Vanishing and Eulerianity of Fourier coefficients of automorphic forms.

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Representation theory seminar, BGU, June 2020
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Following Piatetski-Shapiro–Shalika, Jian-Shu Li, Ginzburg–Rallis–Soudry, Moeglin-Waldspurger, Jiang–Liu–Savin, Gomez, Ahlen, Hundley–Sayag, Green-Miller-Vanhove, Kazhdan–Polishchuk, Bossard–Pioline

Definitions

- $\mathbb{K}$: number field, $\mathbb{A} := \mathbb{A}_{\mathbb{K}}$, $\mathbf{G}$: reductive group over $\mathbb{K}$, $\Gamma := \mathbf{G}(\mathbb{K})$, $G := \mathbf{G}(\mathbb{A})$, $\mathfrak{g} := Lie(\Gamma)$.

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- Let $[N] := (\Gamma \cap N)\backslash N$. For automorphic form η on G, define *Fourier coefficient*

$$\mathcal{F}_{H,f}[\eta](g) := \int_{[N]} \eta(ng)\chi_f(n)^{-1} dn.$$

Two central cases of Fourier coefficients

$[H, f] = -2f$, $\mathfrak{n} = (\mathfrak{g}_1 \cap \mathfrak{g}^f) \oplus \bigoplus_{i>1} \mathfrak{g}_i$, $N = \mathrm{Exp}(\mathfrak{n})(\mathbb{A})$,

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- Neutral Fourier coefficient, coming from $\mathfrak{sl}_2$-triple (e,H,f), *e.g.*:

$$H=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\ f=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}\ \mathfrak{n}=\begin{pmatrix}0&\underline{*}&0&*\\0&0&0&0\\0&*&0&\underline{*}\\0&0&0&0\end{pmatrix}$$

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- Whittaker coefficient $\mathcal{W}_f$, with N maximal unipotent, *e.g.*:

$$H=\begin{pmatrix}3&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-3\end{pmatrix}\quad f=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}\quad \mathfrak{n}=\begin{pmatrix}0&\underline{*}&*&*\\0&0&*&*\\0&0&0&\underline{*}\\0&0&0&0\end{pmatrix}$$

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Coefficients that are both neutral and Whittaker are Eulerian by local uniqueness of Whittaker models.

Examples of Fourier coefficients

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Comparison for $G=\mathrm{GL}_3(\mathbb{A})$:

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Lemma (Root exchange, cf. Ginzburg-Rallis-Soudry)

1. $\mathcal{F}_{H,f}[\eta](g) = \sum_{\gamma \in (U/I^\perp)(\mathbb{K})} \mathcal{F}^I_{H,f}[\eta](\gamma g)$
2. *For any isotropic subspace* $\mathfrak{j} \subset \mathfrak{u}$ *with* $\dim \mathfrak{j} = \dim \mathfrak{i}$ *and* $\mathfrak{j} \cap \mathfrak{i}^\perp = \{0\}$,

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For $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$: $\begin{pmatrix} 0 & \mathfrak{i} & \underline{\mathfrak{n}} \\ 0 & 0 & \mathfrak{j} \\ 0 & 0 & 0 \end{pmatrix}$

Cf. θ, Stone-von-Neumann thm, Poisson summation formula.

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Theorem

Let $(H, f) \succ (S, f)$. Then

1. $\mathcal{F}_{H,f}[\eta]$ *linearly determines* $\mathcal{F}_{S,f}[\eta]$.
2. *If* $\Gamma f \in \mathrm{WO}^{\max}(\eta)$ *and* $\mathfrak{g}^H_1 = \mathfrak{g}^S_1 = 0$ *let* $\mathfrak{v} := \mathfrak{g}^H_{>1} \cap \mathfrak{g}^S_{<1}$. *Then*

$$\mathcal{F}_{H,f}[\eta](g) = \int_{V(\mathbb{A})} \mathcal{F}_{S,f}[\eta](vg)\, dv$$

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2. (ii) *If* $\Gamma f \in \mathrm{WO}^{\max}(\eta)$ *and* $\mathfrak{g}_1^H = \mathfrak{g}_1^S = 0$ *let* $\mathfrak{v} := \mathfrak{g}^H_{>1} \cap \mathfrak{g}^S_{<1}$. *Then*

