Vanishing and Eulerianity of Fourier coefficients of automorphic forms.

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Representation theory seminar, BGU, June 2020
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Following Piatetski-Shapiro–Shalika, Jian-Shu Li, Ginzburg–Rallis–Soudry, Moeglin-Waldspurger, Jiang–Liu–Savin, Gomez, Ahlen, Hundley–Sayag, Green-Miller-Vanhove, Kazhdan–Polishchuk, Bossard–Pioline

Definitions

- $\mathbb{K}$: number field, $\mathbb{A} := \mathbb{A}_{\mathbb{K}}$, $\mathbf{G}$: reductive group over $\mathbb{K}$, $\Gamma := \mathbf{G}(\mathbb{K})$, $G := \mathbf{G}(\mathbb{A})$, $\mathfrak{g} := Lie(\Gamma)$.
- Fix a semisimple $H \in \mathfrak{g}$, and let $\mathfrak{g}_i := \mathfrak{g}_i^H$ denote the eigenspaces of $ad(H)$. Assume that all the eigenvalues i lie in $\mathbb{Q}$.
- Let $f \in \mathfrak{g}_{-2}$. Call $(H, f) \in \mathfrak{g} \times \mathfrak{g}$ a *Whittaker pair*.
- Define $\mathfrak{n} := \mathfrak{n}_{H,f} := (\mathfrak{g}_1 \cap \mathfrak{g}^f) \oplus \bigoplus_{i>1} \mathfrak{g}_i$, $N := \operatorname{Exp}(\mathfrak{n})(\mathbb{A})$.
- Fix a non-trivial unitary additive character $\psi : \mathbb{K}\backslash\mathbb{A} \to \mathbb{C}$ and define $\chi_f : N \to \mathbb{C}$ by $\chi_f(\operatorname{Exp} X) := \psi(\langle f, X\rangle)$.
- Let $[N] := (\Gamma \cap N)\backslash N$. For automorphic form η on G, define *Fourier coefficient*

$$\mathcal{F}_{H,f}[\eta](g) := \int_{[N]} \eta(ng)\chi_f(n)^{-1} dn.$$

Two central cases of Fourier coefficients

$[H,f]=-2f$, $\mathfrak{n}=(\mathfrak{g}_1\cap\mathfrak{g}^f)\oplus\bigoplus_{i>1}\mathfrak{g}_i$, $N=\mathrm{Exp}(\mathfrak{n})(\mathbb{A})$,

$$\mathcal{F}_{H,f}[\eta](g):=\int_{[N]}\eta(ng)\chi_f(n)^{-1}dn.$$

- Neutral Fourier coefficient, coming from $\mathfrak{sl}_2$-triple (e,H,f), *e.g.*:

$$H=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix} f=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix} \mathfrak{n}=\begin{pmatrix}0&\underline{*}&0&*\\0&0&0&0\\0&*&0&\underline{*}\\0&0&0&0\end{pmatrix}$$

- Whittaker coefficient $\mathcal{W}_f$, with N maximal unipotent, *e.g.*:

$$H=\begin{pmatrix}3&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-3\end{pmatrix} f=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix} \mathfrak{n}=\begin{pmatrix}0&\underline{*}&*&*\\0&0&*&*\\0&0&0&\underline{*}\\0&0&0&0\end{pmatrix}$$

Coefficients that are both neutral and Whittaker are Eulerian by local uniqueness of Whittaker models.

Examples of Fourier coefficients

$[H,f]=-2f$, $\mathfrak{n}=(\mathfrak{g}_1\cap\mathfrak{g}^f)\oplus\bigoplus_{i>1}\mathfrak{g}_i$, $N=\mathrm{Exp}(\mathfrak{n})(\mathbb{A})$

$$\mathcal{F}_{H,f}[\eta](g):=\int_{[N]}\eta(ng)\chi_f(n)^{-1}dn.$$

Comparison for $G=\mathrm{GL}_3(\mathbb{A})$:

- Neutral Fourier coefficient:

$$H=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}, f=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}, \mathfrak{n}=\begin{pmatrix}0&0&\underline{*}\\0&0&0\\0&0&0\end{pmatrix}$$

