letter K to denote analytic groups. Also we will use the letters L, V, W to denote vector spaces of all kinds.

- For an algebraic variety X defined over $\mathbb{R}$ we denote by $X_{\mathbb{C}}$ the natural algebraic variety defined over $\mathbb{R}$ such that $X_{\mathbb{C}}(\mathbb{R}) = X(\mathbb{C})$. Note that over $\mathbb{C}$, $X_{\mathbb{C}}$ is isomorphic to $X \times X$.

- An action of a Lie algebra $\mathfrak{g}$ on a (smooth, algebraic, etc) manifold M is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on M. Note that an action of a (Lie, algebraic, etc) group on M defines an action of its Lie algebra on M.

- For a Lie algebra $\mathfrak{g}$ acting on M, an element $\alpha \in \mathfrak{g}$ and a point $x \in M$ we denote by $\alpha(x) \in T_xM$ the value at point x of the vector field corresponding to α. We denote by $\mathfrak{g}x \subset T_xM$ or by $\mathfrak{g}(x)$ the image of the map $\alpha \mapsto \alpha(x)$ and by $\mathfrak{g}_x \subset \mathfrak{g}$ its kernel.

1.1.2 Analytic manifolds

In this paper we consider distributions over l-spaces, smooth manifolds and Nash manifolds. l-spaces are locally compact totally disconnected topological spaces and Nash manifolds are semi-algebraic smooth manifolds.

For basic facts on l-spaces and distributions over them we refer the reader to [BZ76, §1].

For basic facts on Nash manifolds and Schwartz functions and distributions over them see subsection 1.2 and [AG08a]. In this paper we consider only separated Nash manifolds.

We now introduce notation and terminology which allows a uniform treatment of the Archimedean and the non-Archimedean cases.

We will use the notion of an analytic manifold over a local field (see e.g. [Ser64, Part II, Chapter III]). When we say "**analytic manifold**" we always mean analytic manifold over some local field. Note that an analytic manifold over a non-Archimedean field is in particular an l-space and an analytic manifold over an Archimedean field is in particular a smooth manifold.

Definition 1.1.1. *A **B-analytic manifold** is either an analytic manifold over a non-Archimedean local field, or a Nash manifold.*

Example 1.1.2. *If X is a smooth algebraic variety, then $X(F)$ is a B-analytic manifold and $(T_xX)(F) = T_x(X(F))$.*

Notation 1.1.3. *Let M be an analytic manifold and S be an analytic submanifold. We denote by $N_S^M := (T_M|_Y)/T_S$ the* ***normal bundle to** S **in** M. The* ***conormal bundle*** *is defined by $CN_S^M := (N_S^M)^*$. Denote by $\mathrm{Sym}^k(CN_S^M)$ the k-th symmetric power of the conormal bundle. For a point $y \in S$ we denote by $N_{S,y}^M$ the normal space to S in M at the point y and by $CN_{S,y}^M$ the conormal space.*

Now let us introduce the term *"vector system"*. This term allows to formulate statements in wider generality.

Definition 1.1.4. *For an analytic manifold M we define the notions of a* ***vector system*** *and a* ***B-vector system*** *over it.*

For a smooth manifold M, a vector system over M is a pair (E, B) where B is a smooth locally trivial fibration over M and E is a smooth (finite-dimensional) vector bundle over B.

For a Nash manifold M, a B-vector system over M is a pair (E, B) where B is a Nash fibration over M and E is a Nash (finite-dimensional) vector bundle over B.

For an l-space M, a vector system over M (or a B-vector system over M) is a sheaf of complex linear spaces.

In particular, in the case where M is a point, a vector system over M is either a $\mathbb{C}$-vector space if F is non-Archimedean, or a smooth manifold together with a vector bundle in the case where F is Archimedean. The simplest example of a vector system over a manifold M is given by the following.

Definition 1.1.5. *Let $\mathcal{V}$ be a vector system over a point pt. Let M be an analytic manifold. A* ***constant vector system with fiber** $\mathcal{V}$ is the pullback of $\mathcal{V}$ with respect to the map $M \to pt$. We denote it by $\mathcal{V}_M$.*

1.1.3 Distributions

Definition 1.1.6. *Let M be an analytic manifold over F. We define $C_c^\infty(M)$ in the following way.*

If F is non-Archimedean then $C_c^\infty(M)$ is the space of locally constant compactly supported complex valued functions on M. We do not consider any topology on $C_c^\infty(M)$.

If F is Archimedean then $C_c^\infty(M)$ is the space of smooth compactly supported complex valued functions on M, endowed with the standard topology.

For any analytic manifold M, we define the space of distributions $\mathcal{D}(M)$ by $\mathcal{D}(M) := C_c^\infty(M)^*$. We consider the weak topology on it.

Definition 1.1.7. *Let M be a B-analytic manifold. We define $\mathcal{S}(M)$ in the following way.*

If M is an analytic manifold over non-Archimedean field, $\mathcal{S}(M) := C_c^\infty(M)$.

If M is a Nash manifold, $\mathcal{S}(M)$ is the space of Schwartz functions on M, namely smooth functions which are rapidly decreasing together with all their derivatives. See [AG08a] for the precise definition. We consider $\mathcal{S}(M)$ as a Fréchet space.

For any B-analytic manifold M, we define the space of ***Schwartz distributions*** $\mathcal{S}^*(M)$ *by $\mathcal{S}^*(M) := \mathcal{S}(M)^*$. Clearly, $\mathcal{S}(M)^*$ is naturally embedded into $\mathcal{D}(M)$.*

Remark 1.1.8. *Schwartz distributions have the following two advantages over general distributions:*
(i) For a Nash manifold X and an open Nash submanifold $U \subset X$, we have the following exact sequence

$$0 \to \mathcal{S}_X^*(X \setminus U) \to \mathcal{S}^*(X) \to \mathcal{S}^*(U) \to 0.$$

(ii) Fourier transform defines an isomorphism $\mathcal{F} : \mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}^*(\mathbb{R}^n)$.

For a short survey on Schwartz functions and distributions on Nash manifolds see section 1.2, and for more information see [AG08a].

Notation 1.1.9. *Let M be an analytic manifold. For a distribution $\xi \in \mathcal{D}(M)$ we denote by $\operatorname{Supp}(\xi)$ the support of ξ.*

For a closed subset $N \subset M$ we denote

$$\mathcal{D}_M(N) := \{\xi \in \mathcal{D}(M) | \operatorname{Supp}(\xi) \subset N\}.$$

More generally, for a locally closed subset $N \subset M$ we denote

$$\mathcal{D}_M(N) := \mathcal{D}_{M \setminus (\overline{N} \setminus N)}(N).$$

Similarly if M is a B-analytic manifold and N is a locally closed subset we define $\mathcal{S}_M^*(N)$ in a similar vein. [1]

Definition 1.1.10. *Let M be an analytic manifold over F and $\mathcal{E}$ be a vector system over M. We define $C_c^\infty(M, \mathcal{E})$ in the following way.*

If F is non-Archimedean then $C_c^\infty(M, \mathcal{E})$ is the space of compactly supported sections of $\mathcal{E}$.

If F is Archimedean and $\mathcal{E} = (E, B)$ where B is a fibration over M and E is a vector bundle over B, then $C_c^\infty(M, \mathcal{E})$ is the complexification of the space of smooth compactly supported sections of E over B.

If $\mathcal{V}$ is a vector system over a point then we denote $C_c^\infty(M, \mathcal{V}) := C_c^\infty(M, \mathcal{V}_M)$.

We define $\mathcal{D}(M, \mathcal{E})$, $\mathcal{D}_M(N, \mathcal{E})$, $\mathcal{S}(M, \mathcal{E})$, $\mathcal{S}^*(M, \mathcal{E})$ and $\mathcal{S}_M^*(N, \mathcal{E})$ in the natural way.

[1] In the Archimedean case, locally closed is considered with respect to the restricted topology - see section 1.2.

1.2 Schwartz distributions on Nash manifolds

In this section we give a short survey on Schwartz functions and distributions on Nash manifolds. We will also prove some properties of K-equivariant Schwartz distributions. We work in the notation of [AG08a], where one can read about Nash manifolds and Schwartz distributions over them. More detailed references on Nash manifolds are [BCR98] and [Shi87].

Nash manifolds are equipped with the **restricted topology**, in which open sets are open semi-algebraic sets. This is not a topology in the usual sense of the word as infinite unions of open sets are not necessarily open sets in the restricted topology. However, finite unions of open sets are open and therefore in the restricted topology we consider only finite covers. In particular, if $E \to M$ is a Nash vector bundle it means that there exists a finite open cover U_i of M such that $E|_{U_i}$ is trivial.

Notation 1.2.1. *Let M be a Nash manifold. We denote by D_M the Nash bundle of densities on M. It is the natural bundle whose smooth sections are smooth measures. For the precise definition see e.g. [AG08a].*

An important property of Nash manifolds is

Theorem 1.2.2 (Local triviality of Nash manifolds; [Shi87], Theorem I.5.12)**.** *Any Nash manifold can be covered by a finite number of open submanifolds Nash diffeomorphic to $\mathbb{R}^n$.*

Definition 1.2.3. *Let M be a Nash manifold. We denote by $\mathcal{G}(M) := \mathcal{S}^*(M, D_M)$ the* ***space of Schwartz generalized functions*** *on M. Similarly, for a Nash bundle $E \to M$ we denote by $\mathcal{G}(M, E) := \mathcal{S}^*(M, E^* \otimes D_M)$ the* ***space of Schwartz generalized sections*** *of E.*

In the same way, for any smooth manifold M we denote by $C^{-\infty}(M) := \mathcal{D}(M, D_M)$ the ***space of generalized functions*** *on M and for a smooth bundle $E \to M$ we denote by $C^{-\infty}(M, E) := \mathcal{D}(M, E^* \otimes D_M)$ the* ***space of generalized sections*** *of E.*

Usual L^1-functions can be interpreted as Schwartz generalized functions but not as Schwartz distributions. We will need several properties of Schwartz functions from [AG08a].

Property 1.2.4 ([AG08a], Theorem 4.1.3)**.** $\mathcal{S}(\mathbb{R}^n)$ = *Classical Schwartz functions on $\mathbb{R}^n$.*

Property 1.2.5 ([AG08a], Theorem 5.4.3)**.** *Let $U \subset M$ be a (semi-algebraic) open subset, then*

$$\mathcal{S}(U, E) \cong \{\phi \in \mathcal{S}(M, E)| \quad \phi \text{ is 0 on } M \setminus U \text{ with all derivatives}\}.$$

Property 1.2.6 (see [AG08a], §5)**.** *Let M be a Nash manifold. Let $M = \bigcup U_i$ be a finite open cover of M. Then a function f on M is a Schwartz function if and only if it can be written as $f = \sum_{i=1}^{n} f_i$ where $f_i \in \mathcal{S}(U_i)$ (extended by zero to M).*

Moreover, there exists a smooth partition of unity $1 = \sum_{i=1}^{n} \lambda_i$ such that for any Schwartz function $f \in \mathcal{S}(M)$ the function $\lambda_i f$ is a Schwartz function on U_i (extended by zero to M).

Property 1.2.7 (see [AG08a], §5)**.** *Let M be a Nash manifold and E be a Nash bundle over it. Let $M = \bigcup U_i$ be a finite open cover of M. Let $\xi_i \in \mathcal{G}(U_i, E)$ such that $\xi_i|_{U_j} = \xi_j|_{U_i}$. Then there exists a unique $\xi \in \mathcal{G}(M, E)$ such that $\xi|_{U_i} = \xi_i$.*

We will also use the following notation.

Notation 1.2.8. *Let M be a metric space and $x \in M$. We denote by $B(x, r)$ the open ball with center x and radius r.*

1.2.1 Submersion principle

Theorem 1.2.9 ([AG08b], Theorem 2.4.16)**.** *Let M and N be Nash manifolds and $s : M \to N$ be a surjective submersive Nash map. Then locally it has a Nash section, i.e. there exists a finite open cover $N = \bigcup_{i=1}^{k} U_i$ such that s has a Nash section on each U_i.*

Corollary 1.2.10. *An ètale map $\phi : M \to N$ of Nash manifolds is locally an isomorphism. That means that there exists a finite cover $M = \bigcup U_i$ such that $\phi|_{U_i}$ is an isomorphism onto its open image.*

Theorem 1.2.11. *Let $p : M \to N$ be a Nash submersion of Nash manifolds. Then there exist a finite open (semi-algebraic) cover $M = \bigcup U_i$ and isomorphisms $\phi_i : U_i \cong W_i$ and $\psi_i : p(U_i) \cong V_i$ where $W_i \subset \mathbb{R}^{d_i}$ and $V_i \subset \mathbb{R}^{k_i}$ are open (semi-algebraic) subsets, $k_i \leq d_i$ and $p|_{U_i}$ correspond to the standard projections.*

Proof. The problem is local, hence without loss of generality we can assume that $N = \mathbb{R}^k$, M is an equidimensional closed submanifold of $\mathbb{R}^n$ of dimension d, $d \geq k$, and p is given by the standard projection $\mathbb{R}^n \to \mathbb{R}^k$.

Let Ω be the set of all coordinate subspaces of $\mathbb{R}^n$ of dimension d which contain N. For any $V \in \Omega$ consider the projection $pr : M \to V$. Define $U_V = \{x \in M | d_x pr$ is an isomorphism $\}$. It is easy to see that $pr|_{U_V}$ is ètale and $\{U_V\}_{V \in \Omega}$ gives a finite cover of M. The theorem now follows from the previous corollary (Corollary 1.2.10). □

Theorem 1.2.12. *Let $\phi : M \to N$ be a Nash submersion of Nash manifolds. Let E be a Nash bundle over N. Then*
(i) there exists a unique continuous linear map $\phi_* : \mathcal{S}(M, \phi^*(E) \otimes D_M) \to \mathcal{S}(N, E \otimes D_N)$ such that for any $f \in \mathcal{S}(N, E^*)$ and $\mu \in \mathcal{S}(M, \phi^*(E) \otimes D_M)$ we have

$$\int_{x \in N} \langle f(x), \phi_* \mu(x) \rangle = \int_{x \in M} \langle \phi^* f(x), \mu(x) \rangle.$$

In particular, we mean that both integrals converge.
(ii) If ϕ is surjective then ϕ_* is surjective.

Proof.

(i)

Step 1. Proof for the case when $M = \mathbb{R}^n$, $N = \mathbb{R}^k$, $k \leq n$, ϕ is the standard projection and E is trivial.
Fix Haar measure on $\mathbb{R}$ and identify $D_{\mathbb{R}^l}$ with the trivial bundle for any l. Define

$$\phi_*(f)(x) := \int_{y \in \mathbb{R}^{n-k}} f(x, y) dy.$$

Convergence of the integral and the fact that $\phi_*(f)$ is a Schwartz function follows from standard calculus.

Step 2. Proof for the case when $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^k$ are open (semi-algebraic) subsets, ϕ is the standard projection and E is trivial.
Follows from the previous step and Property 1.2.5.

Step 3. Proof for the case when E is trivial.
Follows from the previous step, Theorem 1.2.11 and partition of unity (Property 1.2.6).

Step 4. Proof in the general case.
Follows from the previous step and partition of unity (Property 1.2.6).

(ii) The proof is the same as in (i) except of Step 2. Let us prove (ii) in the case of Step 2. Again, fix Haar measure on $\mathbb{R}$ and identify $D_{\mathbb{R}^l}$ with the trivial bundle for any l. By Theorem 1.2.9 and partition of unity (Property 1.2.6) we can assume that there exists a Nash section $\nu : N \to M$. We can write ν in the form $\nu(x) = (x, s(x))$.

For any $x \in N$ define $R(x) := \sup\{r \in \mathbb{R}_{\geq 0} | B(\nu(x), r) \subset M\}$. Clearly, R is continuous and positive. By Tarski - Seidenberg principle (see e.g. [AG08a, Theorem 2.2.3]) it is semi-algebraic. Hence (by [AG08a, Lemma A.2.1]) there exists a positive Nash function $r(x)$ such that $r(x) < R(x)$. Let $\rho \in \mathcal{S}(\mathbb{R}^{n-k})$ such that ρ is supported in the unit ball and its integral is 1. Now let $f \in \mathcal{S}(N)$. Let $g \in C^\infty(M)$ defined by $g(x, y) := f(x)\rho((y - s(x))/r(x))/r(x)$ where $x \in N$ and $y \in \mathbb{R}^{n-k}$. It is easy to see that $g \in \mathcal{S}(M)$ and $\phi_* g = f$. $\square$

Notation 1.2.13. *Let $\phi : M \to N$ be a Nash submersion of Nash manifolds. Let E be a bundle on N. We denote by $\phi^* : \mathcal{G}(N, E) \to \mathcal{G}(M, \phi^*(E))$ the dual map to ϕ_*.*

Remark 1.2.14. *Clearly, the map $\phi^* : \mathcal{G}(N, E) \to \mathcal{G}(M, \phi^*(E))$ extends to the map $\phi^* : C^{-\infty}(N, E) \to C^{-\infty}(M, \phi^*(E))$ described in [AGS08, Theorem A.0.4].*

Proposition 1.2.15. *Let $\phi : M \to N$ be a surjective Nash submersion of Nash manifolds. Let E be a bundle on N. Let $\xi \in C^{-\infty}(N)$. Suppose that $\phi^*(\xi) \in \mathcal{G}(M)$. Then $\xi \in \mathcal{G}(N)$.*

Proof. It follows from Theorem 1.2.12 and Banach Open Map Theorem (see [Rud73, Theorem 2.11]). □

1.3 Basic tools

This section is based on [BZ76, Ber84, Bar03, AGS08].

Theorem 1.3.1. *Let an l-group K act on an l-space M. Let $M = \bigcup_{i=0}^{l} M_i$ be a K-invariant stratification of M. Let χ be a character of K. Suppose that $\mathcal{S}^*(M_i)^{K,\chi} = 0$. Then $\mathcal{S}^*(M)^{K,\chi} = 0$.*

This theorem is a direct corollary of [BZ76, Corollary 1.9].

Theorem 1.3.2. *Let a Nash group K act on a Nash manifold M. Let N be a locally closed subset. Let $N = \bigcup_{i=0}^{l} N_i$ be a Nash K-invariant stratification of N. Let χ be a character of K. Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq l$,*

$$\mathcal{S}^*(N_i, \mathrm{Sym}^k(CN_{N_i}^M))^{K,\chi} = 0.$$

Then $\mathcal{S}^*_M(N)^{K,\chi} = 0$.

For the proof see e.g. [AGS08, §B.2].

The following proposition sometimes helps to verify the conditions of this theorem.

Proposition 1.3.3. *Let a Nash group K act on a Nash manifold M. Let V be a real finite dimensional representation of K. Suppose that K preserves the Haar measure on V. Let $U \subset V$ be an open non-empty K-invariant subset. Let χ be a character of K. Suppose that $\mathcal{S}^*(M \times U)^{K,\chi} = 0$. Then $\mathcal{S}^*(M, Sym^k(V))^{K,\chi} = 0$.*

For proof see [AGS08, Section B.4].