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Corollary

1. *If* η *is cuspidal then any* $\mathcal{O} \in \mathrm{WO}^{\max}(\eta)$ *is* $\mathbb{K}$*-distinguished.*
   In particular, $\mathcal{O}$ *is totally even for* $G = \mathrm{Sp}_{2n}$, *totally odd for* $G = SO(V)$, *not minimal for* $\mathrm{rk}\, G > 1$,
   and not next-to-minimal for $\mathrm{rk}\, G > 2$, $G \neq F_4$.
2. *Lower bounds for partitions of* $\mathcal{O} \in \mathrm{WO}^{\max}(\eta)$ *with cuspidal* η:
   2^n *for* Sp_{2n}, $3^n 1^n$ *for* $SO(2n, 2n)$, $5 3^{n-1} 1^n$ *for* $SO(2n+1, 2n+1)$,
   $3^n 1^{n+1}$ *for* $SO(2n+1, 2n)$, *and* $(3^{n+1}, 1^n)$ *for* $SO(2n+2, 2n+1)$.
3. *If* $f \notin \mathrm{WO}(\eta)$ *then* $\mathcal{F}_{H,f}(\eta) = 0$ *for any* H.

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1. *If η is cuspidal then any $\mathcal{O} \in \mathrm{WO}^{\max}(\eta)$ is $\mathbb{K}$-distinguished. In particular, $\mathcal{O}$ is totally even for $G = \mathrm{Sp}_{2n}$, totally odd for $G = SO(V)$, not minimal for $\mathrm{rk}G > 1$, and not next-to-minimal for $\mathrm{rk}G > 2$, $G \neq F_4$.*
2. *Lower bounds for partitions of $\mathcal{O} \in \mathrm{WO}^{\max}(\eta)$ with cuspidal η: 2^n for Sp_{2n}, 3^n1^n for $SO(2n,2n)$, $53^{n-1}1^n$ for $SO(2n+1,2n+1)$, 3^n1^{n+1} for $SO(2n+1,2n)$, and $(3^{n+1},1^n)$ for $SO(2n+2,2n+1)$.*
3. *If $f \notin \mathrm{WO}(\eta)$ then $\mathcal{F}_{H,f}(\eta) = 0$ for any H.*

Proof of (i).

Let $\mathfrak{l} \subset \mathfrak{g}$ be Levi subalgebra intersecting $\mathcal{O}$. Let $(e,h,f) \in \mathfrak{l}$ be an $\mathfrak{sl}_2$-triple with $f \in \mathcal{O}$. Let $Z \in \mathfrak{g}$ be a (rational) semi-simple element s.t. $\mathfrak{l} = \mathfrak{g}^Z$. Let $T >> 0 \in \mathbb{Z}$ and let $H := h + TZ$.
Then $\mathcal{F}_{H,f}(\eta) = \mathcal{F}_{H,f}(c_L(\eta))$, where $c_L(\eta)$ denotes the constant term.
Since $\mathcal{F}_{H,f}(\eta) \neq 0$ by the theorem and η is cuspidal, $L = G$. □

Example for the proof of the Theorem

$G := \mathrm{GL}(4, \mathbb{A})$, $f := E_{21} + E_{43}$, $H := \mathrm{diag}(3, 1, -1, -3)$,
$h = \mathrm{diag}(1, -1, 1, -1)$, $Z = H - h = \mathrm{diag}(2, 2, -2, -2)$, $H_t := h + tZ$.

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Then $\mathfrak{n}_0 \subset \mathfrak{n}_{1/4} \oplus \mathfrak{i} \sim \mathfrak{n}_{1/4} \oplus \mathfrak{j} \subset \mathfrak{n}_{3/4} = \mathfrak{n}_1$:

$$\begin{pmatrix} 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & * & a & * \\ 0 & 0 & 0 & a \\ 0 & - & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & * & - & * \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Both $*$ and $-$ denote arbitrary elements. $*$ denotes the entries in $\mathfrak{g}^{H_t}_{>1}$ and $-$ those in $\mathfrak{g}^{H_t}_1$. a denotes equal elements in $\mathfrak{g}^{H_t}_1 \cap \mathfrak{g}^f$.