- Whittaker coefficient:

$$H=\begin{pmatrix}1&0&0\\0&-3&0\\0&0&-1\end{pmatrix}, f=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}, \mathfrak{n}=\begin{pmatrix}0&*&\underline{*}\\0&0&0\\0&*&0\end{pmatrix}$$

Fourier-Jacobi coefficients

- $\mathfrak{u} := \mathfrak{g}_1/(\mathfrak{g}_1 \cap \mathfrak{g}^f)$. $\omega_f(X, Y) := \langle f, [X, Y] \rangle$- symplectic form.
- $\forall$ isotropic subspace $\mathfrak{i} \subset \mathfrak{u}$, let $I := \operatorname{Exp}(\mathfrak{i})(\mathbb{A})$

$$\mathcal{F}^I_{H,f}[\eta](g) := \int_{[I]} \mathcal{F}_{H,f}[\eta](ug)\, du$$

Lemma (Root exchange, cf. Ginzburg-Rallis-Soudry)

(i) $\mathcal{F}_{H,f}[\eta](g) = \sum_{\gamma \in (U/I^\perp)(\mathbb{K})} \mathcal{F}^I_{H,f}[\eta](\gamma g)$

(ii) *For any isotropic subspace* $\mathfrak{j} \subset \mathfrak{u}$ *with* $\dim \mathfrak{j} = \dim \mathfrak{i}$ *and* $\mathfrak{j} \cap \mathfrak{i}^\perp = \{0\}$,

$$\mathcal{F}^J_{H,f}[\eta](g) = \int_{J(\mathbb{A})} \mathcal{F}^I_{H,f}[\eta](ug)\, du$$

For $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}: \quad \begin{pmatrix} 0 & \mathfrak{i} & \underline{\mathfrak{n}} \\ 0 & 0 & \mathfrak{j} \\ 0 & 0 & 0 \end{pmatrix}$

Cf. θ, Stone-von-Neumann thm, Poisson summation formula.

Relating different coefficients

- $\mathrm{WO}(\eta) := \{\mathcal{O} \in \mathcal{N}(\mathfrak{g}) \mid \forall$ neutral (h,f) with $f \in \mathcal{O}$, $\mathcal{F}_{h,f}(\eta) \not\equiv 0\}$.
- Say $(H,f) \succ (S,f)$ if $[H,S]=0$ and $\mathfrak{g}^f \cap \mathfrak{g}^H_{\geq 1} \subseteq \mathfrak{g}^{S-H}_{\geq 0}$.
- f is $\mathbb{K}$-distinguished if $\forall$ Levi $\mathfrak{l} \ni f$ defined over $\mathbb{K}$, $\mathfrak{l} = \mathfrak{g}$.
  Equivalently: the semi-simple part of the centralizer G_f is anisotropic
- (S,f) is called Levi-distinguished if $\exists$ parabolic $\mathfrak{p} = \mathfrak{l}\mathfrak{u}$
  s.t. f is $\mathbb{K}$-distinguished in $\mathfrak{l}$, and $\mathfrak{n}_{S,f} = \mathfrak{l}_{S,f} \oplus \mathfrak{u}$.
- Whittaker coefficients are Levi-distinguished.
- For Whittaker pairs with the same f and commuting H-s,
  neutral $\succ$ any $\succ$ Levi-distinguished.

Theorem

Let $(H,f) \succ (S,f)$. Then

1. *$\mathcal{F}_{H,f}[\eta]$ linearly determines $\mathcal{F}_{S,f}[\eta]$.*
2. *If $\Gamma f \in \mathrm{WO}^{\max}(\eta)$ and $\mathfrak{g}^H_1 = \mathfrak{g}^S_1 = 0$ let $\mathfrak{v} := \mathfrak{g}^H_{>1} \cap \mathfrak{g}^S_{<1}$. Then*

$$\mathcal{F}_{H,f}[\eta](g) = \int_{V(\mathbb{A})} \mathcal{F}_{S,f}[\eta](vg)\, dv$$