Theorem 1.3.4 (Frobenius descent). *Let an analytic group K act on an analytic manifold M. Let N be an analytic manifold with a transitive action of K. Let $\phi : M \to N$ be a K-equivariant map.*

Let $z \in N$ be a point and $M_z := \phi^{-1}(z)$ be its fiber. Let K_z be the stabilizer of z in K. Let Δ_K and Δ_{K_z} be the modular characters of K and K_z.

Let $\mathcal{E}$ be a K-equivariant vector system over M. Then
(i) there exists a canonical isomorphism

$$\mathrm{Fr} : \mathcal{D}(M_z, \mathcal{E}|_{M_z} \otimes \Delta_K|_{K_z} \cdot \Delta_{K_z}^{-1})^{K_z} \cong \mathcal{D}(M, \mathcal{E})^K.$$

In particular, Fr *commutes with restrictions to open sets.*

(ii) For B-analytic manifolds Fr *maps $\mathcal{S}^*(M_z, \mathcal{E}|_{M_z} \otimes \Delta_K|_{K_z} \cdot \Delta_{K_z}^{-1})^{K_z}$ to $\mathcal{S}^*(M, \mathcal{E})^K$.*

For the proof of (i) see [Ber84, §§1.5] and [BZ76, §§2.21 - 2.36] for the case of l-spaces and [AGS08, Theorem 4.2.3] or [Bar03] for smooth manifolds. For the proof of (ii) see [AG08c], Appendix B.

The following fundamental theorem is a combination of [BZ76, Theorem 6.13 and Theorem 6.15].

Theorem 1.3.5. *Suppose that F is non-Archimedean. Let γ be an algebraic action of a linear algebraic group G on an algebraic variety X. Let $\sigma : X(F) \to X(F)$ be a homemorphism such that:*
(i) For each $g \in G(F)$ there exists $G^\sigma \in G(F)$ such that $\gamma(g)\sigma = \sigma\gamma(g^\sigma)$.
(ii) For some number n and $g_0 \in G(F)$, $\sigma^n = \gamma(g_0)$.
(iii) σ preserves all $G(F)$ orbits.

Then each $G(F)$-invariant distribution on $X(F)$ is invariant under σ.

Remark 1.3.6. *This theorem is very powerful since it reduces a statement on invariant distributions to check of simple geometric conditions. Many theorems on invariant distributions over non-Archimedean fields were proven using this theorem.*

However, in more difficult cases, including the ones under consideration in this thesis, σ preserves majority of $G(F)$-orbits but not all of them. In such cases there actually exist $G(F)$-invariant but not σ-invariant distributions defined on some locally closed subsets of $X(F)$ and one has to prove that they cannot be invariantly continued to $X(F)$. For this one has to use some non-geometric tools, like the ones described in sections 1.7 and 1.8.

We will also use the following straightforward proposition.

Proposition 1.3.7. *Let K_i be analytic groups acting on analytic manifolds M_i for $i = 1 \dots n$. Let $\Omega_i \subset K_i$ be analytic subgroups. Let $\mathcal{E}_i \to M_i$ be K_i-equivariant vector systems. Suppose that*

$$\mathcal{D}(M_i, E_i)^{\Omega_i} = \mathcal{D}(M_i, E_i)^{K_i}$$

for all i. Then

$$\mathcal{D}(\prod M_i, \boxtimes E_i)^{\prod \Omega_i} = \mathcal{D}(\prod M_i, \boxtimes E_i)^{\prod K_i},$$

where $\boxtimes$ denotes the external product.

Moreover, if Ω_i, K_i, M_i and $\mathcal{E}_i$ are B-analytic then the analogous statement holds for Schwartz distributions.

For the proof see e.g. [AGS08, proof of Proposition 3.1.5].

1.4 Generalized Harish-Chandra descent

Harish-Chandra developed a technique based on Jordan decomposition that allows to reduce certain statements on conjugation invariant distributions on a reductive group to the set of unipotent elements, provided that the statement is known for certain subgroups (see e.g. [HC99]).

In this section we generalize an aspect of this technique to the setting of a reductive group acting on a smooth affine algebraic variety, using the Luna Slice Theorem. Our technique is oriented towards proving Gelfand property for pairs of reductive groups.

In this section we assume that the characteristic of the local field F is zero. This section is based on [AG08c].

We start with some preliminaries from algebraic geometry and invariant theory.

Categorical quotient

Definition 1.4.1. *Let an algebraic group G act on an algebraic variety X. A pair consisting of an algebraic variety Y and a G-invariant morphism $\pi : X \to Y$ is called* ***the quotient of X by the action of G*** *if for any pair (π', Y'), there exists a unique morphism $\phi : Y \to Y'$ such that $\pi' = \phi \circ \pi$. Clearly, if such pair exists it is unique up to a canonical isomorphism. We will denote it by $(\pi_X, X/G)$.*

Theorem 1.4.2 (cf. [Dre00])**.** *Let a reductive group G act on an affine variety X. Then the quotient X/G exists, and every fiber of the quotient map π_X contains a unique closed orbit. In fact, $X/G := \operatorname{Spec} \mathcal{O}(X)^G$.*

1.4.1 Preliminaries on algebraic geometry over local fields

G-orbits on X and $G(F)$-orbits on $X(F)$

Lemma 1.4.3 ([AG08c], Lemma 2.3.4)**.** *Let G be an algebraic group and let $H \subset G$ be a closed subgroup. Then $G(F)/H(F)$ is open and closed in $(G/H)(F)$.*

Corollary 1.4.4. *Let an algebraic group G act on an algebraic variety X. Let $x \in X(F)$. Then*

$$N^X_{Gx,x}(F) \cong N^{X(F)}_{G(F)x,x}.$$

Proposition 1.4.5. *Let an algebraic group G act on an algebraic variety X. Suppose that $S \subset X(F)$ is a non-empty closed $G(F)$-invariant subset. Then S contains a closed orbit.*

Proof. The proof is by Noetherian induction on X. Choose $x \in S$. Consider $Z := \overline{Gx} - Gx$.

If $Z(F) \cap S$ is empty then $Gx(F) \cap S$ is closed and hence $G(F)x \cap S$ is closed by Lemma 1.4.3. Therefore $G(F)x$ is closed.

If $Z(F) \cap S$ is non-empty then $Z(F) \cap S$ contains a closed orbit by the induction assumption. □

Corollary 1.4.6. *Let an algebraic group G act on an algebraic variety X. Let U be an open $G(F)$-invariant subset of $X(F)$. Suppose that U contains all closed $G(F)$-orbits. Then $U = X(F)$.*

Theorem 1.4.7 ([RR96], §2 fact A, pages 108-109)**.** *Let a reductive group G act on an affine variety X. Let $x \in X(F)$. Then the following are equivalent:*
(i) $G(F)x \subset X(F)$ is closed (in the analytic topology).
(ii) $Gx \subset X$ is closed (in the Zariski topology).

Definition 1.4.8. *Let a reductive group G act on an affine variety X. We call an element $x \in X$ G-**semisimple** if its orbit Gx is closed.*

In particular, in the case where G acts on itself by conjugation, the notion of G-semisimplicity coincides with the usual one.

Notation 1.4.9. *Let V be an F-rational finite-dimensional representation of a reductive group G. We set*

$$Q_G(V) := Q(V) := (V/V^G)(F).$$

Since G is reductive, there is a canonical embedding $Q(V) \hookrightarrow V(F)$. Let $\pi : V(F) \to (V/G)(F)$ be the natural map. We set

$$\Gamma_G(V) := \Gamma(V) := \pi^{-1}(\pi(0)).$$

Note that $\Gamma(V) \subset Q(V)$. We also set

$$R_G(V) := R(V) := Q(V) - \Gamma(V).$$

Notation 1.4.10. *Let a reductive group G act on an affine variety X. For a G-semisimple element $x \in X(F)$ we set*

$$S_x := \{y \in X(F) \,|\, \overline{G(F)y} \ni x\}.$$

Lemma 1.4.11. *Let V be an F-rational finite-dimensional representation of a reductive group G. Then $\Gamma(V) = S_0$.*

This lemma follows from [RR96, fact A on page 108] for non-Archimedean F and [Brk71, Theorem 5.2 on page 459] for Archimedean F.

Example 1.4.12. *Let a reductive group G act on its Lie algebra $\mathfrak{g}$ by the adjoint action. Then $\Gamma(\mathfrak{g})$ is the set of nilpotent elements of $\mathfrak{g}$.*

Proposition 1.4.13. *Let a reductive group G act on an affine variety X. Let $x, z \in X(F)$ be G-semisimple elements which do not lie in the same orbit of $G(F)$. Then there exist disjoint $G(F)$-invariant open neighborhoods U_x of x and U_z of z.*

For the proof of this Proposition see [Lun75] for Archimedean F and [RR96, fact B on page 109] for non-Archimedean F.

Corollary 1.4.14. *Let a reductive group G act on an affine variety X. Suppose that $x \in X(F)$ is a G-semisimple element. Then the set S_x is closed.*

Proof. Let $y \in \overline{S_x}$. By Proposition 1.4.5, $\overline{G(F)y}$ contains a closed orbit $G(F)z$. If $G(F)z = G(F)x$ then $y \in S_x$. Otherwise, choose disjoint open G-invariant neighborhoods U_z of z and U_x of x. Since $z \in \overline{G(F)y}$, U_z intersects $G(F)y$ and hence contains y. Since $y \in \overline{S_x}$, this means that U_z intersects S_x. Let $t \in U_z \cap S_x$. Since U_z is $G(F)$-invariant, $G(F)t \subset U_z$. By the definition of S_x, $x \in \overline{G(F)t}$ and hence $x \in \overline{U_z}$. Hence U_z intersects U_x – contradiction! □

Analytic Luna slices

The Luna Slice Theorem is an important theorem in invariant theory of algebraic reductive groups (see [Lun73, Dre00]). Here we formulate a version of the Luna Slice Theorem for points over local fields. For Archimedean F this was done by Luna himself in [Lun75].

Definition 1.4.15. *Let a reductive group G act on an affine variety X. Let $\pi : X(F) \to (X/G)(F)$ be the natural map. An open subset $U \subset X(F)$ is called **saturated** if there exists an open subset $V \subset (X/G)(F)$ such that $U = \pi^{-1}(V)$.*

We will use the following corollary of the Luna Slice Theorem:

Theorem 1.4.16 ([AG08c], Theorem 2.3.17)**.** *Let a reductive group G act on a smooth affine variety X. Let $x \in X(F)$ be G-semisimple. Consider the natural action of the stabilizer G_x on the normal space $N^X_{Gx,x}$. Then there exist*
(i) an open $G(F)$-invariant B-analytic neighborhood U of $G(F)x$ in $X(F)$ with a G-equivariant B-analytic retract $p : U \to G(F)x$ and
(ii) a G_x-equivariant B-analytic embedding $\psi : p^{-1}(x) \hookrightarrow N^X_{Gx,x}(F)$ with an open saturated image such that $\psi(x) = 0$.

Definition 1.4.17. *In the notation of the previous theorem, denote $S := p^{-1}(x)$ and $N := N^X_{Gx,x}(F)$. We call the quintuple (U, p, ψ, S, N) an* ***analytic Luna slice at*** *x.*

Corollary 1.4.18. *In the notation of the previous theorem, let $y \in p^{-1}(x)$. Denote $z := \psi(y)$. Then*
(i) $(G(F)_x)_z = G(F)_y$
(ii) $N^{X(F)}_{G(F)y,y} \cong N^N_{G(F)_x z,z}$ as $G(F)_y$-spaces
(iii) y is G-semisimple if and only if z is G_x-semisimple.

1.4.2 Generalized Harish-Chandra descent

In this subsection we will prove the following theorem.

Theorem 1.4.19. *Let a reductive group G act on a smooth affine variety X. Let χ be a character of $G(F)$. Suppose that for any G-semisimple $x \in X(F)$ we have*

$$\mathcal{D}(N^X_{Gx,x}(F))^{G(F)_x,\chi} = 0.$$

Then

$$\mathcal{D}(X(F))^{G(F),\chi} = 0.$$

Remark 1.4.20. *In fact, the converse is also true. We will not prove it since we will not use it.*

For the proof of this theorem we will need the following lemma

Lemma 1.4.21. *Let a reductive group G act on a smooth affine variety X. Let χ be a character of $G(F)$. Let $U \subset X(F)$ be an open saturated subset. Suppose that $\mathcal{D}(X(F))^{G(F),\chi} = 0$. Then $\mathcal{D}(U)^{G(F),\chi} = 0$.*

Proof. Consider the quotient X/G. It is an affine algebraic variety. Embed it in an affine space $\mathbb{A}^n$. This defines a map $\pi : X(F) \to F^n$. Since U is saturated, there exists an open subset $V \subset (X/G)(F)$ such that $U = \pi^{-1}(V)$. Clearly there exists an open subset $V' \subset F^n$ such that $V' \cap (X/G)(F) = V$.

Let $\xi \in \mathcal{D}(U)^{G(F),\chi}$. Suppose that ξ is non-zero. Let $x \in \mathrm{Supp}\xi$ and let $y := \pi(x)$. Let $g \in C_c^\infty(V')$ be such that $g(y) = 1$. Consider $\xi' \in \mathcal{D}(X(F))$ defined by $\xi'(f) := \xi(f \cdot (g \circ \pi))$. Clearly, $\mathrm{Supp}(\xi') \subset U$ and hence we can interpret ξ' as an element in $\mathcal{D}(X(F))^{G(F),\chi}$. Therefore $\xi' = 0$. On the other hand, $x \in \mathrm{Supp}(\xi')$. Contradiction. □

Proof of Theorem 1.4.19. Let x be a G-semisimple element. Let $(U_x,p_x,\psi_x, S_x,N_x)$ be an analytic Luna slice at x.

Let $\xi' = \xi|_{U_x}$. Then $\xi' \in \mathcal{D}(U_x)^{G(F),\chi}$. By Frobenius descent it corresponds to $\xi'' \in \mathcal{D}(S_x)^{G_x(F),\chi}$.

The distribution ξ'' corresponds to a distribution $\xi''' \in \mathcal{D}(\psi_x(S_x))^{G_x(F),\chi}$.

However, by the previous lemma the assumption implies that $\mathcal{D}(\psi_x(S_x))^{G_x(F),\chi} = 0$. Hence $\xi' = 0$.

Let $S \subset X(F)$ be the set of all G-semisimple points. Let $U = \bigcup_{x \in S} U_x$. We saw that $\xi|_U = 0$. On the other hand, U includes all the closed orbits, and hence by Corollary 1.4.6 $U = X$. □

The following generalization of this theorem is proven in the same way.

Theorem 1.4.22. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let χ be a character of K. Suppose that for any G-semisimple $x \in X(F)$ we have*

$$\mathcal{D}(N_{Gx,x}^X(F))^{K_x,\chi} = 0.$$

Then

$$\mathcal{D}(X(F))^{K,\chi} = 0.$$

Remark 1.4.23. *If K is an open B-analytic subgroup*[2] *of $G(F)$ then the theorem also holds for Schwartz distributions. Namely, if $\mathcal{S}^*(N_{Gx,x}^X(F))^{K_x} = 0$ for any G-semisimple $x \in X(F)$ then $\mathcal{S}^*(X(F))^K = 0$. The proof is the same.*

The following generalization of Theorem 1.4.19 is proven in the same way.

Theorem 1.4.24. *Let a reductive group G act on smooth affine varieties X and Y. Let χ be a character of $G(F)$. Suppose that for any G-semisimple $x \in X(F)$ we have*

$$\mathcal{S}^*((N_{Gx,x}^X \times Y)(F))^{G(F)_x,\chi} = 0.$$

Then $\mathcal{S}^*(X(F) \times Y(F))^{G(F)_x,\chi} = 0$.

[2]In fact, any open subgroup of a B-analytic group is B-analytic.

1.4.3 A stronger version

In this subsection we provide means to validate the conditions of Theorems 1.4.19 and 1.4.22 based on an inductive argument.

More precisely, the goal of this subsection is to prove the following theorem.

Theorem 1.4.25. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let χ be a character of K. Suppose that for any G-semisimple $x \in X(F)$ such that*

$$\mathcal{D}(R_{G_x}(N^X_{Gx,x}))^{K_x,\chi} = 0$$

we have

$$\mathcal{D}(Q_{G_x}(N^X_{Gx,x}))^{K_x,\chi} = 0.$$

Then for any G-semisimple $x \in X(F)$ we have

$$\mathcal{D}(N^X_{Gx,x}(F))^{K_x,\chi} = 0.$$

Together with Theorem 1.4.22, this theorem gives the following corollary.

Corollary 1.4.26. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let χ be a character of K. Suppose that for any G-semisimple $x \in X(F)$ such that*

$$\mathcal{D}(R(N^X_{Gx,x}))^{K_x,\chi} = 0$$

we have

$$\mathcal{D}(Q(N^X_{Gx,x}))^{K_x,\chi} = 0.$$

Then $\mathcal{D}(X(F))^{K,\chi} = 0$.

From now till the end of the subsection we fix G, X, K and χ. Let us introduce several definitions and notation.

Notation 1.4.27. *Denote*

- *$T \subset X(F)$ the set of all G-semisimple points.*
- *For $x, y \in T$ we say that $x > y$ if $G_x \supsetneq G_y$.*
- *$T_0 := \{x \in T \mid \mathcal{D}(Q(N^X_{Gx,x}))^{K_x,\chi} = 0\} = \{x \in T \mid \mathcal{D}((N^X_{Gx,x}))^{K_x,\chi} = 0\}$.*

Proof of Theorem 1.4.25. We have to show that $T = T_0$. Assume the contrary.

Note that every chain in T with respect to our ordering has a minimum. Hence by Zorn's lemma every non-empty set in T has a minimal element. Let x be a minimal element of $T - T_0$. To get a contradiction, it is enough to show that $\mathcal{D}(R(N^X_{Gx,x}))^{K_x,\chi} = 0$.

Denote $R := R(N^X_{Gx,x})$. By Theorem 1.4.22, it is enough to show that for any $y \in R$ we have

$$\mathcal{D}(N^R_{G(F)_x y,y})^{(K_x)_y,\chi} = 0.$$

Let (U, p, ψ, S, N) be an analytic Luna slice at x.

Since $\psi(S)$ is open and contains 0, we can assume, upon replacing y by λy for some $\lambda \in F^\times$, that $y \in \psi(S)$. Let $z \in S$ be such that $\psi(z) = y$. By Corollary 1.4.18, $G(F)_z = (G(F)_x)_y \subsetneq G(F)_x$ and $N^R_{G(F)_x y,y} \cong N^X_{Gz,z}(F)$. Hence $(K_x)_y = K_z$ and therefore

$$\mathcal{D}(N^R_{G(F)_x y,y})^{(K_x)_y,\chi} \cong \mathcal{D}(N^X_{Gz,z}(F))^{K_z,\chi}.$$

However $z < x$ and hence $z \in T_0$ which means that $\mathcal{D}(N^X_{Gz,z}(F))^{K_z,\chi} = 0$. □

Remark 1.4.28. *One can rewrite this proof such that it will use Zorn's lemma for finite sets only, which does not depend on the axiom of choice.*

Remark 1.4.29. *As before, Theorem 1.4.25 and Corollary 1.4.26 also hold for Schwartz distributions, with a similar proof.*

1.5 Localization Principle

In this section we formulate and prove localization principle. For non-Archimedean local fields, a more general theorem is proven in [Ber84, Section 1.5]. The proofs we give in this section work for local fields of characteristic zero. This section is based on [AG08c, Appendix D].