Corollary (Hidden symmetry)

Let η *be an automorphic form on* G*, and let* (H, f) *be a Whittaker pair with* $\Gamma f \in \mathrm{WO}^{\max}(\eta)$*. Then any unipotent element* u *of the centralizer of the pair* (H, f) *in* G *acts trivially on the Fourier coefficient* $\mathcal{F}_{H,f}[\eta]$ *using the left regular action.*

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Example: $G = GL_4(\mathbb{A}), f = E_{31} + E_{42}, H = \mathrm{diag}(1,1,-1,-1), u = Id + E_{12} + E_{34}, Z = \mathrm{diag}(1,-1,1,-1)$.

$$\begin{pmatrix} 0 & b & \underline{*} & * \\ 0 & 0 & * & \underline{*} \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 0 & * & \underline{*} & * \\ 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & 0 & \underline{*} & * \\ 0 & 0 & * & \underline{*} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Corollary: if $\mathrm{WO}^{\max}(\eta) = \{2^n\}$ then $\mathcal{F}_{H,f}[\eta]$ is Eulerian (Shalika model).

Applications of hidden symmetry

Corollary (Hidden symmetry)

Let η be an automorphic form on G, and let (H,f) be a Whittaker pair with $\Gamma f \in \mathrm{WO}^{\max}(\eta)$. Then any unipotent element u of the centralizer of the pair (H,f) in G acts trivially on the Fourier coefficient $\mathcal{F}_{H,f}[\eta]$ using the left regular action.

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Corollary

If $G = \mathrm{GL}_n(\mathbb{A})$ and $\mathrm{WO}^{\max}(\eta) = \{2^n\}$ or $G \in \{SO(n,n), SO(n+1,n)\}$ and $\mathrm{WO}^{\max}(\eta) = \{31\dots1\}$ then $\mathcal{F}_{H,f}[\eta]$ is Eulerian.

Follows from uniqueness of Shalika and Bessel models.

Fourier-Jacobi periods and the Weil representation

For a symplectic space V over $\mathbb{K}$, let $\mathcal{H}(V) := V \oplus \mathbb{K}$ be the Heisenberg group and $\widetilde{J(V)} := \widetilde{\mathrm{Sp}(V(\mathbb{A}))} \ltimes \mathcal{H}(V(\mathbb{A}))$ be the double cover Jacobi group. It has unique irreducible unitarizable representation ϖ_V with central character χ. It has automorphic realization given by theta functions:

$$\theta_f(g) = \sum_{a \in \mathcal{E}(K)} \omega_\chi(g) f(a), \text{ where } g \in \widetilde{J(V)}, f \in \mathcal{S}(\mathcal{E}(\mathbb{A})), \mathcal{E} \subset V \text{ Lagrangi}$$

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$$f \otimes \eta \mapsto \int_{U(K) \backslash U(\mathbb{A})} f(u\tilde{g}) \theta_\eta(\ell(u, \tilde{g})) du$$

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If $\Gamma \cdot f \in \mathrm{WO}^{\max}(\pi)$ then $\widetilde{M}$ acts on the image of FJ by ± 1.

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If $\Gamma \cdot f \in \mathrm{WO}^{\max}(\pi)$ then the cover $\widetilde{M}$ splits.

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Corollary

If $\Gamma \cdot f \in \mathrm{WO}^{\max}(\pi)$ and G is classical then the orbit of f is special.

Eulerianity

Lemma

Let (S, f) *and* (H, f') *be two Whittaker pairs such that* $\Gamma f = \Gamma f' \in \mathrm{WO}^{\max}(\eta)$. *Suppose that a Fourier–Jacobi coefficient* $\mathcal{F}^{l}_{S,f}[\eta]$ *is Eulerian. Then any Fourier–Jacobi coefficient* $\mathcal{F}^{l'}_{H,\psi}[\eta]$ *is also Eulerian.*

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Verified for:

1. Discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$.
2. Minimal representations of most split simply-laced groups
3. Next-to-minimal Eisenstein series of most split simply-laced groups

Expressing forms through their Whittaker coefficients

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Any $\mathcal{F}_{H,f}$ is linearly determined by all Levi-distinguished Fourier coefficients $\mathcal{F}_{S,F}$ with $\Gamma F \geq \Gamma f$.