Theorem

Let $(H, f) \succ (S, f)$. *Then*

1. $\mathcal{F}_{H,f}[\eta]$ *linearly determines* $\mathcal{F}_{S,f}[\eta]$.
2. *If* $\Gamma f \in \mathrm{WO}^{\max}(\eta)$ *and* $\mathfrak{g}_1^H = \mathfrak{g}_1^S = 0$ *let* $\mathfrak{v} := \mathfrak{g}_{>1}^H \cap \mathfrak{g}_{<1}^S$. *Then*

$$\mathcal{F}_{H,f}[\eta](g) = \int_{V(\mathbb{A})} \mathcal{F}_{S,f}[\eta](vg)\, dv$$

Corollary

1. *If* η *is cuspidal then any* $\mathcal{O} \in \mathrm{WO}^{\max}(\eta)$ *is* $\mathbb{K}$*-distinguished. In particular,* $\mathcal{O}$ *is totally even for* $G = \mathrm{Sp}_{2n}$, *totally odd for* $G = SO(V)$, *not minimal for* $\mathrm{rk}\, G > 1$, *and not next-to-minimal for* $\mathrm{rk}\, G > 2$, $G \neq F_4$.
2. *Lower bounds for partitions of* $\mathcal{O} \in \mathrm{WO}^{\max}(\eta)$ *with cuspidal* η*:* 2^n *for* Sp_{2n}, $3^n 1^n$ *for* $SO(2n, 2n)$, $53^{n-1}1^n$ *for* $SO(2n+1, 2n+1)$, $3^n 1^{n+1}$ *for* $SO(2n+1, 2n)$, *and* $(3^{n+1}, 1^n)$ *for* $SO(2n+2, 2n+1)$.
3. *If* $f \notin \mathrm{WO}(\eta)$ *then* $\mathcal{F}_{H,f}(\eta) = 0$ *for any* H.

Corollary

1. *If η is cuspidal then any $\mathcal{O} \in \mathrm{WO}^{\max}(\eta)$ is $\mathbb{K}$-distinguished. In particular, $\mathcal{O}$ is totally even for $G = \mathrm{Sp}_{2n}$, totally odd for $G = SO(V)$, not minimal for $\mathrm{rk} G > 1$, and not next-to-minimal for $\mathrm{rk} G > 2$, $G \neq F_4$.*
2. *Lower bounds for partitions of $\mathcal{O} \in \mathrm{WO}^{\max}(\eta)$ with cuspidal η: 2^n for Sp_{2n}, 3^n1^n for $SO(2n, 2n)$, $53^{n-1}1^n$ for $SO(2n+1, 2n+1)$, 3^n1^{n+1} for $SO(2n+1, 2n)$, and $(3^{n+1}, 1^n)$ for $SO(2n+2, 2n+1)$.*
3. *If $f \notin \mathrm{WO}(\eta)$ then $\mathcal{F}_{H,f}(\eta) = 0$ for any H.*

Proof of (i).

Let $\mathfrak{l} \subset \mathfrak{g}$ be Levi subalgebra intersecting $\mathcal{O}$. Let $(e, h, f) \in \mathfrak{l}$ be an $\mathfrak{sl}_2$-triple with $f \in \mathcal{O}$. Let $Z \in \mathfrak{g}$ be a (rational) semi-simple element s.t. $\mathfrak{l} = \mathfrak{g}^Z$. Let $T >> 0 \in \mathbb{Z}$ and let $H := h + TZ$.
Then $\mathcal{F}_{H,f}(\eta) = \mathcal{F}_{H,f}(c_L(\eta))$, where $c_L(\eta)$ denotes the constant term.
Since $\mathcal{F}_{H,f}(\eta) \neq 0$ by the theorem and η is cuspidal, $L = G$. $\square$