Theorem 1.5.1 (Localization Principle)**.** *Let a reductive group G act on a smooth algebraic variety X. Let Y be an algebraic variety and $\phi : X \to Y$ be an affine algebraic G-invariant map. Let χ be a character of $G(F)$. Suppose that for any $y \in Y(F)$ we have $\mathcal{D}_{X(F)}((\phi^{-1}(y))(F))^{G(F),\chi} = 0$. Then $\mathcal{D}(X(F))^{G(F),\chi} = 0$.*

Proof. Clearly, it is enough to prove the theorem for the case when X and Y are affine. In this case there exists a categorical quotient X/G (see Theorem 1.4.2). By the definition of categorical quotient ϕ is a composition of $\pi_X : X \to X/G$ and some morphism $X/G \to Y$. Since every fiber of π_X is included in some fiber of Y, it is enough to prove the theorem for the case when $Y = X/G$ and $\phi = \pi_X$.

By the Generalized Harish-Chandra Descent (Corollary 1.4.26), it is enough to prove that for any G-semisimple $x \in X(F)$, we have

$$\mathcal{D}_{N^X_{Gx,x}(F)}(\Gamma(N^X_{Gx,x}))^{G_x(F),\chi} = 0.$$

Let (U, p, ψ, S, N) be an analytic Luna slice at x. Clearly,

$$\mathcal{D}_{N^X_{Gx,x}(F)}(\Gamma(N^X_{Gx,x}))^{G_x(F),\chi} \cong \mathcal{D}_{\psi(S)}(\Gamma(N^X_{Gx,x}))^{G_x(F),\chi} \cong \mathcal{D}_S(\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G_x(F),\chi}.$$

By Frobenius descent (Theorem 1.3.4),

$$\mathcal{D}_S(\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G_x(F),\chi} = \mathcal{D}_U(G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G(F),\chi}.$$

By Lemma 1.4.11,

$$G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})) = \{y \in X(F) | x \in \overline{G(F)y}\}.$$

Hence by Corollary 1.4.14, $G(F)\psi^{-1}(\Gamma(N^X_{Gx,x}))$ is closed in $X(F)$. Hence

$$\mathcal{D}_U(G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G(F),\chi} = \mathcal{D}_{X(F)}(G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G(F),\chi}.$$

Now,

$$G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})) \subset \pi_X(F)^{-1}(\pi_X(F)(x))$$

and we are given

$$\mathcal{D}_{X(F)}(\pi_X(F)^{-1}(\pi_X(F)(x)))^{G(F),\chi} = 0$$

for any G-semisimple x. □

Remark 1.5.2. *An analogous statement holds for Schwartz distributions and the proof is the same.*

Corollary 1.5.3. *Let a reductive group G act on a smooth algebraic variety X. Let Y be an algebraic variety and $\phi : X \to Y$ be an affine algebraic G-invariant submersion. Suppose that for any $y \in Y(F)$ we have $\mathcal{S}^*(\phi^{-1}(y))^{G(F),\chi} = 0$. Then $\mathcal{S}^*(X(F))^{G(F),\chi} = 0$.*

Proof. For any $y \in Y(F)$, denote $X(F)_y := (\phi^{-1}(y))(F)$. Since ϕ is a submersion, for any $y \in Y(F)$ the set $X(F)_y$ is a smooth manifold. Moreover, $d\phi$ defines an isomorphism between $N^{X(F)}_{X(F)_y,z}$ and $T_{Y(F),y}$ for any $z \in X(F)_y$. Hence the bundle $CN^{X(F)}_{X(F)_y}$ is a trivial $G(F)$-equivariant bundle.

We know that

$$\mathcal{S}^*(X(F)_y)^{G(F),\chi} = 0.$$

Therefore for any k, we have

$$\mathcal{S}^*(X(F)_y, \mathrm{Sym}^k(CN^{X(F)}_{X(F)_y}))^{G(F),\chi} = 0.$$

Thus by Theorem 1.3.2,

$$\mathcal{S}^*_{X(F)}(X(F)_y)^{G(F),\chi} = 0.$$

Now, the Localization Principle implies that

$$\mathcal{S}^*(X(F))^{G(F),\chi} = 0.$$

$\square$

Remark 1.5.4. *Theorem 1.5.1 and Corollary 1.5.3 admit obvious generalizations to constant vector systems. The same proofs hold.*

1.6 Distributions versus Schwartz distributions

This section is relevant only for Archimedean F.

In this section we show that if there are no $G(F)$-equivariant Schwartz distributions on $X(F)$ then there are no $G(F)$-equivariant distributions on $X(F)$. This section is based on [AG08c].

Theorem 1.6.1. *Let a reductive group G act on a smooth affine variety X. Let V be a finite-dimensional algebraic representation of $G(F)$. Suppose that*

$$\mathcal{S}^*(X(F), V)^{G(F)} = 0.$$

Then

$$\mathcal{D}(X(F), V)^{G(F)} = 0.$$

For the proof we will need the following definition and theorem.

Definition 1.6.2. *(i) Let a topological group K act on a topological space M. We call a closed K-invariant subset $C \subset M$* ***compact modulo*** *K if there exists a compact subset $C' \subset M$ such that $C \subset KC'$.*

(ii) Let a Nash group K act on a Nash manifold M. We call a closed K-invariant subset $C \subset M$ ***Nashly compact modulo*** *K if there exist a compact subset $C' \subset M$ and semi-algebraic closed subset $Z \subset M$ such that $C \subset Z \subset KC'$.*

Remark 1.6.3. *Let a reductive group G act on a smooth affine variety X. Let $K := G(F)$ and $M := X(F)$. Then it is easy to see that the notions of compact modulo K and Nashly compact modulo K coincide.*

Theorem 1.6.4. *Let a Nash group K act on a Nash manifold M. Let E be a K-equivariant Nash bundle over M. Let $\xi \in \mathcal{D}(M, E)^K$ be such that* $\mathrm{Supp}(\xi)$ *is Nashly compact modulo K. Then $\xi \in \mathcal{S}^*(M, E)^K$.*

For the proof see the next subsection.

Proof of Theorem 1.6.1. Fix any $y \in (X/G)(F)$ and denote $M := \pi_X^{-1}(y)(F)$.

By the Localization Principle (Theorem 1.5.1 and Remark 1.5.4), it is enough to prove that

$$\mathcal{S}^*_{X(F)}(M, V)^{G(F)} = \mathcal{D}_{X(F)}(M, V)^{G(F)}.$$

Choose $\xi \in \mathcal{D}_{X(F)}(M, V)^{G(F)}$. M has a unique closed stable G-orbit and hence a finite number of closed $G(F)$-orbits. By Theorem 1.6.4, it is enough to show that M is Nashly compact modulo $G(F)$. Clearly M is semi-algebraic. Choose representatives x_i of the closed $G(F)$-orbits in M. Choose compact neighborhoods C_i of x_i. Let $C' := \bigcup C_i$. By Corollary 1.4.6, $G(F)C' \supset M$. ☐

1.6.1 K-invariant distributions compactly supported modulo K.

In this subsection we prove Theorem 1.6.4.

For the proof we will need the following lemmas.

Lemma 1.6.5. *Let M be a Nash manifold. Let $C \subset M$ be a compact subset. Then there exists a relatively compact open (semi-algebraic) subset $U \subset M$ that includes C.*

Proof. For any point $x \in C$ choose an affine chart, and let U_x be an open ball with center at x inside this chart. Those U_x give an open cover of C. Choose a finite subcover $\{U_i\}_{i=1}^n$ and let $U := \bigcup_{i=1}^n U_i$. ☐

Lemma 1.6.6. *Let M be a Nash manifold. Let E be a Nash bundle over M. Let $U \subset M$ be a relatively compact open (semi-algebraic) subset. Let $\xi \in \mathcal{D}(M, E)$. Then $\xi|_U \in \mathcal{S}^*(U, E|_U)$.*

Proof. It follows from the fact that extension by zero $ext : \mathcal{S}(U, E|_U) \to C_c^\infty(M, E)$ is a continuous map. ☐

Proof of Theorem 1.6.4. Let $Z \subset M$ be a semi-algebraic closed subset and $C \subset M$ be a compact subset such that $Supp(\xi) \subset Z \subset KC$.

Let $U \supset C$ be as in Lemma 1.6.5. Let $\xi' := \xi|_{KU}$. Since $\xi|_{M-Z} = 0$, it is enough to show that ξ' is Schwartz.

Consider the surjective submersion $m_U : K \times U \to KU$. Let

$$\xi'' := m_U^*(\xi') \in \mathcal{D}(K \times U, m_U^*(E))^K.$$

By Proposition 1.2.15, it is enough to show that

$$\xi'' \in \mathcal{S}^*(K \times U, m_U^*(E)).$$

By Frobenius descent, ξ'' corresponds to $\eta \in \mathcal{D}(U, E)$. It is enough to prove that $\eta \in \mathcal{S}^*(U, E)$. Consider the submersion $m : K \times M \to M$ and let

$$\xi''' := m^*(\xi) \in \mathcal{D}(K \times M, m^*(E)).$$

By Frobenius descent, ξ''' corresponds to $\eta' \in \mathcal{D}(M, E)$. Clearly $\eta = \eta'|_U$. Hence by Lemma 1.6.6, $\eta \in \mathcal{S}^*(U, E)$. □

1.7 Fourier transform and Homogeneity Theorems

This section is based on [AG08c].

Let G be a reductive group and V be a finite-dimensional F-rational representation of G. Let χ be a character of $G(F)$. In this section we provide some tools to verify that $\mathcal{S}^*(Q(V))^{G(F),\chi} = 0$ provided that $\mathcal{S}^*(R(V))^{G(F),\chi} = 0$.

1.7.1 Formulation

For this subsection let B be a non-degenerate bilinear form on a finite-dimensional vector space V over F. We also fix an additive character κ of F. If F is Archimedean we take $\kappa(x) := e^{2\pi \mathrm{i} \operatorname{Re}(x)}$.

Notation 1.7.1. *We identify V and V^* via B and endow V with the self-dual Haar measure with respect to ψ. Denote by $\mathcal{F}_B : \mathcal{S}^*(V) \to \mathcal{S}^*(V)$ the Fourier transform. For any B-analytic manifold M over F we also denote by $\mathcal{F}_B : \mathcal{S}^*(M \times V) \to \mathcal{S}^*(M \times V)$ the partial Fourier transform.*

Notation 1.7.2. *Consider the homothety action of $F^\times$ on V given by $\rho(\lambda)v := \lambda^{-1}v$. It gives rise to an action ρ of $F^\times$ on $\mathcal{S}^*(V)$.*

Let $|\cdot|$ denote the normalized absolute value. Recall that for $F = \mathbb{R}$, $|\lambda|$ is equal to the classical absolute value but for $F = \mathbb{C}$, $|\lambda| = (\operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2$.

Notation 1.7.3. *We denote by $\gamma(B)$ the Weil constant. For its definition see e.g. [Gel76, §2.3] for non-Archimedean F and [RS78, §1] for Archimedean F.*

For any $t \in F^\times$ denote $\delta_B(t) = \gamma(B)/\gamma(tB)$.

Note that $\gamma(B)$ is an 8-th root of unity and if $\dim V$ is odd and $F \neq \mathbb{C}$ then δ_B is **not** a multiplicative character.

Notation 1.7.4. *We denote*

$$Z(B) := \{x \in V \mid B(x, x) = 0\}.$$

Theorem 1.7.5 (non-Archimedean homogeneity)**.** *Suppose that F is **non-Archimedean**. Let M be a B-analytic manifold over F. Let $\xi \in \mathcal{S}^*_{V\times M}(Z(B) \times M)$ be such that $\mathcal{F}_B(\xi) \in \mathcal{S}^*_{V\times M}(Z(B) \times M)$. Then for any $t \in F^\times$, we have $\rho(t)\xi = \delta_B(t)|t|^{\dim V/2}\xi$ and $\xi = \gamma(B)^{-1}\mathcal{F}_B(\xi)$. In particular, if $\dim V$ is odd then $\xi = 0$.*

For the proof see e.g. [RS07, §§8.1] or [JR96, §§3.1].

For the Archimedean version of this theorem we will need the following definition.

Definition 1.7.6. *Let M be a B-analytic manifold over F. We say that a distribution $\xi \in \mathcal{S}^*(V \times M)$ is **adapted to** B if either*
(i) for any $t \in F^\times$ we have $\rho(t)\xi = \delta(t)|t|^{\dim V/2}\xi$ and ξ is proportional to $\mathcal{F}_B\xi$ or
(ii) F is Archimedean and for any $t \in F^\times$ we have $\rho(t)\xi = \delta(t)t|t|^{\dim V/2}\xi$.

Note that if $\dim V$ is odd and $F \neq \mathbb{C}$ then every B-adapted distribution is zero.

Theorem 1.7.7 (Homogeneity Theorem)**.** *Let M be a Nash manifold. Let $L \subset \mathcal{S}^*_{V\times M}(Z(B) \times M)$ be a non-zero subspace such that for all $\xi \in L$ we have $\mathcal{F}_B(\xi) \in L$ and $B \cdot \xi \in L$ (here B is viewed as a quadratic function).*

Then there exists a non-zero distribution $\xi \in L$ which is adapted to B.

For Archimedean F we prove this theorem in subsection 1.7.3. For non-Archimedean F it follows from Theorem 1.7.5.

We will also use the following trivial observation.

Lemma 1.7.8. *Let a B-analytic group K act linearly on V and preserving B. Let M be a B-analytic K-manifold over F. Let $\xi \in \mathcal{S}^*(V \times M)$ be a K-invariant distribution. Then $\mathcal{F}_B(\xi)$ is also K-invariant.*

1.7.2 Applications

The following two theorems easily follow form the results of the previous subsection.

Theorem 1.7.9. *Suppose that F is non-Archimedean. Let G be a reductive group. Let V be a finite-dimensional F-rational representation of G. Let χ be character of $G(F)$. Suppose that $\mathcal{S}^*(R(V))^{G(F),\chi} = 0$. Let $V = V_1 \oplus V_2$ be a G-invariant decomposition of V. Let B be a G-invariant symmetric non-degenerate bilinear form on V_1. Consider the action ρ of $F^\times$ on V by homothety on V_1.*

Then any $\xi \in \mathcal{S}^*(Q(V))^{G(F),\chi}$ satisfies $\rho(t)\xi = \delta_B(t)|t|^{\dim V_1/2}\xi$ and $\xi = \gamma(B)\mathcal{F}_B\xi$. In particular, if $\dim V_1$ is odd then $\xi = 0$.

Theorem 1.7.10. *Let G be a reductive group. Let V be a finite-dimensional F-rational representation of G. Let χ be character of $G(F)$. Suppose that $\mathcal{S}^*(R(V))^{G(F),\chi} = 0$. Let $Q(V) = W \oplus (\bigoplus_{i=1}^k V_i)$ be a G-invariant decomposition of $Q(V)$. Let B_i be G-invariant symmetric non-degenerate bilinear forms on V_i. Suppose that any $\xi \in \mathcal{S}^*_{Q(V)}(\Gamma(V))^{G(F),\chi}$ which is adapted to each B_i is zero.*

Then $\mathcal{S}^*(Q(V))^{G(F),\chi} = 0$.

Remark 1.7.11. *One can easily generalize Theorems 1.7.10 and 1.7.9 to the case of constant vector systems.*

1.7.3 Proof of the Archimedean Homogeneity Theorem

The goal of this subsection is to prove Theorem 1.7.7 for Archimedean F. We fix V and B.

We will need some facts about the Weil representation. For a survey on the Weil representation in the Archimedean case we refer the reader to [RS78, §1].

1. There exists a unique (infinitesimal) action π of $\mathfrak{sl}_2(F)$ on $\mathcal{S}^*(V)$ such that
   (i) $\pi(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})\xi = -\mathrm{i}\pi Re(B)\xi$ and $\pi(\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix})\xi = -\mathcal{F}_B^{-1}(\mathrm{i}\pi Re(B)\mathcal{F}_B(\xi))$.
   (ii) If $F = \mathbb{C}$ then $\pi(\begin{pmatrix} 0 & \mathrm{i} \\ 0 & 0 \end{pmatrix}) = \pi(\begin{pmatrix} 0 & 0 \\ -\mathrm{i} & 0 \end{pmatrix}) = 0$
2. It can be lifted to an action of the metaplectic group $\mathrm{Mp}(2, F)$.
   We will denote this action by Π.
3. In case $F = \mathbb{C}$ we have $\mathrm{Mp}(2, F) = \mathrm{SL}_2(F)$ and in case $F = \mathbb{R}$ the group $\mathrm{Mp}(2, F)$ is a connected 2-fold covering of $\mathrm{SL}_2(F)$. We will denote by $\varepsilon \in \mathrm{Mp}(2, F)$ the central element of order 2 satisfying $\mathrm{SL}_2(F) = \mathrm{Mp}(2, F)/\{1, \varepsilon\}$.
4. In case $F = \mathbb{R}$ we have $\Pi(\varepsilon) = (-1)^{\dim V}$ and therefore if $\dim V$ is even then Π factors through $\mathrm{SL}_2(F)$ and if $\dim V$ is odd then no nontrivial subrepresentation of Π factors through $\mathrm{SL}_2(F)$. In particular if $\dim V$ is odd then Π has no nontrivial finite-dimensional representations, since every finite-dimensional representation of $\mathrm{Mp}(2, F)$ factors through $\mathrm{SL}_2(F)$.
5. In case $F = \mathbb{C}$ or in case $\dim V$ is even we have $\Pi(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})\xi = \delta^{-1}(t)|t|^{-\dim V/2}\rho(t)\xi$ and $\Pi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\xi = \gamma(B)^{-1}\mathcal{F}_B\xi$.

We also need the following straightforward lemma.

Lemma 1.7.12. *Let* (Λ, L) *be a continuous finite-dimensional representation of* $\mathrm{SL}_2(\mathbb{R})$*. Then there exists a non-zero* $\xi \in L$ *such that either*

$$\Lambda(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})\xi = \xi \text{ and } \Lambda(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\xi \text{ is proportional to } \xi$$

or

$$\Lambda(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})\xi = t\xi,$$

for all t*.*

Now we are ready to prove the theorem.

Proof of Theorem 1.7.7. Without loss of generality assume $M = pt$.

Let $\xi \in L$ be a non-zero distribution. Let $L' := U_{\mathbb{C}}(\mathfrak{sl}_2(\mathbb{R}))\xi \subset L$. Here, $U_{\mathbb{C}}$ means the complexified universal enveloping algebra.