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2. *If all $\mathcal{O} \in \mathrm{WO}(\eta)$ admit Whittaker coefficients then η is linearly determined by its Whittaker coefficients.*

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2. *If all* $\mathcal{O} \in \mathrm{WO}(\eta)$ *admit Whittaker coefficients then* η *is linearly determined by its Whittaker coefficients.*
3. *If* G *is split and simply-laced, and* η *is minimal or next-to-minimal then all Fourier coefficients of* η *are linearly determined by Whittaker coefficients.*

Explanation for GL_n (PS-Shalika, Ahlen–Gustafsson-Liu-Kleinschmidt-Persson)

Let $\eta \in C^\infty(\Gamma \backslash \mathrm{GL}_n(\mathbb{A}))$. Restrict to the last column and decompose to Fourier series. All non-trivial characters are conjugate by $\mathrm{GL}_{n-1}(\mathbb{K})$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Example: Sp(4)

$$\mathfrak{sp}_4 = \left\{ \begin{pmatrix} A & B = B^t \\ C = C^t & -A^t \end{pmatrix} \right\}.$$

Let $\mathfrak{n}$ be the Borel nilradical, and $\mathfrak{u} \subset \mathfrak{n}$ be the Siegel nilradical, spanned by B. Characters given by $\bar{\mathfrak{u}} \cong \mathrm{Sym}^2(\mathbb{K}^2)$. Restricting η to B and decomposing into Fourier series we obtain $\eta = \sum_{f \in \bar{\mathfrak{u}}} \mathcal{F}_{\mathfrak{u},f}[\eta]$.

1. Constant term $\mathcal{F}_{\mathfrak{u},0}[\eta]$: Restrict to the Siegel Levi $L \cong \mathrm{GL}_2(\mathbb{A})$, and decompose to Fourier series on the abelian group $N \cap L$:

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2. Any f of rank one is conjugate under L to $f_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

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3. Split non-degenerate forms are conjugate to $f_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   Using Weyl group conjugation (24) and root exchange, we express $\mathcal{F}_{\mathfrak{u},f_2}$ through $\mathcal{F}_{\mathfrak{u}',e_{21}}$, where $\mathfrak{u}' = Span(e_{12} - e_{43}, e_{13}, e_{24}) \subset \mathfrak{n}$. Fourier expansion by the remaining coordinate of $e_{14} + e_{23} \in \mathfrak{n}$:

$$\mathcal{F}_{\mathfrak{u},f}[\eta](g) = \int_{x \in \mathbb{A}} \mathcal{W}_{1,a}[\eta]((Id + xe_{24})wg).$$

X:=set of anisotropic 2×2 forms. For $f \in X$, we cannot simplify $\mathcal{F}_{\mathfrak{u},f}[\eta]$. Summarizing, for any η on $G = \mathrm{Sp}_4(\mathbb{A})$ we have

$$\eta(g) = \sum_{f \in X} \mathcal{F}_{\mathfrak{u},\varphi}[\eta](g) + \sum_{a \in \mathbb{K}} \Big(\sum_{\gamma \in L/O(1,1)} \int_{x \in \mathbb{A}} \mathcal{W}_{1,a}[\eta](v_x w \gamma g) + \sum_{\gamma \in L/(N \cap L)} \mathcal{W}_{a,1}[\eta](\gamma g) + \mathcal{W}_{a,0}[\eta](g) \Big)$$

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X:=set of anisotropic 2×2 forms. For $f\in X$, we cannot simplify $\mathcal{F}_{\mathfrak{u},f}[\eta]$. Summarizing, for any η on $G=\mathrm{Sp}_4(\mathbb{A})$ we have

$$\eta(g)=\sum_{f\in X}\mathcal{F}_{\mathfrak{u},\varphi}[\eta](g)+\sum_{a\in\mathbb{K}}\Big(\sum_{\gamma\in L/O(1,1)}\int_{x\in\mathbb{A}}\mathcal{W}_{1,a}[\eta](v_x w\gamma g)+\sum_{\gamma\in L/(N\cap L)}\mathcal{W}_{a,1}[\eta](\gamma g)+\mathcal{W}_{a,0}[\eta](g)\Big)$$

If η is cuspidal then $\mathcal{W}_{0,a}[\eta]=\mathcal{W}_{a,0}[\eta]=0$. If η is non-generic η, then $\mathcal{W}_{1,a}[\eta]=\mathcal{W}_{a,1}[\eta]=0$, unless $a=0$. Thus

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3. If η is cuspidal and non-generic then $\eta=\sum_{f\in X}\mathcal{F}_{\mathfrak{u},f}[\eta]$.

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- All the theorems above have local analogues with similar proofs.

Wave front set and wave-front cycle

Let π be smooth, admissible and finitely generated.

Theorem (Howe, Harish-Chandra 70s)

Near $e \in G$, the character distribution equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

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- *If $f \in \mathrm{WF}^{\max}(\pi)$ then $\dim \pi_{H,f} = c_f$.*