Example for the proof of the Theorem

$G := \mathrm{GL}(4, \mathbb{A})$, $f := E_{21} + E_{43}$, $H := \mathrm{diag}(3, 1, -1, -3)$,
$h = \mathrm{diag}(1, -1, 1, -1)$, $Z = H - h = \mathrm{diag}(2, 2, -2, -2)$, $H_t := h + tZ$.
Then $\mathfrak{n}_0 \subset \mathfrak{n}_{1/4} \oplus \mathfrak{i} \sim \mathfrak{n}_{1/4} \oplus \mathfrak{j} \subset \mathfrak{n}_{3/4} = \mathfrak{n}_1$:

$$\begin{pmatrix} 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & * & a & * \\ 0 & 0 & 0 & a \\ 0 & - & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & * & - & * \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\subset \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & - & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Both $*$ and $-$ denote arbitrary elements. $*$ denotes the entries in $\mathfrak{g}^{H_t}_{>1}$ and $-$ those in $\mathfrak{g}^{H_t}_1$. a denotes equal elements in $\mathfrak{g}^{H_t}_1 \cap \mathfrak{g}^f$.

Corollary (Hidden symmetry)

Let η be an automorphic form on G, and let (H, f) be a Whittaker pair with $\Gamma f \in \mathrm{WO}^{\max}(\eta)$. Then any unipotent element u of the centralizer of the pair (H, f) in G acts trivially on the Fourier coefficient $\mathcal{F}_{H,f}[\eta]$ using the left regular action.

Proof.

Want to show that $\mathcal{F}_{H,f}[\eta - \eta^u] = 0$. By the theorem, enough to show $\mathcal{F}_{H+Z,f}[\eta - \eta^u] = 0$ for some Z. Find Z such that $u \in N_{H+Z,f}$. □

Example: $G = GL_4(\mathbb{A}), f = E_{31} + E_{42}, H = \mathrm{diag}(1, 1, -1, -1), u = Id + E_{12} + E_{34}, Z = \mathrm{diag}(1, -1, 1, -1)$.

$$\begin{pmatrix} 0 & b & \underline{*} & * \\ 0 & 0 & * & \underline{*} \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & * & \underline{*} & * \\ 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & 0 & \underline{*} & * \\ 0 & 0 & * & \underline{*} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\underline{*}$: non-zero pairing with f. b: entries of u.

Corollary: if $\mathrm{WO}^{\max}(\eta) = \{2^n\}$ then $\mathcal{F}_{H,f}[\eta]$ is Eulerian (Shalika model).

Applications of hidden symmetry

Corollary (Hidden symmetry)

Let η be an automorphic form on G, and let (H, f) be a Whittaker pair with $\Gamma f \in \mathrm{WO}^{\max}(\eta)$. Then any unipotent element u of the centralizer of the pair (H, f) in G acts trivially on the Fourier coefficient $\mathcal{F}_{H,f}[\eta]$ using the left regular action.

Proof.

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Corollary

If $G = \mathrm{GL}_n(\mathbb{A})$ and $\mathrm{WO}^{\max}(\eta) = \{2^n\}$ or $G \in \{SO(n,n), SO(n+1,n)\}$ and $\mathrm{WO}^{\max}(\eta) = \{31\ldots1\}$ then $\mathcal{F}_{H,f}[\eta]$ is Eulerian.

Follows from uniqueness of Shalika and Bessel models.

Fourier-Jacobi periods and the Weil representation

For a symplectic space V over $\mathbb{K}$, let $\mathcal{H}(V) := V \oplus \mathbb{K}$ be the Heisenberg group and $\widetilde{J(V)} := \widetilde{\mathrm{Sp}(V(\mathbb{A}))} \ltimes \mathcal{H}(V(\mathbb{A}))$ be the double cover Jacobi group. It has unique irreducible unitarizable representation ϖ_V with central character χ. It has automorphic realization given by theta functions:

$$\theta_f(g) = \sum_{a \in \mathcal{E}(K)} \omega_\chi(g) f(a), \text{ where } g \in \widetilde{J(V)}, f \in \mathcal{S}(\mathcal{E}(\mathbb{A})), \mathcal{E} \subset V \text{ Lagrangi}$$

For a Whittaker pair (H, f) let $\mathfrak{u} := \mathfrak{g}^H_{\geq 1}$ and $V := \mathfrak{u}/\mathfrak{n}_{H,f}$, with symplectic form $\omega_f(A, B) := \langle f, [A, B] \rangle$. Then we have a natural map $\ell : U \rtimes \widetilde{G_{H,f}} \to \widetilde{J(V)}$. Define $FJ : \pi \otimes \varpi_V \to C^\infty(\Gamma \backslash \widetilde{G_{H,f}})$ by

$$f \otimes \eta \mapsto \int_{U(K)\backslash U(\mathbb{A})} f(u\tilde{g}) \theta_\eta(\ell(u, \tilde{g})) du$$

M:=split semi-simple part of the centralizer $G_{H,f}$.