We are given that $\xi, \mathcal{F}_B(\xi) \in \mathcal{S}^*_V(Z(B))$. By Lemma 1.7.13 below this implies that $L' \subset \mathcal{S}^*(V)$ is finite-dimensional. Clearly, L' is also a subrepresentation of Π. Therefore by Fact (4), $F = \mathbb{C}$ or $\dim V$ is even. Hence Π factors through $\mathrm{SL}_2(F)$.

Now by Lemma 1.7.12 there exists $\xi' \in L'$ which is B-adapted. □

Lemma 1.7.13. *Let* V *be a representation of* $\mathfrak{sl}_2$*. Let* $v \in V$ *be a vector such that* $e^k v = f^n v = 0$ *for some* n, k*. Then the representation generated by* v *is finite-dimensional.*[3]

This lemma is probably well-known. Since we have not found any reference we include the proof.

Proof. The proof is by induction on k.

Base k=1:
It is easy to see that

$$e^l f^l v = l!(\prod_{i=0}^{l-1}(h-i))v$$

for all l. This can be checked by direct computation, and also follows from the fact that $e^l f^l$ is of weight 0, hence it acts on the singular vector v by its Harish-Chandra projection which is

$$\mathrm{HC}(e^l f^l) = l! \prod_{i=0}^{l-1}(h-i).$$

Therefore $(\prod_{i=0}^{n-1}(h-i))v = 0$.

[3]For our purposes it is enough to prove this lemma for k=1.

Hence $W := U_{\mathbb{C}}(h)v$ is finite-dimensional and h acts on it semi-simply. Here, $U_{\mathbb{C}}(h)$ denotes the universal enveloping algebra of h. Let $\{v_i\}_{i=1}^m$ be an eigenbasis of h in W. It is enough to show that $U_{\mathbb{C}}(\mathfrak{sl}_2)v_i$ is finite-dimensional for any i. Note that $e|_W = f^n|_W = 0$. Now, $U_{\mathbb{C}}(\mathfrak{sl}_2)v_i$ is finite-dimensional by the Poincare-Birkhoff-Witt Theorem.

Induction step:
Let $w := e^{k-1}v$. Let us show that $f^{n+k-1}w = 0$. Consider the element $f^{n+k-1}e^{k-1} \in U_{\mathbb{C}}(\mathfrak{sl}_2)$. It is of weight $-2n$, hence by the Poincare-Birkhoff-Witt Theorem it can be rewritten as a combination of elements of the form $e^a h^b f^c$ such that $c - a = n$ and hence $c \geq n$. Therefore $f^{n+k-1}e^{k-1}v = 0$.

Now let $V_1 := U_{\mathbb{C}}(\mathfrak{sl}_2)v$ and $V_2 := U_{\mathbb{C}}(\mathfrak{sl}_2)w$. By the base of the induction V_2 is finite-dimensional, by the induction hypotheses V_1/V_2 is finite-dimensional, hence V_1 is finite-dimensional. □

1.8 Singular support

In this section we assume that F has zero characteristic. This section is based on [AG08d], [Aiz08] and [Hef]. In this section we will introduce the notion of singular support of a distribution and prove some of its properties.

The most important property states that the singular support of a distribution on X is a (weakly) coisotropic subvariety of T^*X. Therefore we dedicate the first subsection to coisotropic and weakly coisotropic varieties.

1.8.1 Coisotropic varieties

Definition 1.8.1. *Let X be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset X$ be an algebraic subvariety. We call it X-**coisotropic** if one of the following equivalent conditions holds.*
(i) The ideal sheaf of regular functions that vanish on $\overline{Z}$ is closed under Poisson bracket.
(ii) At every smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^{\perp}$. Here, $(T_zZ)^{\perp}$ denotes the orthogonal complement to T_zZ in T_zX with respect to ω.
(iii) For a generic smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^{\perp}$.

If there is no ambiguity, we will call Z a coisotropic variety.

Note that every non-empty X-coisotropic variety is of dimension at least $\frac{1}{2}\dim X$.

Notation 1.8.2. *For a smooth algebraic variety X we always consider the standard symplectic form on T^*X. Also, we denote by $p_X : T^*X \to X$ the standard projection.*

Definition 1.8.3. *Let (V, ω) be a symplectic vector space with a fixed Lagrangian subspace $L \subset V$. Let $p : V \to V/L$ be the standard projection. Let $W \subset V$ be a linear subspace. We call it V-weakly coisotropic with respect to L if one of the following equivalent conditions holds.*
(i)$Z^{\perp} \cap L \subset Z \cap L$
Here, $Z^{\perp}$ denotes the orthogonal complement with respect to ω.
(ii) $p(Z^{\perp}) \subset p(Z)$.
(iii) $p(Z)^{\perp} \subset Z \cap L$.
Here, $p(Z)^{\perp}$ denotes the orthogonal complement in L under the identification $L \cong (V/L)^*$.

Definition 1.8.4. *Let X be a smooth algebraic variety. Let $Z \subset T^*X$ be an algebraic subvariety. We call it* T^*X**-weakly coisotropic** *if one of the following equivalent conditions holds.*
(i) At every smooth point $z \in Z$ the space $T_z(Z)$ is $T_z(T^*(X))$ -weakly coisotropic with respect to $Ker(dp_X)$.
(ii)For a generic smooth point $z \in Z$ the space $T_z(Z)$ is $T_z(T^*(X))$ -weakly coisotropic with respect to $Ker(dp_X)$.
(iii) For a generic smooth point $x \in Z$ and for a generic smooth point $y \in p_X^{-1}(x) \cap Z$ we have

$$CN^X_{p_X(Z),x} \subset T_y(p_X^{-1}(x) \cap Z).$$

(iv) For any smooth point $x \in p_X(Z)$ the fiber $p_X^{-1}(x) \cap Z$ is locally invariant with respect to shifts by $CN^X_{p_X(Z),x}$. I.e. for any point $y \in p_X^{-1}(x)$ the intersection

$$(y + CN^X_{p_X(Z),x}) \cap (p_X^{-1}(x) \cap Z)$$

is Zariski open in $y + CN_{p_X(Z)}$.

If there is no ambiguity, we will call Z a weakly coisotropic variety.

Note that every non-empty T^*X-weakly coisotropic variety is of dimension at least $\dim X$.

The following lemma is straightforward.

Lemma 1.8.5. *Any T^*X-coisotropic variety is T^*X-weakly coisotropic.*

Proposition 1.8.6. *Let X be a smooth algebraic variety with a symplectic form on it. Let $R \subset T^*X$ be an algebraic subvariety. Then there exists a maximal T^*X-weakly coisotropic subvariety of R i.e. a T^*X-weakly coisotropic subvariety $T \subset R$ that includes all T^*X-weakly coisotropic subvarieties of R.*

Proof. Let T' be the union of all smooth T^*X-weakly coisotropic subvarieties of R. Let T be the Zariski closure of T' in R. It is easy to see that T is the maximal T^*X-weakly coisotropic subvariety of R. □

The following lemma is trivial.

Lemma 1.8.7. *Let X be a smooth algebraic variety. Let a group G act on X. This induces an action on T^*X. Let $S \subset T^*X$ be a G-invariant subvariety. Then the maximal T^*X-weakly coisotropic subvariety of S is also G-invariant.*

Notation 1.8.8. *Let Y be a smooth algebraic variety. Let $Z \subset Y$ be a smooth subvariety and $R \subset T^*Y$ be any subvariety. We define* **the restriction** $R|_Z \subset T^*Z$ **of** R **to** Z *by $R|_Z := i^*(R)$, where $i : Z \to Y$ is the embedding.*

Lemma 1.8.9. *Let Y be a smooth algebraic variety. Let $Z \subset Y$ be a smooth subvariety and $R \subset T^*Y$ be a weakly coisotropic subvariety. Assume that any smooth point $z \in Z \cap p_Y(R)$ is also a smooth point of $p_Y(R)$ and we have $T_z(Z \cap p_Y(R)) = T_z(Z) \cap T_z(p_Y(R))$.*

Then $R|_Z$ is T^*Z-weakly coisotropic.

Proof. Let $x \in Z$, let $M := p_Y^{-1}(x) \cap R \subset p_Y^{-1}(x)$ and $L := CN^Y_{p_Y(R),x} \subset p_Y^{-1}(x)$. We know that M is locally invariant with respect to shifts in L. Let $M' := p_Z^{-1}(x) \cap R|_Z \subset p_Z^{-1}(x)$ and $L' := CN^Y_{p_Z(R|_Z),x} \subset p_Z^{-1}(x)$. We want to show that M' is locally invariant with respect to shifts in L'. Let $q : p_Y^{-1}(x) \to p_Z^{-1}(x)$ be the standard projection. Note that $M' = q(M)$ and $L' = q(L)$. Now clearly M' is locally invariant with respect to shifts in L'. □

Corollary 1.8.10. *Let Y be a smooth algebraic variety. Let an algebraic group H act on Y. Let $q : Y \to B$ be an H-equivariant morphism. Let $O \subset B$ be an orbit. Consider the natural action of G on T^*Y and let $R \subset T^*Y$ be an H-invariant subvariety. Suppose that $p_Y(R) \subset q^{-1}(O)$. Let $x \in O$. Denote $Y_x := q^{-1}(x)$. Then*

- *if R is T^*Y-weakly coisotropic then $R|_{Y_x}$ is $T^*(Y_x)$-weakly coisotropic.*

Corollary 1.8.11. *In the notation of the previous corollary, if $R|_{Y_x}$ has no (non-empty) $T^*(Y_x)$-weakly coisotropic subvarieties then R has no (non-empty) $T^*(Y)$-weakly coisotropic subvarieties.*

Remark 1.8.12. *The results on weakly coistropic varieties that we presented here have versions for coistropic varieties, see [AG08d, section 5.1].*

1.8.2 Definition and properties of singular support

Notation 1.8.13.

Let M, N be (smooth, algebraic, etc) manifolds. Let $S \subset (T^*(N))$. Let $\phi : M \to N$ be a morphism. We denote $\phi^*(S) := d(\phi)^*(S \times_N M)$.

Let V be a linear space. For a point $x = (v, \phi) \in V \times V^*$ we denote $\widehat{x} = (\phi, -v) \in V^* \times V$, similarly for subset $X \subset V \times V^*$ we define $\widehat{X}$. for a (smooth, algebraic, etc) manifold and a subset $X \subset T^*(M \times V)$ we denote $\widehat{X}_V \subset T^*(M \times V^*)$ in a similar way.

Let B be a non-degenerate bilinear form on V. This gives an identification between V and V^* and therefore, by the previous notation, defines maps $F_B : V \times V \to V \times V$ and $F_B : T^*M \times V \times V \to T^*M \times V \times V$. If there is bo ambiguity we will denote it by F_V.

Definition 1.8.14. *Let F be non-Archimedean. Let $U \subset F^n$ be an open subset and $\xi \in \mathcal{S}^*(U)$ be a distribution. We say that ξ is smooth at $(x_0, v_0) \in T^*U$ if there are open neighborhoods A of x_0 and B of v_0 such that for any $\phi \in \mathcal{S}(A)$ there is an $N_\phi > 0$ for which for any $\lambda \in F$ satisfying $\lambda > N_\phi$ we have $\widehat{(\phi\xi)}|_{\lambda B} = 0$. The complement in T^*U of the set of smooth pairs (x_0, v_0) of ξ is called the wave front set of ξ and denoted by $WF(\xi)$.*

Remark 1.8.15. *Sometimes in the literature (cf. [Hef]) the wave front set is defined to be a subset of $T^*U - U \times 0$. In our notation this subset will be $WF(\xi) - U \times 0$.*

We will use the following proposition.

Proposition 1.8.16 ([Aiz08], Proposition 2.1.19)**.** *Let F be non-Archimedean. Let $V, U \subset F^n$ be open subsets and $f : V \to U$ be an analytic isomorphism. Then for any $\xi \in \mathcal{S}^*(V)$ we have $WF(f^*(\xi)) = f^*(WF(\xi))$.*

Corollary 1.8.17. *Let F be non-Archimedean. Let X be an F-analytic manifold. Then we can define the the wave front set of any element in $\mathcal{S}^*(X)$.*

Definition 1.8.18. *Let X be a smooth algebraic variety let $\xi \in \mathcal{S}^*(X(F))$. We will now define the singular support of ξ, it is an algebraic subvariety of T^*X and we will denote it by $SS(\xi)$.*

In the case when F is non-Archimedean we define it to be the Zariski closure of $WF(\xi)$. In the case when F is Archimedean we define it to be the singular support of the D_X-module generated by ξ.

Remark 1.8.19. *For Archimedean F one can also define the wave front set in a way similar to the above. In this case the singular support always includes the Zariski closure of $WF(\xi)$ but is sometimes bigger.*

For readers who are not familiar with the theory of D-modules, the next theorem summarizes the only properties of singular support that we use. We will give more details on the algebraic theory of D-modules in subsection 1.8.4. For a good introduction to the algebraic theory of D-modules we refer the reader to [Ber] and [Bor87].

Theorem 1.8.20. *Let X be a smooth algebraic variety. Then*

1. *Let $\xi \in \mathcal{S}^*(X(F))$. Then $\overline{\mathrm{Supp}(\xi)}_{Zar} = p_X(SS(\xi))(F)$, where $\overline{\mathrm{Supp}(\xi)}_{Zar}$ denotes the Zariski closure of* $\mathrm{Supp}(\xi)$.
2. *Let an algebraic group G act on X. Let $\mathfrak{g}$ denote the Lie algebra of G. Let $\xi \in \mathcal{S}^*(X(F))^{G(F)}$. Then*
$$SS(\xi) \subset \{(x, \phi) \in T^*X \mid \forall \alpha \in \mathfrak{g}\, \phi(\alpha(x)) = 0\}.$$
3. *Let V be a linear space. Let $Z \subset X \times V$ be a closed subvariety, invariant with respect to homotheties in V. Suppose that* $\mathrm{Supp}(\xi) \subset Z(F)$. *Then $SS(\mathcal{F}_V(\xi)) \subset F_V(p_{X\times V}^{-1}(Z))$.*
4. *Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(F))$. Then $SS(\xi)$ is weakly coisotropic.*

For non-Archimedean fields these properties are proven in [Aiz08, Section 4]. For non-Archimedean fields, properties 1 - 3 are proven in subsection 1.8.4. Property 4, which is crucial for us, follows from the following theorem.

Theorem 1.8.21 (Integrability Theorem)**.** *Let X be a smooth algebraic variety. Let $\mathcal{M}$ be a finitely generated D_X-module. Then $SS(\mathcal{M})$ is a T^*X-coisotropic variety.*

This is a special case of Theorem I in [Gab81]. For similar versions see also [KKS73, Mal79].

Remark 1.8.22. *Evidently, in the Archimedean case the Integrability Theorem implies a stronger property. Namely it implies that $SS(\xi)$ is coisotropic and not only weakly coisotropic. We conjecture that this holds for the non-Archimedean case also.*

1.8.3 Distributions on non distinguished nilpotent orbits

Definition 1.8.23. *Let V be an algebraic finite dimensional representation over F of a reductive group G. Suppose that there is a finite number of G orbits in $\Gamma(V)$. Let $x \in \Gamma(V)$. We will call it G-distinguished, if $CN_{Gx,x}^{Q(V)} \subset \Gamma(V^*)$. We will call a G orbit G-distinguished if all (or equivalently one of) its elements are G- distinguished.*

If there is no ambiguity we will omit the "G-".

Example 1.8.24. *For the case of a semisimple group acting on its Lie algebra, the notion of G-distinguished element coincides with the standard notion of distinguished nilpotent element. In particular, in the case when $G = SL_n$ and $V = sl_n$ the set of G-distinguished elements is exactly the set of regular nilpotent elements.*

Proposition 1.8.25. *Let V be an algebraic finite dimensional representation over F of a reductive group G. Suppose that there is a finite number of G orbits on $\Gamma(V)$. Let $W := Q(V)$, let A be the set of non-distinguished elements in $\Gamma(V)$. Then there are no non-empty $W \times W^*$-weakly coisotropic subvarieties of $A \times \Gamma(V^*)$.*

The proof is clear.

Corollary 1.8.26. *Let $\xi \in \mathcal{S}^*(W)$ and suppose that* $\mathrm{Supp}(\xi) \subset \Gamma(V)$ *and supp*$(\widehat{\xi}) \subset \Gamma(V^*)$. *Then the set of distinguished elements in* $\mathrm{Supp}(\xi)$ *is dense in* $\mathrm{Supp}(\xi)$.

Remark 1.8.27. *In the same way one can prove an analogous result for distributions on $W \times M$ for any B-analytic manifold M.*

1.8.4 D-modules and proof of properties 1 - 3 in the Archimedean case

In this subsection X denotes a smooth affine variety defined over $\mathbb{R}$. All the statements of this section extend automatically to general smooth algebraic varieties defined over $\mathbb{R}$. In this thesis we use only the case when X is an affine space.

Definition 1.8.28. *Let $D(X)$ denote the algebra of polynomial differential operators on X. We consider the filtration $F^{\leq i}D(X)$ on $D(X)$ given by the order of differential operator.*

Definition 1.8.29. *We denote by* $\mathrm{Gr}\, D(X)$ *the associated graded algebra of $D(X)$.*

Define the symbol map $\sigma : D(X) \to \mathrm{Gr}\, D(X)$ in the following way. Let $d \in D(X)$. Let i be the minimal index such that $d \in F^{\leq i}$. We define $\sigma(d)$ to be the image of d in $(F^{\leq i}D(X))/(F^{\leq i-1}D(X))$

Proposition 1.8.30. $\mathrm{Gr}\, D(X) \cong \mathcal{O}(T^*X)$.

For proof see e.g. [Bor87].

Notation 1.8.31. *Let (V, B) be a quadratic space.*
(i) We define a morphism of algebras $\Phi_V^D : D(X \times V) \to D(X \times V)$ in the following way.

Consider B as a map $B : V \to V^*$. For any $f \in V^*$ we set $\Phi_V^D(f) := \partial_{B^{-1}(f)}$. For any $v \in V$ we set $\Phi_V^D(\partial_v) := -B(v)$ and for any $d \in D(X)$ we set $\Phi_V^D(d) := d$.
(ii) It defines a morphism of algebras $\Phi_V^O : \mathcal{O}(T^*X) \to \mathcal{O}(T^*X)$.

The following lemma is straightforward.

Lemma 1.8.32. *Let f be a homogeneous polynomial. Consider it as a differential operator. Then $\sigma(\Phi_V^D(f)) = \Phi_V^O(\sigma(f))$.*

The D-modules we use in the paper are right D-modules. The difference between right and left D-modules is not essential (see e.g. section VI.3 in [Bor87]). We will use the notion of good filtration on a D-module, see e.g. section II.4 in [Bor87]. Let us now remind the definition of singular support of a module and a distribution.