Theorem

If $\Gamma \cdot f \in \mathrm{WO}^{\max}(\pi)$ then $\widetilde{M}$ acts on the image of FJ by ± 1.

Fourier-Jacobi periods and the Weil representation

For a Whittaker pair (H, f) let $\mathfrak{u} := \mathfrak{g}^H_{\geq 1}$ and $V := \mathfrak{u}/\mathfrak{n}_{H,f}$, with symplectic form $\omega_\varphi(A, B) := \langle f, [A, B] \rangle$. Then we have a natural map $\ell : U \rtimes \widetilde{G_\gamma} \to \widetilde{J(V)}$. Define $FJ : \pi \otimes \varpi_V \to C^\infty(\Gamma \backslash \widetilde{G_{H,f}})$ by

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M:=split semi-simple part of the centralizer $G_{H,f}$.

Theorem

If $\Gamma \cdot f \in \mathrm{WO}^{\max}(\pi)$ then $\widetilde{M}$ acts on the image of FJ by ± 1.

Since the Weil representation ϖ_V is genuine, obtain:

Corollary

If $\Gamma \cdot f \in \mathrm{WO}^{\max}(\pi)$ then the cover $\widetilde{M}$ splits.

Corollary

If $\Gamma \cdot f \in \mathrm{WO}^{\max}(\pi)$ and G is classical then the orbit of f is special.

Eulerianity

Lemma

Let (S, f) and (H, f') be two Whittaker pairs such that $\Gamma f = \Gamma f' \in \mathrm{WO}^{\max}(\eta)$. Suppose that a Fourier–Jacobi coefficient $\mathcal{F}^{l}_{S,f}[\eta]$ is Eulerian. Then any Fourier–Jacobi coefficient $\mathcal{F}^{l'}_{H,\psi}[\eta]$ is also Eulerian.

Question

Is any Fourier-Jacobi coefficient $\mathcal{F}^{l}_{S,f}[\eta]$ with $\Gamma f \in \mathrm{WO}^{\max}(\eta)$ Eulerian for any spherical η that generates an irreducible representation?

Verified for:

1. Discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$.
2. Minimal representations of most split simply-laced groups
3. Next-to-minimal Eisenstein series of most split simply-laced groups

Expressing forms through their Whittaker coefficients

Theorem

Any $\mathcal{F}_{H,f}$ is linearly determined by all Levi-distinguished Fourier coefficients $\mathcal{F}_{S,F}$ with $\Gamma F \geq \Gamma f$.

Corollary

1. *Any η is linearly determined by all its Levi-distinguished Fourier coefficients.*
2. *If all $\mathcal{O} \in \mathrm{WO}(\eta)$ admit Whittaker coefficients then η is linearly determined by its Whittaker coefficients.*
3. *If G is split and simply-laced, and η is minimal or next-to-minimal then all Fourier coefficients of η are linearly determined by Whittaker coefficients.*

Explanation for GL_n (PS-Shalika, Ahlen–Gustafsson-Liu-Kleinschmidt-Persson)

Let $\eta \in C^\infty(\Gamma\backslash \mathrm{GL}_n(\mathbb{A}))$. Restrict to the last column and decompose to Fourier series. All non-trivial characters are conjugate by $\mathrm{GL}_{n-1}(\mathbb{K})$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Explanation for GL_n

Let $\eta \in C^\infty(\Gamma \backslash \mathrm{GL}_n(\mathbb{A}))$. Restrict to the last column and decompose to Fourier series. All non-trivial characters are conjugate by $\mathrm{GL}_{n-1}(\mathbb{K})$. Conjugate, restrict to the next column and continue

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \ast \\ 0 & 0 & 0 & 0 & \ast \\ 0 & 0 & 0 & 0 & \ast \\ 0 & 0 & 0 & 0 & \underline{\ast} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & 0 & \underline{\ast} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & \underline{\ast} & \ast \\ 0 & 0 & 0 & 0 & \underline{\ast} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \cdots$$

$$\begin{pmatrix} 0 & 0 & \ast & \ast & \ast \\ 0 & 0 & \underline{\ast} & \ast & \ast \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \ast & \ast & \ast \\ 0 & 0 & \underline{\ast} & \ast & \ast \\ 0 & 0 & 0 & 0 & \ast \\ 0 & 0 & 0 & 0 & \ast \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \ast & \ast & \ast & \ast \\ 0 & 0 & \underline{\ast} & \ast & \ast \\ 0 & 0 & 0 & 0 & \ast \\ 0 & 0 & 0 & 0 & \underline{\ast} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \ldots$$

Example: Sp(4)

$$\mathfrak{sp}_4 = \left\{ \begin{pmatrix} A & B = B^t \\ C = C^t & -A^t \end{pmatrix} \right\}.$$

Let $\mathfrak{n}$ be the Borel nilradical, and $\mathfrak{u} \subset \mathfrak{n}$ be the Siegel nilradical, spanned by B. Characters given by $\bar{\mathfrak{u}} \cong \mathrm{Sym}^2(\mathbb{K}^2)$. Restricting η to B and decomposing into Fourier series we obtain $\eta = \sum_{f \in \bar{\mathfrak{u}}} \mathcal{F}_{\mathfrak{u},f}[\eta]$.

1. Constant term $\mathcal{F}_{\mathfrak{u},0}[\eta]$: Restrict to the Siegel Levi $L \cong \mathrm{GL}_2(\mathbb{A})$, and decompose to Fourier series on the abelian group $N \cap L$:

$$\mathcal{F}_{\mathfrak{u},0}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a,0}[\eta].$$

2. Any f of rank one is conjugate under L to $f_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

   Decomposing $\mathcal{F}_{\mathfrak{u},f_1}[\eta]$ on $N \cap L$:

$$\mathcal{F}_{\mathfrak{u},f_1}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a,1}[\eta].$$

$$\mathfrak{sp}_4 = \left\{ \begin{pmatrix} A & B = B^t \\ C = C^t & -A^t \end{pmatrix} \right\}; \quad \eta = \sum_{f \in \overline{\mathfrak{u}}} \mathcal{F}_{\mathfrak{u},f}[\eta] \,.$$

1. Constant term $\mathcal{F}_{\mathfrak{u},0}[\eta]$: Decompose to Fourier series on $N \cap L$:
$$\mathcal{F}_{\mathfrak{u},0}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a,0}[\eta] \,.$$
2. Any f of rank one is conjugate under L to $f_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
Decomposing $\mathcal{F}_{\mathfrak{u},f_1}[\eta]$ on $N \cap L$:
$$\mathcal{F}_{\mathfrak{u},f_1}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a,1}[\eta] \,.$$
3. Split non-degenerate forms are conjugate to $f_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
Using Weyl group conjugation (24) and root exchange, we express $\mathcal{F}_{\mathfrak{u},f_2}$ through $\mathcal{F}_{\mathfrak{u}',e_{21}}$, where $\mathfrak{u}' = Span(e_{12} - e_{43}, e_{13}, e_{24}) \subset \mathfrak{n}$. Fourier expansion by the remaining coordinate of $e_{14} + e_{23} \in \mathfrak{n}$:
$$\mathcal{F}_{\mathfrak{u},f}[\eta](g) = \int_{x \in \mathbb{A}} \mathcal{W}_{1,a}[\eta]((Id + xe_{24})wg).$$