Notation 1.8.33. *Let M be a $D(X)$-module. Let $\alpha \in M$ be an element. Then we denote by $Ann_{D(X)}$ the annihilator of α.*

Definition 1.8.34. *Let M be a $D(X)$-module. Choose a good filtration on M. Consider grM as a module over $\operatorname{Gr} D(X) \cong \mathcal{O}(T^*X)$. We define*

$$SS(M) := \operatorname{Supp}(\operatorname{Gr} M) \subset T^*X.$$

This does not depend on the choice of the good filtration on M (see e.g. [Bor87], section II.4).

For a distribution $\xi \in S^*(X(\mathbb{R}))$ we define $SS(\xi)$ to be the singular support of the module of distributions generated by ξ.

The following proposition is trivial.

Proposition 1.8.35. *Let $I < D(X)$ be a right ideal. Consider the induced filtrations on I and $D(X)/I$. Then $\operatorname{Gr}(D(X)/I) \cong \operatorname{Gr}(D(X))/\operatorname{Gr}(I)$.*

Corollary 1.8.36. *Let $\xi \in S^*(X)$. Then $SS(\xi)$ is the zero set of $\operatorname{Gr}(Ann_{D(X)}\xi)$.*

Corollary 1.8.37. *Let $I < \mathcal{O}(T^*X)$ be the ideal generated by $\{\sigma(d) \,|\, d \in Ann_{D(X)}(\xi)\}$. Then $SS(\xi)$ is the zero set of I.*

Corollary 1.8.38. *Property 2 holds.*

Lemma 1.8.39. *Let $\xi \in S^*(X)$. Let $Z \subset X$ be a closed subvariety such that $\operatorname{Supp}(\xi) \subset Z(\mathbb{R})$. Let $f \in \mathcal{O}(X)$ be a polynomial that vanishes on Z. Then there exists $k \in \mathbb{N}$ such that $f^k\xi = 0$.*

Proof.

Step 1. Proof for the case when X is affine space and f is a coordinate function. This follows from the proof of Corollary 5.5.4 in [AG08a].

Step 2. Proof for the general case.

Embed X into an affine space A^N such that f will be a coordinate function and consider ξ as distribution on A^N supported in X. By Step 1, $f^k\xi = 0$ for some k. □

Corollary 1.8.40. *Property 1 holds.*

Proposition 1.8.41. *Property 3 holds. Namely:*

Let (V, B) be a quadratic space. Let $Z \subset X \times V$ be a closed subvariety, invariant with respect to homotheties in V. Suppose that $\operatorname{Supp}(\xi) \subset Z(\mathbb{R})$. Then $SS(\mathcal{F}_V(\xi)) \subset F_V(p_{X\times V}^{-1}(Z))$.

Proof. Let $f \in \mathcal{O}(X \times V)$ be homogeneous with respect to homotheties in V. Suppose that f vanishes on Z. Then $\Phi_V^D(f^k) \in Ann_{D(X)}(\mathcal{F}_V(\xi))$. Therefore $\sigma(\Phi_V^D(f^k))$ vanishes on $SS(\mathcal{F}_V(\xi))$. On the other hand, $\sigma(\Phi_V^D(f^k)) = \Phi_V^O(\sigma(f^k)) = (\Phi_V^O(\sigma(f)))^k$. Hence $SS(\mathcal{F}_V(\xi))$ is included in the zero set of $\Phi_V^O(\sigma(f))$. Intersecting over all such f we obtain the required inclusion. □

Chapter 2

Gelfand Pairs and Gelfand-Kazhdan criterion

This chapter is based on [GK75, AGS08, AG08c, AG08d].

2.1 Gelfand Pairs

Definition 2.1.1. *Let G be a reductive group. By an* ***admissible representation of*** *G we mean an admissible smooth representation of $G(F)$ if F is non-Archimedean (see [BZ76]) and admissible smooth Fréchet representation of $G(F)$ if F is Archimedean (see section 2.2).*

We now introduce three a-priori distinct notions of Gelfand pair.

Definition 2.1.2. *Let $H \subset G$ be a pair of reductive groups.*

- *We say that (G, H) satisfy* **GP1** *if for any irreducible admissible representation (π, E) of G we have*
$$\dim \operatorname{Hom}_{H(F)}(E, \mathbb{C}) \leq 1.$$
- *We say that (G, H) satisfy* **GP2** *if for any irreducible admissible representation (π, E) of G we have*
$$\dim \operatorname{Hom}_{H(F)}(E, \mathbb{C}) \cdot \dim \operatorname{Hom}_{H}(\widetilde{E}, \mathbb{C}) \leq 1.$$
- *We say that (G, H) satisfy* **GP3** *if for any irreducible* **unitary** *representation $(\pi, \mathcal{H})$ of $G(F)$ on a Hilbert space $\mathcal{H}$ we have*
$$\dim \operatorname{Hom}_{H(F)}(\mathcal{H}^{\infty}, \mathbb{C}) \leq 1.$$

We will call pairs that satisfy GP1 ***Gelfand pairs*** *and pairs that satisfy GP3* ***generalized Gelfand pairs****.*

Property GP1 was established by Gelfand and Kazhdan in certain p-adic cases (see [GK75]). Property GP2 was introduced in [Gro91] in the p-adic setting. Property GP3 was studied extensively by various authors both in the real and p-adic settings (see e.g. [vDP90], [vD86], [BvD94]).

We have the following straightforward proposition.

Proposition 2.1.3. *$GP1 \Rightarrow GP2 \Rightarrow GP3$.*

Remark 2.1.4. *It is not known whether some of these notions are equivalent.*

The most powerful tool for proving Gelfand property is the following theorem.

Theorem 2.1.5 (Gelfand-Kazhdan criterion)**.** *Let $H \subset G$ be reductive groups and let σ be an involutive anti-automorphism of G and assume that $\sigma(H) = H$. Suppose $\sigma(\xi) = \xi$ for all bi $H(F)$-invariant Schwartz distributions ξ on $G(F)$. Then (G, H) satisfies GP2.*

For non-Archimedean F this theorem is proven in [GK75]. We adapt their proof to the Archimedean case in the next subsection.

Remark 2.1.6. *This theorem is the main reason for introducing the property $GP2$.*

Remark 2.1.7. *In [Tho84] it is proven that if σ preserves for all bi $H(F)$-invariant positive definite distributions on $G(F)$ then (G, H) satisfies GP3.*

In some cases, GP2 is known to be equivalent to GP1. For example, see Corollary 2.1.9 below.

Theorem 2.1.8. *Let G be a reductive group and let σ be an anti-involution of G. Let θ be the involution of G defined by $\theta(g) := \sigma(g^{-1})$. Let (π, E) be an irreducible admissible representation of G. Suppose that either σ preserves all conjugacy classes in $G(F)$ or $\operatorname{char} F = 0$ and σ preserves all semisimple conjugacy classes in $G(F)$.*

Then $\widetilde{E} \cong E^{\theta}$, where $\widetilde{E}$ denotes the smooth contragredient representation and E^{θ} is E twisted by θ.

Proof. Since the representations $\widetilde{E}$ and E^{θ} are irreducible, they are isomorphic if and only if their characters are identical (see e.g. [Wall88, Theorem 8.1.5]).

If F is non-archimedean and σ preserves all conjugacy classes in $G(F)$ then the identity of characters follows from Theorem 1.3.5. If $\operatorname{char} F = 0$ and σ preserves all semisimple conjugacy classes in $G(F)$ then this follows from [AG08c, Corollary 7.6.3]. $\square$

Corollary 2.1.9. *Let $H \subset G$ be reductive groups and let σ be an anti-involution of G that preserves all conjugacy classes and such that $\sigma(H) = H$. Then $GP1$ is equivalent to $GP2$ for the pair (G, H).*

Choosing $\sigma(g) := g^t$ we obtain

Corollary 2.1.10. *$GP1$ is equivalent to $GP2$ for the pairs $(\mathrm{GL}_{n+1}, \mathrm{GL}_n)$ and $(O(V \oplus F), O(V))$.*

2.2 Smooth Fréchet representations and proof of Theorem 2.1.5 in the Archimedean case

The theory of representations in the context of Fréchet spaces is developed in [Cas89b] and [Wall92]. We present here a well-known slightly modified version of that theory. In this section the field F is Archimedean.

Definition 2.2.1. *Let V be a complete locally convex topological vector space. A representation (π, V, G) is a continuous map $G \times V \to V$. A representation is called* ***Fréchet*** *if there exists a countable family of semi-norms ρ_i on V defining the topology of V and such that the action of G is continuous with respect to each ρ_i. We will say that V is* ***smooth Fréchet*** *representation if, for any $X \in \mathfrak{g}$ the differentiation map $v \mapsto \pi(X)v$ is a continuous linear map from V to V.*

An important class of examples of smooth Fréchet representations is obtained from continuous Hilbert representations (π, H) by considering the subspace of smooth vectors H^∞ as a Fréchet space (see [Wall88, section 1.6] and [Wall92, section 11.5]).

We will consider mostly smooth Fréchet representations.

Remark 2.2.2. *In the language of [Wall92] and [Cas89a] the representations above are called smooth Fréchet representations of moderate growth.*

Recall that a smooth Fréchet representation is called *admissible* if its underlying $(\mathfrak{g}, K)$-module is admissible (in particular finitely generated). In what follows *admissible representation* will always refer to admissible smooth Fréchet representation.

For an admissible (smooth Fréchet) representation (π, E) we denote by $(\widetilde{\pi}, \widetilde{E})$ the contragredient representation.

We will require the following corollary of the globalization theorem of Casselman and Wallach (see [Wall92] , chapter 11).

Theorem 2.2.3. *Let E be an admissible Fréchet representation, then there exists a continuous Hilbert space representation (π, H) such that $E = H^\infty$.*

This theorem follows easily from the embedding theorem of Casselman combined with Casselman-Wallach globalization theorem.

Fréchet representations of G can be lifted to representations of $\mathcal{S}(G)$, the Schwartz space of G.

For a Fréchet representation (π, E) of G, the algebra $\mathcal{S}(G)$ acts on E through

$$\pi(\phi) = \int_G \phi(g)\pi(g)dg \tag{2.2.0.1}$$

(see [Wall88], section 8.1.1).
The following lemma is straightforward:

Lemma 2.2.4. *Let* (π, E) *be an admissible Fréchet representation of* G *and let* $\lambda \in E^*$*. Then* $\phi \to \pi(\phi)\lambda$ *is a continuous map* $\mathcal{S}(G) \to \widetilde{E}$*.*

The following proposition follows from Schur's lemma for $(\mathfrak{g}, K)$ modules (see [Wall88] page 80) in light of Casselman-Wallach theorem.

Proposition 2.2.5. *Let* G *be a real reductive group. Let* W *be a Fréchet representation of* G *and let* E *be an irreducible admissible representation of* G*. Let* $T_1, T_2 : W \hookrightarrow E$ *be two embeddings of* W *into* E*. Then* T_1 *and* T_2 *are proportional.*

Now we are ready to prove Theorem 2.1.5.

Proof of Theorem 2.1.5. Let (π, E) be an irreducible admissible Fréchet representation. If E or $\widetilde{E}$ are not distinguished by H we are done. Thus we can assume that there exists a non-zero $\lambda : E \to \mathbb{C}$ which is H-invariant. Now let ℓ_1, ℓ_2 be two non-zero H-invariant functionals on $\widetilde{E}$. We wish to show that they are proportional. For this we define two distributions D_1, D_2 as follows

$$D_i(\phi) = \ell_i(\pi(\phi)\lambda)$$

for $i = 1, 2$. Here $\phi \in \mathcal{S}(G)$. Note that D_i are also Schwartz distributions. Both distributions are bi-H-invariant and hence, by the assumption, both distributions are σ invariant. Now consider the bilinear forms on $\mathcal{S}(G)$ defined by

$$B_i(\phi_1, \phi_2) = D_i(\phi_1 * \phi_2).$$

Since E is irreducible, the right kernel of B_1 is equal to the right kernel of B_2. We now use the fact that D_i are σ invariant. Denote by J_i the left kernels of B_i. Then $J_1 = J_2$ which we denote by J. Consider the Fréchet representation $W = \mathcal{S}(G)/J$ and define the maps $T_i : \mathcal{S}(G) \to \widetilde{\widetilde{E}} \cong E$ by $T_i(\phi) = \pi(\phi)\ell_i$. These are well defined by Lemma 2.2.4 and we use the same letters to denote the induced maps $T_i : W \to E$. By Proposition 2.2.5, T_1 and T_2 are proportional and hence ℓ_1 and ℓ_2 are proportional and the proof is complete. $\square$

2.3 Strong Gelfand Pairs

Like in the previous section, there are several notions of strong Gelfand pair.

Definition 2.3.1. *Let* $H \subset G$ *be a pair of reductive groups.*

- *We say that* (G, H) *that is a strong Gelfand pair or that* (G, H) *satisfy* **SGP1** *if for any irreducible admissible representations* π *of* G *and* τ *of* H *we have*

$$\dim \mathrm{Hom}_{H(F)}(\pi, \tau) \leq 1.$$

- *We say that (G, H) satisfy* **SGP2** *if for any irreducible admissible representations π of G and τ of H we have*

$$\dim \mathrm{Hom}_{H(F)}(\pi, \tau) \cdot \dim \mathrm{Hom}_H(\widetilde{\pi}, \widetilde{\tau}) \leq 1.$$

- *We say that (G, H) is a generalized strong Gelfand pair or that (G, H) satisfy* **SGP3** *if for any irreducible* **unitary** *representations π of G and τ of H we have*

$$\dim \mathrm{Hom}_{H(F)}(\pi, \tau) \leq 1.$$

We will use the following two known theorems.

Theorem 2.3.2. *Let G and H be reductive groups. Let π and τ be irreducible admissible smooth representations of G and H respectively. Then $\pi \otimes \tau$ is irreducible representation $G \times H$.*

Is theorem is proven by reduction to an analogous statement on finite-dimensional representations of algebras with 1. For the statement see [Bou58, Ch. VIII, §7, Proposition 8].

For the non-Archimedean case this reduction is done in [BZ76, Proposition 2.16]. For the Archimedean case see e.g. [AG08d, Appendix A].

Theorem 2.3.3. *Let H be a reductive group. Let π and τ be smooth (Fréchet) representations of H. Suppose that τ is admissible.*

Then $Hom(\pi, \tau)$ is canonically isomorphic to $Hom(\pi \otimes \widetilde{\tau}, \mathbb{C})$.

Sketch of proof. Consider the canonical embeddings

$$Hom(\pi, \tau) \hookrightarrow Hom(\pi \otimes \widetilde{\tau}, \mathbb{C}) \hookrightarrow Hom(\pi, \widetilde{\widetilde{\tau}}).$$

Since for admissible representations $\widetilde{\widetilde{\tau}} \cong \tau$, those embeddings are isomorphisms. □

Corollary 2.3.4. *Let G and H be real reductive groups. The pair (G, H) is a strong Gelfand pair (respectively satisfies SGP2) if and only if the pair $(G \times H, \Delta H)$ is a Gelfand pair (respectively satisfies GP2).*

Using Corollary 2.1.9 we obtain

Corollary 2.3.5. *Let $H \subset G$ be reductive groups and let σ be an anti-involution of G that preserves all conjugacy classes and such that $\sigma(H) = H$.*

Then $SGP1$ is equivalent to $SGP2$ for the pair (G, H). In particular $SGP1$ is equivalent to $SGP2$ for the pair $(\mathrm{GL}_{n+1}, \mathrm{GL}_n)$.

Corollary 2.3.4 also enables to formulate the following analog of Gelfand-Kazhdan criterion for strong Gelfand pairs.

Theorem 2.3.6. *Let $H \subset G$ be reductive groups and let σ be an involutive anti-automorphism of G and assume that $\sigma(H) = H$. Suppose $\sigma(\xi) = \xi$ for all $AdH(F)$-invariant Schwartz distributions ξ on $G(F)$. Then (G, H) satisfies SGP2.*

Proof. Define $\sigma' : G \times H \to G \times H$ by $\sigma'(g, h) := (\sigma(g), \sigma(h))$. Let $\Delta H < G \times H$ be the diagonal. Consider the projection $G \times H \to H$. By Frobenius descent (Theorem 1.3.4), the assumption implies that any ΔH-bi-invariant distribution on $G \times H$ is invariant with respect to σ'.

Hence by Theorem 2.1.5 $(G \times H, \Delta H)$ satisfies GP2 and hence by the previous corollary (G, H) satisfies SGP2. □

Corollary 2.3.7. *Theorem B implies Theorem A.*

Chapter 3

Gelfand property for the pairs $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ and $(\mathrm{O}_{n+1}(F), \mathrm{O}_n(F))$

For fields of characteristic zero the results of this chapter are much weaker then the main results. I decided to include this chapter for two reasons. First, the proofs in this chapter work over any local field, while the analogs of the main results for fields of positive characteristic are not known. Second, the proofs in this chapter are shorter and clearly show how to use some of the tools. The first section of this chapter is based on [AGS08] and the second on [AGS09].

3.1 The pair $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$

Consider the standard imbedding $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F)$. We consider the two-sided action of $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ on $\mathrm{GL}_{n+1}(F)$ defined by $(g_1, g_2)h := g_1 h g_2^{-1}$. The goal of this section is to prove the following theorem.

Theorem 3.1.1. *Any* $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ *invariant distribution on* $\mathrm{GL}_{n+1}(F)$ *is invariant with respect to transposition.*

By Theorem 2.1.5 and Corollary 2.1.10 this implies

Theorem 3.1.2. *The pair* $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ *is a Gelfand pair.*

Since any character of $\mathrm{GL}_n(F)$ can be extended to $\mathrm{GL}_{n+1}(F)$, we obtain

Corollary 3.1.3. *Let* (π, E) *be an irreducible admissible representation of* $\mathrm{GL}_{n+1}(F)$ *and let* χ *be a character of* $\mathrm{GL}_n(F)$*. Then*

$$\dim Hom_{\mathrm{GL}_n(F)}(\pi, \chi) \leq 1.$$

For the proof of Theorem 3.1.1 we will use the following notation.

Notation 3.1.4. *Denote* $H := H_n := \mathrm{GL}_n$. *Denote*

$$G := G_n := \{(h_1, h_2) \in \mathrm{GL}_n \times \mathrm{GL}_n | \, det(h_1) = det(h_2)\}.$$

We consider H *to be diagonally embedded to* G.

Consider the action of the 2-element group S_2 *on* G *given by the involution* $(h_1, h_2) \mapsto (h_2^{-1^t}, h_1^{-1^t})$. *It defines a semidirect product* $\widetilde{G} := \widetilde{G}_n := G \rtimes S_2$. *Denote also* $\widetilde{H} := \widetilde{H}_n := H_n \rtimes S_2$.

Let $V = F^n$ *and* $X := X_n := \mathrm{gl}_n \times V \times V^*$.

The group $\widetilde{G}$ *acts on* X *by*

$$(h_1, h_2)(A, v, \phi) := (h_1 A h_2^{-1}, h_1 v, h_2^{-1^t}\phi) \text{ and}$$

$$\sigma(A, v, \phi) := (A^t, \phi^t, v^t)$$

where $(h_1, h_2) \in G$ *and* σ *is the generator of* S_2. *Note that* $\widetilde{G}$ *acts separately on* gl_n *and on* $V \times V^*$. *Define a character* χ *of* $\widetilde{G}$ *by* $\chi(g, s) := sign(s)$.

We will show that the following theorem implies Theorem 3.1.1.

Theorem 3.1.5. $\mathcal{S}^*(X(F))^{\widetilde{G}(F),\chi} = 0$.

3.1.1 Proof that Theorem 3.1.5 implies Theorem 3.1.1

We will divide this reduction to several propositions.
Consider the action of $\widetilde{G}_n$ on GL_{n+1} and on gl_{n+1}, where G_n acts by the two-sided action and the generator of S_2 acts by transposition.

Proposition 3.1.6. *If* $\mathcal{D}(\mathrm{GL}_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$ *then Theorem 3.1.1 holds.*

The proof is straightforward.

Proposition 3.1.7. *If* $\mathcal{S}^*(\mathrm{GL}_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$ *then* $\mathcal{D}(\mathrm{GL}_{n+1(F)})^{\widetilde{G}_n(F),\chi} = 0$.

Follows from Theorem 1.6.1.

Proposition 3.1.8. *If* $\mathcal{S}^*(\mathrm{gl}_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$ *then* $\mathcal{S}^*(\mathrm{GL}_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$.

Proof. [1] Let $\xi \in \mathcal{S}^*(\mathrm{GL}_{n+1}(F))^{\widetilde{G}_n(F),\chi}$. We have to prove $\xi = 0$. Assume the contrary. Take $p \in \mathrm{Supp}(\xi)$. Let $t = \det(p)$. Let $f \in C_c^\infty(F)$ be such that f vanishes in a neighborhood of zero and $f(t) \neq 0$. Consider the determinant map $\det : \mathrm{GL}_{n+1}(F) \to F$. Consider $\xi' := (f \circ \det) \cdot \xi$. It is easy to check that $\xi' \in \mathcal{S}^*(\mathrm{GL}_{n+1}(F))^{\widetilde{G}_n(F),\chi}$ and $p \in \mathrm{Supp}(\xi')$. However, we can extend ξ' by zero to $\xi'' \in \mathcal{S}^*(\mathrm{gl}_{n+1}(F))^{\widetilde{G}_n(F),\chi}$, which is zero by the assumption. Hence ξ' is also zero. Contradiction. □

[1] This proposition is an adaption of a statement in [Ber84], section 2.2.

Proposition 3.1.9. *If* $\mathcal{S}^*(X_n(F))^{\widetilde{G}_n(F),\chi} = 0$ *then* $\mathcal{S}^*(\mathrm{gl}_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$.

Proof. Note that gl_{n+1} is isomorphic as a $\widetilde{G}_n$-equivariant space to $X_n \times F$ where the action on F is trivial. This isomorphism is given by

$$\begin{pmatrix} A_{n\times n} & v_{n\times 1} \\ \phi_{1\times n} & t \end{pmatrix} \mapsto ((A, v, \phi), t).$$

The proposition now follows from Proposition 1.3.7. □

This finishes the proof that Theorem 3.1.5 implies Theorem 3.1.1.

3.1.2 Proof of Theorem 3.1.5

We will now stratify $X(= gl_n \times V \times V^*)$ and deal with each strata separately.

Notation 3.1.10. *Denote* $W := W_n := V_n \oplus V_n^*$. *Denote by* $Q^i := Q^i_n \subset \mathrm{gl}_n$ *the set of all matrices of rank* i. *Denote* $Z^i := Z^i_n := Q^i_n \times W_n$.

Note that $X = \bigcup Z^i$. Hence by Theorems 1.3.1 and 1.3.2, it is enough to prove the following proposition.

Proposition 3.1.11.
(i) If F *is non-Archimedean then* $\mathcal{S}^*(Z^i(F))^{\widetilde{G}(F),\chi} = 0$ *for any* i.
(ii) If F *is Archimedean then*

$$\mathcal{S}^*(Z^i(F), Sym^k(CN^X_{Z^i})(F))^{\widetilde{G}(F),\chi} = 0$$

for any k *and* i.

We will use the following important lemma.

Lemma 3.1.12. $\mathcal{S}^*(W(F))^{\widetilde{H}(F),\chi} = 0$.

For proof see subsection 3.1.3 below.

Corollary 3.1.13. *Proposition 3.1.11 holds for* $i = n$.

Proof. Clearly, one can extend the actions of $\widetilde{G}$ on Q^n and on Z^n to actions of $\widetilde{GL_n \times GL_n} := (GL_n \times GL_n) \rtimes S_2$ in the obvious way.

Step 1. $\mathcal{S}^*(Z^n(F))^{\widetilde{GL_n \times GL_n}(F),\chi} = 0$.
Consider the projection on the first coordinate from Z^n to the transitive $\widetilde{GL_n \times GL_n}$-space $Q^n = GL_n$. Choose the point $Id \in Q^n$. Its stabilizer is $\widetilde{H}$ and its fiber is W. Hence by Frobenius descent (Theorem 1.3.4),

$\mathcal{S}^*(Z^n(F))^{GL_n(\widetilde{F)\times GL_n}(F),\chi} \cong \mathcal{S}^*(W(F))^{\widetilde{H}(F),\chi}$ which is zero by the key lemma.

Step 2. $\mathcal{S}^*(Z^n(F))^{\widetilde{G}(F),\chi} = 0$.
Consider the space $Y := Z^n \times F^\times$ and let the group $GL_n \times GL_n$ act on it by $(h_1, h_2)(z, \lambda) := ((h_1, h_2)z, \det h_1 \det h_2^{-1}\lambda)$. Extend this action to action of $\widetilde{GL_n \times GL_n}$ by $\sigma(z, \lambda) := (\sigma(z), \lambda)$. Consider the projection $Z^n \times F^\times \to F^\times$. By Frobenius descent (Theorem 1.3.4),

$$\mathcal{S}^*(Y(F))^{GL_n(\widetilde{F)\times GL_n}(F),\chi} \cong \mathcal{S}^*(Z^n(F))^{\widetilde{G},\chi}.$$

Let Y' be equal to Y as an l-space and let $\widetilde{GL_n \times GL_n}$ act on Y' by $(h_1, h_2)(z, \lambda) := ((h_1, h_2)z, \lambda)$ and $\sigma(z, \lambda) := (\sigma(z), \lambda)$. Now Y is isomorphic to Y' as a $\widetilde{GL_n \times GL_n}$ space by $((A, v, \phi), \lambda) \mapsto ((A, v, \phi), \lambda \det A^{-1})$.

Since $\mathcal{S}^*(Z^n(F))^{GL_n(\widetilde{F)\times GL_n}(F),\chi} = 0$, Proposition 1.3.7 implies that $\mathcal{S}^*(Y'(F))^{GL_n(\widetilde{F)\times GL_n}(F),\chi} = 0$ and hence $\mathcal{S}^*(Y(F))^{GL_n(\widetilde{F)\times GL_n}(F),\chi} = 0$ and thus $\mathcal{S}^*(Z^n(F))^{\widetilde{G}_n(F),\chi} = 0$. □

Till the end of the subsection we assume that F is Archimedean, since the argument in the non-Archimedean case is similar but much easier.

Corollary 3.1.14. $\mathcal{S}^*(W_n(F), Sym^k(\mathrm{gl}_n^*)(F))^{\widetilde{G}(F),\chi} = 0$.

Proof. Consider the Killing form $K : \mathrm{gl}_n^* \to \mathrm{gl}_n$. Let $U := K^{-1}(Q_n^n)$. In the same way as in the previous corollary one can show that $\mathcal{S}^*(W_n(F) \times U(F))^{\widetilde{G}(F),\chi} = 0$. Hence by Proposition 1.3.3, $\mathcal{S}^*(W_n(F), Sym^k(\mathrm{gl}_n^*)(F))^{\widetilde{G}(F),\chi} = 0$. □

Corollary 3.1.15. *We have*

$$\mathcal{S}^*(W_i(F) \times W_{n-i}(F), Sym^k(0 \times \mathrm{gl}_{n-i}^*)(F))^{H_i(F)\times G_{n-i}(F)} =$$
$$= \mathcal{S}^*(W_i(F) \times W_{n-i}(F), Sym^k(0 \times \mathrm{gl}_{n-i}^*)(F))^{\widetilde{H}_i(F)\times \widetilde{G}_{n-i}(F)}.$$

Proof. It follows from Lemma 3.1.12, the last corollary and Proposition 1.3.7. □

Now we are ready to prove Proposition 3.1.11.

Proof of Proposition 3.1.11. Fix $i < n$. Consider the projection $pr_1 : Z^i \to Q^i$. It is easy to see that the action of $\widetilde{G}$ on Q^i is transitive. Denote

$$A_i := \begin{pmatrix} Id_{i\times i} & 0 \\ 0 & 0 \end{pmatrix} \in Q^i.$$

Denote by $G_{A_i} := \mathrm{Stab}_G(A_i)$ and $\widetilde{G}_{A_i} := \mathrm{Stab}_{\widetilde{G}}(A_i)$. Note that they are unimodular. By Frobenius descent (Theorem 1.3.4),

$$\mathcal{S}^*(Z^i(F), Sym^k(CN_{Z^i}^X)(F))^{\widetilde{G}(F),\chi} = \mathcal{S}^*(W(F), Sym^k(CN_{Q^i,A_i}^{\mathrm{gl}_n})(F))^{\widetilde{G}_{A_i}(F),\chi}.$$

Hence it is enough to show that

$$\mathcal{S}^*(W(F), Sym^k(CN_{Q^i,A_i}^{\mathrm{gl}_n})(F))^{G_{A_i}(F)} = \mathcal{S}^*(W(F), Sym^k(CN_{Q^i,A_i}^{\mathrm{gl}_n})(F))^{\widetilde{G}_{A_i}(F)}.$$

It is easy to check by explicit computation that

- $H_i \times G_{n-i}$ is canonically embedded into G_{A_i},
- W is isomorphic to $W_i \times W_{n-i}$ as $H_i \times G_{n-i}$-spaces
- $CN_{Q^i,A_i}^{\mathrm{gl}_n}$ is isomorphic to $0 \times \mathrm{gl}_{n-i}^*$ as $H_i \times G_{n-i}$ representations.

Let $\xi \in \mathcal{S}^*(W, Sym^k(CN_{Q^i,A_i}^{\mathrm{gl}_n})(F))^{G_{A_i}(F)}$. By the previous corollary, ξ is $\widetilde{H}_i(F) \times \widetilde{G}_{n-i}(F)$-invariant. Since ξ is also $G_{A_i}(F)$-invariant, it is $\widetilde{G}_{A_i}(F)$-invariant. □

3.1.3 Proof of Lemma 3.1.12

Proposition 3.1.16. *It is enough to prove the key lemma for $n = 1$.*

Proof. Consider the subgroup $T_n \subset H_n$ consisting of diagonal matrices, and $\widetilde{T}_n := T_n \rtimes S_2 \subset \widetilde{H}_n$. It is enough to prove $\mathcal{S}^*(W_n(F))^{\widetilde{T}_n(F),\chi} = 0$.

Now by Proposition 1.3.7 it is enough to prove $\mathcal{S}^*(W_1(F))^{\widetilde{H}_1(F),\chi} = 0$. □

From now on we fix $n := 1$, $H := H_1$, $\widetilde{H} := \widetilde{H}_1$ and $W := W_1$. Note that $H(F) = F^\times$ and $W(F) = F^2$. The action of $H(F)$ is given by $\rho(\lambda)(x, y) := (\lambda x, \lambda^{-1} y)$ and extended to the action of $\widetilde{H}(F)$ by the involution $\sigma(x, y) = (y, x)$.

Let $Y := \{(x, y) \in F^2 | \, xy = 0\} \subset W$ be the **cross** and $Y' := Y \setminus \{0\}$.

Lemma 3.1.17. *Every $(\widetilde{H}(F), \chi)$-equivariant distribution on W is supported inside the cross Y.*

Proof. Denote $U := W \setminus Y$. We have to show $\mathcal{S}^*(U(F))^{\widetilde{H}(F),\chi} = 0$. Consider the coordinate change $U(F) \cong F^\times \times F^\times$ given by $(x, y) \mapsto (xy, x/y)$. It is an isomorphism of $\widetilde{H}(F)$-spaces where the action of $\widetilde{H}(F)$ on $F^\times \times F^\times$ is only on the second coordinate, and given by $\lambda(w) = \lambda^2 w$ and $\sigma(w) = w^{-1}$. Clearly, $\mathcal{S}^*(F^\times)^{\widetilde{H}(F),\chi} = 0$ and hence by Proposition 1.3.7 $\mathcal{S}^*(F^\times \times F^\times)^{\widetilde{H}(F),\chi} = 0$. □

Lemma 3.1.18.
(i) $\mathcal{S}^*(W(F) \setminus Y(F))^{\widetilde{H}(F),\chi} = 0$.
(ii) Any distribution $\xi \in \mathcal{S}^*(Y'(F))^{\widetilde{H}(F),\chi}$ is invariant with respect to homotheties.

Moreover, if F is Archimedean then for all $k \in \mathbb{Z}_{\geq 0}$, any distribution $\xi \in \mathcal{S}^*(Y'(F), Sym^k(CN_{Y'}^W)(F))^{\widetilde{H}(F),\chi}$ is $\mathbb{R}$-homogeneous of type α_k where $\alpha_k(\lambda) := \lambda^{-2k}$.
(iii) $\mathcal{S}^*(\{0\})^{\widetilde{H}(F),\chi} = 0$. Moreover, if F is Archimedean then $\mathcal{S}^*(\{0\}, Sym^k(CN_{\{0\}}^W)(F))^{\widetilde{H}(F),\chi} = 0$.

If F is non-Archimedean this proposition is clear. Let us prove it for Archimedean F.

Proof. We have proven (i) in the proof of the previous lemma.
(ii) Fix $x_0 := (1, 0) \in Y'(F)$. Now we want to use Frobenius descent (Theorem 1.3.4) for the group $\widetilde{H}(F) \times \mathbb{R}^\times$ and character $\chi \times \alpha_k$. Note that $Stab_{\widetilde{H}}(x_0)$ is trivial and $Stab_{\widetilde{H}(F)\times\mathbb{R}^\times}(x_0) \cong \mathbb{R}^\times$. Note that $N_{Y',x_0}^W(F) \cong F$ and $Stab_{\widetilde{H}(F)\times\mathbb{R}^\times}(x_0)$ acts on it by $\rho(\lambda)a = \lambda^2 a$. So we have

$$Sym^k(N_{Y',x_0}^W(F)) = Sym^k(N_{Y',x_0}^W(F))^{\mathbb{R}^\times,\alpha_k^{-1}}.$$

So by Frobenius descent any distribution $\xi \in \mathcal{S}^*(Y'(F), Sym^k(CN_{Y'}^W)(F))^{\widetilde{H}(F),\chi}$ is $\mathbb{R}$-homogeneous of type α_k.
(iii) is a simple computation. Also, it can be deduced from (i) using Proposition 1.3.3. □

Proof of Lemma 3.1.12. Let $\xi \in \mathcal{S}^*(W(F))^{\widetilde{H}(F),\chi}$. Consider the quadratic form on $W = F^2$ given by $B(x, y) := xy$. It defines a Fourier transform on $\mathcal{S}^*(W(F))$. By Lemma 1.7.8, $\mathcal{F}(\xi) \in \mathcal{S}^*(W(F))^{\widetilde{H}(F),\chi}$.

Hence by Lemma 3.1.17, $\mathrm{Supp}\xi, \mathrm{Supp}\mathcal{F}(\xi) \subset Y$. By Homogeneity Theorem (Theorem 1.7.10) this implies that ξ is B-adapted.

Finally, the previous lemma and Theorem 1.3.2 imply that all B-adapted distributions in $\mathcal{S}^*(W(F))^{\widetilde{H}(F),\chi}$ are 0. □

3.2 The pair $(\mathrm{O}_{n+1}(F), \mathrm{O}_n(F))$

Let (W, Q) be a quadratic space defined over F and fix $e \in W$ a unit vector. Consider the quadratic space $V = e^\perp$ with $q = Q|_V$. Define the standard imbedding $\mathrm{O}(V) \hookrightarrow \mathrm{O}(W)$ and consider the two-sided action of $\mathrm{O}(V) \times \mathrm{O}(V)$ on $\mathrm{O}(W)$ defined by $(g_1, g_2)h := g_1 h g_2^{-1}$. We also consider the anti-involution σ of O_Q given by $\sigma(g) = g^{-1}$. In this paper we prove the following Theorem

Theorem 3.2.1. *Any $\mathrm{O}(V) \times \mathrm{O}(V)$ invariant distribution on $\mathrm{O}(W)$ is invariant under σ.*

By Theorem 2.1.5 and Corollary 2.1.10 this theorem implies

Theorem 3.2.2. *The pair* $(\mathrm{O}(W), \mathrm{O}(V))$ *is a Gelfand pair.*

Remark 3.2.3. *In fact, our proof can be easily adapted to prove also an analogous theorem for unitary groups.*

Remark 3.2.4. *This result was earlier proven in a bit different form in [vD86] for* $F = \mathbb{R}$*, in [AvD06] for* $F = \mathbb{C}$ *and in [BvD94] for non-Archimedean* F *of characteristic zero. Here we give a different proof.*

Also, an analogous theorem for unitary groups over $\mathbb{R}$ *is proven in [vD09].*

For the proof we need some further notations.

- $O_q = O(V, q)$ is the group of isometries of the quadratic space (V, q).
- $G_q = O(V, q) \times O(V, q)$.
- $\Delta : O_q \to G_q$ the diagonal. $H_q = \Delta(O_q) \subset G_q$.
- $\sigma(g_1, g_2) = (g_2, g_1)$.
- $\widetilde{G_q} = G_q \rtimes \{1, \sigma\}$, same for $\widetilde{H_q}$
- $\chi : \widetilde{G_q} \to \{+1, -1\}$ the non trivial character with $\chi(G_q) = 1$.
- $\widetilde{G_Q}$ acts on O_Q by $(g_1, g_2)x = g_1 x g_2^{-1}$ and $\sigma(x) = x^{-1}$.

By Theorem 1.6.1, Theorem 3.2.1 follows from the following theorem:

Theorem 3.2.5.

$$\mathcal{S}^*(O_Q)^{\widetilde{G_q}, \chi} = 0.$$

3.2.1 Proof of Theorem 3.2.5

We denote by $\Gamma = \{w \in W : Q(w) = 1\}$. Note that by Witt's theorem Γ is an O_Q transitive set and therefore $\Gamma \times \Gamma$ is a transitive $\widetilde{G}_Q$ set where the action of G_Q is the standard action on $W \oplus W$ and σ acts by flip.