X:=set of anisotropic 2×2 forms. For $f\in X$, we cannot simplify $\mathcal{F}_{\mathfrak{u},f}[\eta]$. Summarizing, for any η on $G=\mathrm{Sp}_4(\mathbb{A})$ we have

$$\eta(g)=\sum_{f\in X}\mathcal{F}_{\mathfrak{u},\varphi}[\eta](g)+\sum_{a\in\mathbb{K}}\Big(\sum_{\gamma\in L/O(1,1)}\int_{x\in\mathbb{A}}\mathcal{W}_{1,a}[\eta](v_x w\gamma g)+\sum_{\gamma\in L/(N\cap L)}\mathcal{W}_{a,1}[\eta](\gamma g)+\mathcal{W}_{a,0}[\eta](g)\Big)$$

If η is cuspidal then $\mathcal{W}_{0,a}[\eta]=\mathcal{W}_{a,0}[\eta]=0$. If η is non-generic η, then $\mathcal{W}_{1,a}[\eta]=\mathcal{W}_{a,1}[\eta]=0$, unless $a=0$. Thus

1. If η is cuspidal then $\eta(g)=\sum_{f\in X}\mathcal{F}_{\mathfrak{u},f}[\eta](g)+$
$$\sum_{a\in\mathbb{K}^\times}\Big(\sum_{\gamma\in L/\mathrm{O}(1,1)}\int_{x\in\mathbb{A}}\mathcal{W}_{1,a}[\eta](v_x w\gamma g)+\sum_{\gamma\in L/(N\cap L)}\mathcal{W}_{a,1}[\eta](\gamma g)\Big)$$
2. If η is non-generic then $\eta(g)=\sum_{f\in X}\mathcal{F}_{\mathfrak{u},f}[\eta](g)+$
$$\sum_{\gamma\in L/\mathrm{O}(1,1)}\int_{x\in\mathbb{A}}\mathcal{W}_{1,0}[\eta](v_x w\gamma g)+\sum_{\gamma\in L/(N\cap L)}\mathcal{W}_{0,1}[\eta](\gamma g)+\sum_{a\in\mathbb{K}}\mathcal{W}_{a,0}[\eta](\cdots$$
3. If η is cuspidal and non-generic then $\eta=\sum_{f\in X}\mathcal{F}_{\mathfrak{u},f}[\eta]$.

Local picture

- $F :=$ a p-adic local field, $G := \mathbf{G}(F)$, $\mathfrak{g} := Lie(G)$,
- $(H,f) \in \mathfrak{g} \times \mathfrak{g}$ Whittaker pair $\mathfrak{u} := \mathfrak{g}^H_{\geq 1}$, $\mathfrak{n}_{H,f} := (\mathfrak{g}^H_1 \cap \mathfrak{g}^f) \oplus \mathfrak{g}^H_{>1}$.
- $\mathfrak{u}/\mathfrak{n}_{H,f}$ is a symplectic space, and its Heisenberg group $\mathcal{H}$ is a quotient of U.
- $\varpi_{H,f} :=$ oscillator representation of $\mathcal{H}$ lifted to $\mathfrak{u}$. $\mathcal{W}_{H,f} := ind^G_U \varpi_{H,f}$
- $\forall$ smooth representation π, define its (H,f)-Whittaker quotient by

$$\pi_{H,f} := \mathcal{W}_{H,f} \otimes_G \pi \simeq \pi_{I,\chi}.$$

- All the theorems above have local analogues with similar proofs.

Wave front set and wave-front cycle

Let π be smooth, admissible and finitely generated.

Theorem (Howe, Harish-Chandra 70s)

Near $e \in G$, the character distribution equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

$$\exp^*(\chi_\pi) = \sum c_{\mathcal{O}} \mathcal{F}(\mu_{\mathcal{O}})$$

- Let $\mathcal{N} \subset \mathfrak{g}$ denote the nilpotent cone.
- $\mathrm{WF}(\pi) := \cup\{\overline{\mathcal{O}} \,|\, c_{\mathcal{O}} \neq 0\} \subset \mathcal{N}$.
- $\mathrm{WF}^{\max}(\pi) :=$ union of maximal orbits in $\mathrm{WF}(\pi)$.

Theorem (Moeglin-Waldspurger, 87')

- *If $\pi_{H,f} \neq 0$ then $f \in \mathrm{WF}(\pi)$.*
- *If $f \in \mathrm{WF}^{\max}(\pi)$ then* $\dim \pi_{H,f} = c_f$.