Applying Frobenuis descent (Theorem 1.3.4) to projections of $O_Q \times \Gamma \times \Gamma$ first on $\Gamma \times \Gamma$ and then on O_Q we have

$$\mathcal{S}^*(O_Q)^{\widetilde{G_q}, \chi} = \mathcal{S}^*(O_Q \times \Gamma \times \Gamma)^{\widetilde{G_Q}, \chi}$$

and also that

$$\mathcal{S}^*(O_Q \times \Gamma \times \Gamma)^{\widetilde{G_Q}, \chi} = \mathcal{S}^*(\Gamma \times \Gamma)^{\widetilde{H_Q}, \chi}$$

In what follows we will abuse notation and write $Q(u, v)$ for the bilinear form defined by Q. Define a map $D : \Gamma \times \Gamma \to Z$ where $Z = \{(v, u) \in W \oplus W : Q(v, u) = 0, Q(v + u) = 4\}$ by

$$D(x, y) = (x + y, x - y).$$

D defines an $\widetilde{G}_Q$-equivariant homeomorphism and thus we need to show that

$$\mathcal{S}^*(Z)^{\widetilde{H_Q},\chi} = 0$$

Here, the action of G_Q on $Z \subset W \oplus W$ is the restriction of its action on $W \oplus W$ while the action of σ is given by $\sigma(v,u) = (v,-u)$.

Now we cover $Z = U_1 \cup U_2$ where

$$U_1 = \{(v,u) \in Z : Q(v) \neq 0\}$$

and

$$U_2 = \{(v,u) \in Z : Q(u) \neq 0\}$$

We will show $\mathcal{S}^*(U_1)^{\widetilde{H_Q},\chi} = 0$, and the proof for U_2 is analogous. This will finish the proof.

Lemma 3.2.6. $\mathcal{S}^*(U_1)^{\widetilde{H_Q},\chi} = 0$

Proof for non-archimedean F. Consider $\ell_1 : U_1 \to F - \{0\}$ defined as $\ell_1(v,u) = Q(v)$. By the Localization Principle (Corollary 1.5.3), it is enough to show $\mathcal{S}^*(U_1^\alpha)^{\widetilde{H_Q},\chi} = 0$ where $U_1^\alpha = \ell_1^{-1}(\alpha)$, for any $\alpha \in F - \{0\}$. But

$$U_1^\alpha = \{(v,u)|Q(v) = \alpha, Q(u) = 4-\alpha, Q(v,u) = 0\}$$

Let $W^\alpha = \{w \in W|Q(w) = \alpha\}$ and let $p_1 : U_1^\alpha \to W^\alpha$ be given by $p_1(v,u) = v$.

On W^α our group acts transitively. Fix a vector $v_0 \in W^\alpha$.

Denote $H(v_0) := H_{(Q|_{v_0^\perp})}$ and $\widetilde{H}(v_0) := \widetilde{H}_{(Q|_{v_0^\perp})}$.

The stabilizer in $\widetilde{H}_Q$ of v_0 is $\widetilde{H}(v_0)$. The fiber $p_1^{-1}(v_0) = \{a \in v_0^\perp|Q(a) = 4-\alpha\}$. Frobenius descent implies that

$$\mathcal{S}^*(U_1^\alpha)^{\widetilde{H_Q},\chi} = \mathcal{S}^*(p_1^{-1}(v_0))^{\widetilde{H}(v_0),\chi}$$

But clearly $\mathcal{S}^*(p_1^{-1}(v_0))^{\widetilde{H}(v_0),\chi} = 0$ as $-Id \in H(v_0)$. □

Proof for archimedean F. Now let us consider the archimedean case. Define $U := \{(v,u) \in U_1|u \neq 0\}$. Note that the map $\ell_1|_U$ is a submersion, so the same argument as in the non-archimedean case shows that $\mathcal{S}^*(U)^{\widetilde{H_Q},\chi} = 0$. Let $Y := \{(v \in W|Q(v) = 4\} \times \{0\}$ be the complement to U in U_1. By Theorem 1.3.2, it is enough to prove

$$\mathcal{S}^*(Y, Sym^k(CN_Y^{U_1}))^{\widetilde{H_Q},\chi} = 0.$$

Note that the action of $\widetilde{H}_Q$ on Y is transitive, and fix a point $(v,0) \in Y$. The stabilizer in $\widetilde{H}_Q$ of $(v,0)$ is $\widetilde{H}(v)$, and the normal space to Y at $(v,0)$ is $v^\perp$. So Frobenius descent (Theorem 1.3.4) implies that

$$\mathcal{S}^*(Y, Sym^k(CN_Y^{U_1}))^{\widetilde{H_Q},\chi} = Sym^k(v^\perp)^{\widetilde{H}(v),\chi}.$$

But clearly $Sym^k(v^\perp)^{\widetilde{H}(v),\chi} = 0$ as $-Id \in H(v)$. $\square$

Chapter 4

Proof of the main results

In this chapter we assume that F has zero characteristic. This chapter is based on [AG08d, Aiz08].

Consider the standard imbedding $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F)$. We consider the action of $\mathrm{GL}_n(F)$ on $\mathrm{GL}_{n+1}(F)$ by conjugation. In this chapter we prove Theorem B, namely

Theorem. *Any* $\mathrm{GL}_n(F)$ *- invariant distribution on* $\mathrm{GL}_{n+1}(F)$ *is invariant with respect to transposition.*

By Corollary 2.3.7 it implies Theorem A, namely

Theorem. $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ *is a strong Gelfand pair.*

4.1 Structure of the proof

We will now briefly sketch the main ingredients of our proof of Theorem B.

First we show that we can switch to the following problem. The group $\mathrm{GL}_n(F)$ acts on a certain linear space X_n and σ is an involution of X_n. We have to prove that every $\mathrm{GL}_n(F)$-invariant distribution on X_n is also σ-invariant. We do that by induction on n. Using the Harish-Chandra descent method we show that the induction hypothesis implies that this holds for distributions on the complement to a certain small closed subset $S \subset X_n$. We call this set *the singular set*.

Next we assume the contrary: there exists a non-zero $\mathrm{GL}_n(F)$-invariant distribution ξ on X which is anti-invariant with respect to σ.

We use the notion of singular support of a distribution (see section 1.8) . Let $T \subset T^*X$ denote the singular support of ξ. Using Fourier transform and the fact any such distribution is supported in S we obtain that T is contained in $\check{S}$ where $\check{S}$ is a certain small subset in T^*X.

Then we use a powerful tool, Theorem 1.8.20, which states that the singular support of a distribution is a weakly coisotropic variety in the cotangent bundle. This enables us to show, using a complicated but purely geometric argument, that the support of ξ is contained in a much smaller subset of S.

Finally it remains to prove that any $\mathrm{GL}_n(F)$-invariant distribution that is supported on this subset together with its Fourier transform is zero. This is proven using Homogeneity Theorem (Theorem 1.7.7) which in turn uses Weil representation.

Now let us describe the chapter section by section.

In section 4.2 we introduce notation that we will use in our proof.

In section 4.3 we use the Harish-Chandra descent method.

In subsection 4.3.1 we linearize the problem to a problem on the linear space $X = \mathrm{sl}(V) \times V \times V^*$, where $V = F^n$.

In subsection 4.3.2 we perform the Harish-Chandra descent on the $\mathrm{sl}(V)$-coordinate and $V \times V^*$ coordinate separately and then use non-linear automorphisms ν_λ of X to descend further to the singular set S. The automorphisms ν_λ of X first appeared in the first proof of Theorem B for non-Archimedean fields, in [AGRS07], and were essential part of the proof. In the proof we give here they are used mainly to shorten computations. However, in the proof of an analogous theorem for orthogonal groups over Archimedean fields in [SZ08] they are again used in an essential way.

In section 4.4 we reduce Theorem B to the following geometric statement: any coisotropic subvariety of $\check{S}$ is contained in a certain set $\check{C}_{X\times X}$. The reduction is done using the fact that the singular support of a distribution has to be coisotropic, and the following proposition: any $\mathrm{GL}(V)$-invariant distribution on X such that it and its Fourier transform are supported on $\mathrm{sl}(V) \times (V \times 0 \cup 0 \times V^*)$ is zero.

In subsection 4.4.1 we prove this proposition using Homogeneity theorem.

In section 4.5 we prove the geometric statement. On one hand, this section is rather complicated. On the other hand, it is purely geometric and involves no distributions. Using the fact that the only non-distinguished orbit in sl_n is the regular orbit, we reduce the geometric statement to the Key Lemma that states that a certain subset R_A of $V \times V^* \times V \times V^*$ contains no non-empty coisotropic subvarieties. This subset has Lagrangian dimension, i.e. $\dim R_A = 2n$. Hence in order to prove the Key Lemma it is enough to exhibit one additional equation on R_A. We write this equation explicitly.

4.2 Notation

In this chapter we will use the following notation

- Let $V := V_n$ be the standard n-dimensional linear space defined over F.
- Let $\mathrm{sl}(V)$ denote the Lie algebra of operators with zero trace.
- Denote $X := X_n := \mathrm{sl}(V_n) \times V_n \times V_n^*$
- $G := G_n := \mathrm{GL}(V_n)$
- $\mathfrak{g} := \mathfrak{g}_n := \mathrm{Lie}(G_n) = \mathrm{gl}(V_n)$
- $\widetilde{G} := \widetilde{G}_n := G_n \rtimes \{1, \sigma\}$, where the action of the 2-element group $\{1, \sigma\}$ on G is given by the involution $g \mapsto g^{t^{-1}}$.
- We define a character χ of $\widetilde{G}$ by $\chi(G) = \{1\}$ and $\chi(\widetilde{G} - G) = \{-1\}$.
- Let G_n act on G_{n+1}, $\mathfrak{g}_{n+1}$ and on $\mathrm{sl}(V_n)$ by $g(A) := gAg^{-1}$.
- Let G act on $V \times V^*$ by $g(v, \phi) := (gv, (g^*)^{-1}\phi)$. This gives rise to an action of G on X.
- Extend the actions of G to actions of $\widetilde{G}$ by $\sigma(A) := A^t$ and $\sigma(v, \phi) := (\phi^t, v^t)$.
- We consider the standard scalar products on $\mathrm{sl}(V)$ and $V \times V^*$. They give rise to a scalar product on X.
- We identify the cotangent bundle T^*X with $X \times X$ using the above scalar product.
- Let $\mathcal{N} := \mathcal{N}_n \subset \mathrm{sl}(V_n)$ denote the cone of nilpotent operators.
- $C := (V \times 0) \cup (0 \times V^*) \subset V \times V^*$.
- $\check{C} := (V \times 0 \times V \times 0) \cup (0 \times V^* \times 0 \times V^*) \subset V \times V^* \times V \times V^*$.
- $\check{C}_{X \times X} := (\mathrm{sl}(V) \times V \times 0 \times \mathrm{sl}(V) \times V \times 0) \cup (\mathrm{sl}(V) \times 0 \times V^* \times \mathrm{sl}(V) \times 0 \times V^*) \subset X \times X$.
- $S := \{(A, v, \phi) \in X_n | A^n = 0 \text{ and } \phi(A^i v) = 0 \text{ for any } 0 \le i \le n\}$.
-
$$\check{S} := \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in X \times X \,|\, \forall i, j \in \{1, 2\} \\ (A_i, v_j, \phi_j) \in S \text{ and } \forall \alpha \in \mathrm{gl}(V), \alpha(A_1, v_1, \phi_1) \perp (A_2, v_2, \phi_2)\}$$
.
- Note that
$$\check{S} = \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in X \times X \,|\, \forall i, j \in \{1, 2\} \\ (A_i, v_j, \phi_j) \in S \text{ and } [A_1, A_2] + v_1 \otimes \phi_2 - v_2 \otimes \phi_1 = 0\}.$$
- $\check{S}' := \check{S} - \check{C}_{X \times X}$.
- $\Gamma := \{(v, \phi) \in V \times V^* \,|\, \phi(V) = 0\}$.

- For any $\lambda \in F$ we define $\nu_\lambda : X \to X$ by $\nu_\lambda(A, v, \phi) := (A+\lambda v\otimes\phi-\lambda\frac{\langle\phi,v\rangle}{n}\mathrm{Id}, v, \phi)$.
- It defines $\check{\nu}_\lambda : X \times X \to X \times X$. It is given by

$$\check{\nu}_\lambda((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) = \\ = ((A_1 + \lambda v_1 \otimes \phi_1 - \lambda\frac{\langle \phi_1, v_1\rangle}{n}\mathrm{Id}, v_1, \phi_1), (A_2, v_2 - \lambda A_2 v_1, \phi_2 - \lambda A_2^* \phi_1)).$$

4.3 Harish-Chandra descent

4.3.1 Linearization

In this subsection we reduce Theorem A to the following one

Theorem 4.3.1. $\mathcal{S}^*(X(F))^{\widetilde{G}(F),\chi} = 0$.

We will divide this reduction to several propositions.

Proposition 4.3.2. *If* $\mathcal{D}(G_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$ *then Theorem A holds.*

The proof is straightforward.

Proposition 4.3.3. *If* $\mathcal{S}^*(G_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$ *then* $\mathcal{D}(G_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$.

Follows from Theorem 1.6.1.

Proposition 4.3.4. *If* $\mathcal{S}^*(\mathfrak{g}_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$ *then* $\mathcal{S}^*(G_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$.

Proof. Let $\xi \in \mathcal{S}^*(G_{n+1}(F))^{\widetilde{G}_n(F),\chi}$. We have to prove $\xi = 0$. Assume the contrary. Take $p \in \mathrm{Supp}(\xi)$. Let $t = \det(p)$. Let $f \in \mathcal{S}(F)$ be such that f vanishes in a neighborhood of 0 and $f(t) \neq 0$. Consider the determinant map $\det : G_{n+1}(F) \to F$. Consider $\xi' := (f \circ \det) \cdot \xi$. It is easy to check that $\xi' \in \mathcal{S}^*(G_{n+1}(F))^{\widetilde{G}_n(F),\chi}$ and $p \in \mathrm{Supp}(\xi')$. However, we can extend ξ' by zero to $\xi'' \in \mathcal{S}^*(\mathfrak{g}_{n+1}(F))^{\widetilde{G}_n(F),\chi}$, which is zero by the assumption. Hence ξ' is also zero. Contradiction. □

Proposition 4.3.5. *If* $\mathcal{S}^*(X_n(F))^{\widetilde{G}_n(F),\chi} = 0$ *then* $\mathcal{S}^*(\mathfrak{g}_{n+1}(F))^{\widetilde{G}_n(F),\chi} = 0$.

Proof. The $\widetilde{G}_n(F)$-space $\mathrm{gl}_{n+1}(F)$ is isomorphic to $X_n(F) \times F \times F$ with trivial action on $F \times F$. This isomorphism is given by

$$\begin{pmatrix} A_{n\times n} & v_{n\times 1} \\ \phi_{1\times n} & \lambda \end{pmatrix} \mapsto ((A - \frac{\mathrm{Tr}\, A}{n}\mathrm{Id}, v, \phi), \lambda, \mathrm{Tr}\, A).$$

□

4.3.2 Harish-Chandra descent

Now we start to prove Theorem 4.3.1. The proof is by induction on n. Till the end of the paper we will assume that Theorem 4.3.1 holds for all $k < n$ for both archimedean local fields.
The theorem obviously holds for $n = 0$. Thus from now on we assume $n \geq 1$. The goal of this subsection is to prove the following theorem.
Theorem 4.3.6. $\mathcal{S}^*(X(F) - S(F))^{\widetilde{G}(F),\chi} = 0$.
In fact, one can prove this theorem directly using Theorem 1.4.24. However, this will require long computations. Thus, we will divide the proof to several steps and use some tricks to avoid part of those computations.
Proposition 4.3.7. $\mathcal{S}^*(X(F) - (\mathcal{N} \times V \times V^*)(F))^{\widetilde{G}(F),\chi} = 0$.

Proof. By Theorem 1.4.24 it is enough to prove that for any semisimple $A \in \mathrm{sl}(V)$ we have

$$\mathcal{S}^*((N_{GA,A}^{\mathrm{sl}(V)} \times (V \times V^*))(F))^{\widetilde{G}(F)_A,\chi} = 0.$$

Now note that $\widetilde{G}(F)_A \cong \prod \widetilde{G}_{n_i}(F_i)$ where $n_i < n$ and F_i are some field extensions of F. Note also that

$$(N_{GA,A}^{\mathrm{sl}(V)} \times V \times V^*)(F) \cong \mathrm{sl}(V)_A \times (V \times V^*)(F) \cong \prod X_{n_i}(F_i) \times \mathcal{Z}(\mathrm{sl}(V)_A)(F),$$

where $\mathcal{Z}(\mathrm{sl}(V)_A)$ is the center of $\mathrm{sl}(V)_A$. Clearly, $\widetilde{G}_A$ acts trivially on $\mathcal{Z}(\mathrm{sl}(V)_A)$. Now by Proposition 1.3.7 the induction hypothesis implies that

$$\mathcal{S}^*(\prod X_{n_i}(F_i) \times \mathcal{Z}(\mathrm{sl}(V)_A)(F))^{\prod \widetilde{G}_{n_i}(F_i),\chi} = 0.$$

□

In the same way we obtain the following proposition.
Proposition 4.3.8. $\mathcal{S}^*(X(F) - (\mathrm{sl}(V) \times \Gamma)(F))^{\widetilde{G}(F),\chi} = 0$.
Corollary 4.3.9. $\mathcal{S}^*(X(F) - (\mathcal{N} \times \Gamma)(F))^{\widetilde{G}(F),\chi} = 0$.
Lemma 4.3.10. *Let $A \in \mathrm{sl}(V)$, $v \in V$ and $\phi \in V^*$. Suppose $A + \lambda v \otimes \phi$ is nilpotent for all $\lambda \in F$. Then $\phi(A^i v) = 0$ for any $i \geq 0$.*

Proof. Since $A + \lambda v \otimes \phi$ is nilpotent, we have $tr(A + \lambda v \otimes \phi)^k = 0$ for any $k \geq 0$ and $\lambda \in F$. By induction on i this implies that $\phi(A^i v) = 0$. □

Proof of Theorem 4.3.6. By the previous lemma, $\bigcap_{\lambda \in F} \nu_\lambda(\mathcal{N} \times \Gamma) \subset S$. Hence $\bigcup_{\lambda \in F} \nu_\lambda(X - \mathcal{N} \times \Gamma) \supset X - S$.
By Corollary 4.3.9 $\mathcal{S}^*(X(F) - (\mathcal{N} \times \Gamma)(F))^{\widetilde{G}(F),\chi} = 0$. Note that ν_λ commutes with the action of $\widetilde{G}$. Thus $\mathcal{S}^*(\nu_\lambda(X(F) - (\mathcal{N} \times \Gamma)(F)))^{\widetilde{G}(F),\chi} = 0$ and hence $\mathcal{S}^*(X(F) - S(F))^{\widetilde{G}(F),\chi} = 0$. □

Remark 4.3.11. *The automorphisms ν_λ of X first appeared in the first proof of Theorem B for non-Archimedean fields, in [AGRS07], and were essential part of the proof. In the proof we give here they are used mainly to shorten computations. However, in the proof of an analogous theorem for orthogonal groups over Archimedean fields in [SZ08] they are again used in an essential way.*

Remark 4.3.12. *The main ingredient of ν_λ, i.e. the map $(A, v, \phi) \mapsto A + v \otimes \phi$ is the moment map corresponding to the action of $\mathcal{G}$ on $\mathfrak{g} \times V \times V^*$.*

4.4 Reduction to the geometric statement

In this section coisotropic variety means $X \times X$-coisotropic variety.
The goal of this section is to reduce Theorem 4.3.1 to the following geometric statement.

Theorem 4.4.1 (geometric statement)**.** *For any coisotropic subvariety of $T \subset \check{S}$ we have $T \subset \check{C}_{X \times X}$.*

Till the end of this section we will assume the geometric statement.

Proposition 4.4.2. *Let $\xi \in \mathcal{S}^*(X(F))^{\widetilde{G}(F),\chi} = 0$. Then* $\mathrm{Supp}(\xi) \subset (\mathrm{sl}(V) \times C)(F)$*.*

Proof for the case $F = \mathbb{R}$.
Step 1. $SS(\xi) \subset \check{S}$.
We know that

$$\mathrm{Supp}(\xi), \mathrm{Supp}(\mathcal{F}^{-1}_{\mathrm{sl}(V)}\xi), \mathrm{Supp}(\mathcal{F}^{-1}_{V \times V^*}(\xi)), \mathrm{Supp}(\mathcal{F}^{-1}_X(\xi)) \subset S(F).$$

By property (3) of the singular support this implies that

$$SS(\xi) \subset (S \times X) \cap F_{\mathrm{sl}(V)}(S \times X) \cap F_{V \times V^*}(S \times X) \cap F_X(S \times X).$$

On the other hand, ξ is $G(F)$-invariant and hence by property (2) of the singular support

$$SS(\xi) \subset \{((x_1, x_2) \in X \times X \,|\, \forall g \in \mathfrak{g}, g(x_1) \perp x_2\}.$$

Thus $SS(\xi) \subset \check{S}$.
Step 2. $SS(\xi) \subset \check{C}_{X \times X}$.
By Corollary 1.8.21, $SS(\xi)$ is $X \times X$-coisotropic and hence by the geometric statement $SS(\xi) \subset \check{C}_{X \times X}$.
Step 3. $\mathrm{Supp}(\xi) \subset (\mathrm{sl}(V) \times C)(F)$.
Follows from the previous step by property (1) of the singular support. □

The case $F = \mathbb{C}$ is proven in the same way using the following corollary of the geometric statement.

Proposition 4.4.3. *Any $(X\times X)_{\mathbb{C}}$-coisotropic subvariety of $\check{S}_{\mathbb{C}}$ is contained in $(\check{C}_{X\times X})_{\mathbb{C}}$.*
Now it is left to prove the following proposition.
Proposition 4.4.4. *Let $\xi \in \mathcal{S}^*(X(F))^{\widetilde{G}(F),\chi}$ be such that*

$$\mathrm{Supp}(\xi), \mathrm{Supp}(\mathcal{F}_{V\times V^*}(\xi)) \subset (\mathrm{sl}(V)\times C)(F).$$

Then $\xi = 0$.

4.4.1 Proof of Proposition 4.4.4

Proposition 4.4.4 follows from the following lemma.
Lemma 4.4.5. *Let $F^\times$ act on $V\times V^*$ by $\lambda(v,\phi) := (\lambda v, \frac{\phi}{\lambda})$. Let $\xi \in \mathcal{S}^*((V\times V^*)(F))^{F^\times}$ be such that*

$$\mathrm{Supp}(\xi), \mathrm{Supp}(\mathcal{F}_{V\times V^*}(\xi)) \subset C(F).$$

Then $\xi = 0$.
By Homogeneity Theorem (Theorem 1.7.7) it is enough to prove the following lemma.
Lemma 4.4.6. *Let μ be a character of $F^\times$ given by $||\cdot||^n u$ or $||\cdot||^{n+1}u$ where u is some unitary character. Let $F^\times \times F^\times$ act on $V\times V^*$ by $(x,y)(v,\phi) = (\frac{y}{x}v, \frac{1}{xy}\phi)$. Then $\mathcal{S}^*_{(V\times V^*)(F)}(C(F))^{F^\times\times F^\times,\mu\times 1} = 0$.*
We will prove this lemma for Archimedean F, since the proof for non-Archimedean is similar but simpler.
By Theorem 1.3.2 this lemma follows from the following one.
Lemma 4.4.7. *For any $k \geq 0$ we have*
(i) $\mathcal{S}^*(((V-0)\times 0)(F), Sym^k(CN^{V\times V^*}_{(V-0)\times 0}(F)))^{F^\times\times F^\times,\mu\times 1} = 0$.
(ii) $\mathcal{S}^*((0\times (V^*-0))(F), Sym^k(CN^{V\times V^*}_{0\times (V^*-0)}(F)))^{F^\times\times F^\times,\mu\times 1} = 0$.
(iii) $\mathcal{S}^*(0, Sym^k(CN^{V\times V^*}_0(F)))^{F^\times\times F^\times,\mu\times 1} = 0$.

Proof.
(i) Cover $V-0$ by standard affine open sets $V_i := \{x_i \neq 0\}$. It is enough to show that $\mathcal{S}^*((V_i\times 0)(F), Sym^k(CN^{V\times V^*}_{(V_i\times 0)(F)}(F)))^{F^\times\times F^\times,\mu\times 1} = 0$.
Note that V_i is isomorphic as an $F^\times\times F^\times$-manifold to $F^{n-1}\times F^\times$ with the action given by $(x,y)(v,\alpha) = (v, \frac{y}{x}\alpha)$. Note also that the bundle $Sym^k(CN^{V\times V^*}_{(V_i\times 0)(F)}(F))$ is a constant bundle with fiber $Sym^k(V)$.
Hence by Proposition 1.3.7 it is enough to show that $\mathcal{S}^*(F^\times, Sym^k(V))^{F^\times\times F^\times,\mu\times 1} = 0$. Let $H := (F^\times\times F^\times)_1 = \{(t,t) \in F^\times\times F^\times\}$.

Now by Frobenius descent (Theorem 1.3.4) it is enough to show that $(Sym^k(V^*(F))\otimes_{\mathbb{R}}\mathbb{C})^{H,\mu\times 1|_H}=0$. This is clear since (t,t) acts on $(Sym^k(V^*(F))$ by multiplication by t^{-2k}.
(ii) is proven in the same way.
(iii) is equivalent to the statement $((Sym^k(V\times V^*)(F))\otimes_{\mathbb{R}}\mathbb{C})^{F^\times\times F^\times,\mu\times 1}=0$. This is clear since $(t,1)$ acts on $Sym^k(V\times V^*)(F)$ by multiplication by t^{-k}. □

4.5 Proof of the geometric statement

Notation 4.5.1. *Denote* $\check{S}'':=\{((A_1,v_1,\phi_1),(A_2,v_2,\phi_2))\in\check{S}'|A_1^{n-1}=0\}$.
By Theorem 1.8.25 and Example 1.8.24 there are no non-empty $X\times X$-weakly coisotropic subvarieties of $\check{S}''$. Therefore it is enough to prove the following Key proposition.

Proposition 4.5.2 (Key proposition)**.** *There are no non-empty* $X\times X$*-weakly coisotropic subvarieties of* $\check{S}'-\check{S}''$.

Notation 4.5.3. *Let* $A\in sl(V)$ *be a nilpotent Jordan block. Denote*

$$R_A:=(\check{S}'-\check{S}'')|_{\{A\}\times V\times V^*}.$$

By Proposition 1.8.11 the Key proposition follows from the following Key Lemma.

Lemma 4.5.4 (Key Lemma)**.** *There are no non-empty* $V\times V^*\times V\times V^*$*-weakly coisotropic subvarieties of* R_A.

Proof. Denote $Q_A=\bigcup_{i=1}^{n-1}(KerA^i)\times(Ker(A^*)^{n-i})$. It is easy to see that $R_A\subset Q_A\times Q_A$ and

$$Q_A\times Q_A=\bigcup_{i,j=0}^{n}(KerA^i)\times(Ker(A^*)^{n-i})\times(KerA^j)\times(Ker(A^*)^{n-j}).$$

Denote $L_{ij}:=(KerA^i)\times(Ker(A^*)^{n-i})\times(KerA^j)\times(Ker(A^*)^{n-j})$.
It is easy to see that any weakly coisotropic subvariety of $Q_A\times Q_A$ is contained in $\bigcup_{i=1}^{n-1}L_{ii}$. Hence it is enough to show that for any $0<i<n$, we have $\dim R_A\cap L_{ii}<2n$.
Let $f\in\mathcal{O}(L_{ii})$ be the polynomial defined by

$$f(v_1,\phi_1,v_2,\phi_2):=(v_1)_i(\phi_2)_{i+1}-(v_2)_i(\phi_1)_{i+1},$$

where $(\cdot)_i$ means the i-th coordinate. It is enough to show that $f(R_A\cap L_{ii})=\{0\}$.

Let $(v_1, \phi_1, v_2, \phi_2) \in L_{ii}$. Let $M := v_1 \otimes \phi_2 - v_2 \otimes \phi_1$. Clearly, M is of the form

$$M = \begin{pmatrix} 0_{i\times i} & * \\ 0_{(n-i)\times i} & 0_{(n-i)\times(n-i)} \end{pmatrix}.$$

Note also that $M_{i,i+1} = f(v_1, \phi_1, v_2, \phi_2)$.
It is easy to see that any B satisfying $[A, B] = M$ is upper triangular. On the other hand, we know that there exists a nilpotent B satisfying $[A, B] = M$. Hence this B is upper nilpotent, which implies $M_{i,i+1} = 0$ and hence $f(v_1, \phi_1, v_2, \phi_2) = 0$.
To sum up, we have shown that $f(R_A \cap L_{ii} = \{0\}$, hence $dim(R_A \cap L_{ii}) < 2n$. Hence every coisotropic subvariety of R_A has dimension less than $2n$ and therefore is empty. □

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תודות

קודם כל ברצוני להודות למנחיי, **יוסף ברנשטיין וסטפן גלברט** על כך שהנחו אותי בפרויקט זה.

אני מודה מקרב לב ל**יוסף ברנשטיין** על כך שלימד אותי את רוב המתמטיקה שאני יודע ועל כך שגילה לי את גישתו למתמטיקה ולמחקר.

ברצוני להודות להוריי, **לנה ואיליה**, על כך שלימדו אותי את גישתם לחיים ולעבודה ועל תמיכתם בקריירה שלי ובכל הבחירות שעשיתי בחיי.

לעונג לי להודות לאשתי **אנה** על עתותיה, עזרתה ותמיכתה בי בחיים בכלל ובמתמטיקה בפרט.

אני מודה מקרב לב לילדינו **מרק וגלינה** על הסבלנות שגילו כלפיי בזמן שעבדתי על תזה זו.

אני שמח להודות לחברי הטוב **אברהם אייזנבוד** שאתו רוב העבודה הזאת נעשתה בשיתוף פעולה. למדנו וחקרנו ביחד משנת 1997. חברותו תמיד הייתה לי למעוז גם בזמנים השמחים וגם בזמנים הקשים של חיי.

ברצוני להודות ל"אחי הגדול המדעי" **איתן סייג** על העידוד והעצות שלו במשך 13 השנים האחרונות – כמחצית מחיי. מזמן היה לי חלום לשתף פעולה איתו, ואני שמח שהחלום התגשם במהלך לימודי הדוקטורט שלי.

אני מודה מקרב לב ל-**Gerard Schiffmann** ו-**Stephen Rallis** על כך שפרסמו את רעיונותיהם על הנושא בעבודתם [RS07] . זה אפשר לנו לכתוב את עבודתנו [AG07]. מאמרי הראשון בנושא, [AGRS07], היה סינתזה של שתי עבודות אלה. כמו כן אני מודה ל**דוד סודרי** שהראה לנו את [RS07] בדיוק בזמן הנכון.

ברצוני להודות ל**דוד קשדן** שגרם לי להתעניין בנושא בכך שהציע לנו להוכיח את המקרה הארכימדי של משפט ג'.

במשך לימודיי לדוקטורט נהניתי משיחות על מתמטיקה עם אנשים רבים. לא אוכל למנות את כולם כאן, אבל ברצוני להודות ל**עומר אופן**, **משה ברוך**, **ולדימיר ברקוביץ**, **נדיה גורביץ**, **מריה גורליק**, **יעקוב ורשבסקי**, **איליה טיומקין**, **ארז לפיד**, **דוד סודרי**, **יובל פליקר**, **דוד קשדן**, **אנדריי רזניקוב**, **יבגני שוסטין**, **Bernhard Kroetz ,Herve Jacquet , Gerrit van Dijk ,Patrick Delorme**, **Oksana** -ו **Wee Teck Gan ,Nolan Wallach ,Yiannis Sakellaridis ,Siddhartha Sahi** **Yakimova** על שיחות פוריות.

כמו כן ברצוני להודות ל**סמיון אלסקר**, **ארז לפיד** ו- **Binyong Sun** על הערות שימושיות.

אני מודה ל**ולדימיר ברקוביץ**, **מריה גורליק**, **סרגיי יעקובנקו** ו**דמיטרי נוביקוב** על עידודם, עצותיהם ותמיכתם.

לבסוף, ברצוני להודות למזכירת המחלקה למתמטיקה עיונית **ג'יזל מימון** ולקודמה **טרי דבש** שהגנו עלי מפני סוגים שונים של ביורוקרטיה.

במהלך העבודה על פרוייקט זה ביקרתי פעמים במרכז **Hausdorff** ובמכון **Max Planck** למתמטיקה ב-Bonn . חלק חשוב של עבודה זו נעשה במהלך ביקורים אלה. ברצוני להודות לשני המוסדות האלה על האווירה היצירתית ועל תנאיי העבודה.

במהלך העבודה על תזה זו קיבלתי תמיכה חלקית מ**הקרן הדו-לאומית למדע**, **הקרן לשיתוף פעולה מדעי בין ישראל לגרמניה** והמרכז למצוינות של **הקרן הישראלית למדע**.

משפט ד. כל התפלגות על $O(V\oplus F)$ שנשמרת על ידי הצמדה ב- $O(V)$ נשמרת גם על ידי שחלוף.
כמו כן, לכל הצגה מותרת בלתי פריקה π של $O(V\oplus F)$ ולכל הצגה מותרת בלתי פריקה ρ של $O(V)$ מתקיים

$$\dim \mathrm{Hom}_{O(V)}(\pi|_{O(V)}, \rho)\leq 1$$

עבור F לא ארכימדי משפט ד' הוכח ב-[AGRS] ועבור F לא ארכימדי משפט ד' הוכח ב-[SZ]. שתי ההוכחות השתמשו במשפט א'.

בהוכחות שלנו עבור F לא ארכימדי השתמשנו בכלים העוסקים עם התפלגויות אינווריאנטיות שפותחו ב-
[BZ], [Ber] ו-[JR]. בשביל F ארכימדי פיתחנו מקבילים ארכימדיים של כלים אלה (ראה פרק 1).

המקרה הארכימדי של משפט א' הרבה יותר קשה ולכן הזדקקנו לכלי נוסף כדי לפתור אותו. הכלי המכריע היה התורה של D-מודולים. בפרט השתמשנו במשפט עמוק על התומך הסינגולרי של D-מודול. כדי שההוכחות שלנו יהיו אחידות לשדות ארכימדיים ולא ארכימדיים השתמשנו במאמר [Aiz] שמפתח כלי מקביל למקרה הלא ארכימדי.

סיכום

יהי F שדה מקומי מאופיין אפס, לדוגמא שדה המספרים הממשיים או שדה המספרים הp-אדיים. נתבונן בשיכון הטבעי $GL(n,F)<GL(n+1,F)$. נתבונן בפעולת ההצמדה של $GL(n,F)$ על $GL(n+1,F)$. התוצאה העיקרית של תזה זו היא המשפט הבא.

משפט א. כל התפלגות על $GL(n+1,F)$, אשר נשמרת על ידי פעולת ההצמדה של $GL(n,F)$ נשמרת גם על ידי שחלוף.

הוכחנו זאת ב- [AGRS] עבור F לא ארכימדי וב-[AG08e] עבור F ארכימדי. עבור F ארכימדי, המשפט הוכח גם ב-[SZ] בו זמנית, באופן בלתי תלוי ובדרך אחרת. ההוכחה שאנו מביאים בתזה זו היא שילוב של שלושת ההוכחות הנ"ל ועובדת בצורה אחידה עבור שדות ארכימדיים ולא ארכימדיים.

למשפט חשיבות בתורת ההצגות משום שהוא גורר את משפט חוסר ריבוי הבא

משפט ב. תהי π הצגה מותרת בלתי פריקה של $GL(n+1,F)$ ו ρ הצגה מותרת בלתי פריקה של $GL(n,F)$. אזי

$$\dim \mathrm{Hom}_{GL(n,F)}(\pi|_{GL(n,F)}, \rho)\leq 1$$

מסקנה נוספת של משפט א' היא

משפט ג. נסמן ב $P(n,F) < GL(n,F)$ את התת-חבורה המורכבת ממטריצות שהשורה האחרונה שלהן היא $(0,\ldots,0,1)$. אזי כל התפלגות על $GL(n,F)$ שנשמרת על ידי הצמדה ב- $P(n,F)$ נשמרת על ידי הצמדה ב- $GL(n,F)$.

משפט ג' גורר את השארת Kirillov שטוענת כי כל הצגה אוניטרית בלתי פריקה של $GL(n,F)$ נשארת בלתי פריקה כהצגה אוניטרית לאחר צמצום ל- $P(n,F)$. בדרך זו השארת Kirillov הוכחה במקור, ב-[Ber] למקרה הלא ארכימדי וב-[Bar] למקרה הארכימדי.

משפטים דומים למשפטים א' וב' מתקיימים גם עבור חבורות אורתוגונליות. יהי V מרחב וקטורי ממימד סופי מעל F ותהי q תבנית ריבועית לא מנוונת על V. נרחיב את q בדרך הטבעית ל- $V\oplus F$ ונתבונן בשיכון $O(V) < O(V\oplus F)$.

לזכר הסבים שלי

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WEIZMANN INSTITUTE OF SCIENCE
מכון ויצמן למדע

Thesis for the degree
Doctor of Philosophy

עבודת גמר (תזה) לתואר
דוקטור לפילוסופיה

Submitted to the Scientific Council of the
Weizmann Institute of Science
Rehovot, Israel

מוגשת למועצה המדעית של
מכון ויצמן למדע
רחובות, ישראל

במתכונת "רגילה"
In a "Regular" Format

By
Dmitry Gourevitch

מאת
דמיטרי גורביץ

משפטי חוסר ריבוי והתפלגויות אינווריאנטיות
Multiplicity One Theorems and Invariant Distributions

Advisors:
Prof. Stephen Gelbart and
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May 2009

סיוון תשס"ט