Wave-front sets of distinguished representations

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CIRM workshop Relative Aspects of the Langlands Program, L-Functions and Beyond Endoscopy
j.w. Eitan Sayag
following Harris-Weich, Prasad, Sakellaridis, Vogan
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Conjecture

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Theorem (G.-Sayag 2020)

1. *If F is Archimedean and $\mathbf{H} \subset \mathbf{G}$ is spherical then*

$$\mathbf{G}(WF^0(\pi))|_{\mathfrak{h}} \ni 0.$$

2. *If F is p-adic then $WF^o(\pi) \subset \overline{G\mathfrak{h}_0^\perp} \subset \mathfrak{g}_0^*$*

$\mathbf{H}$ is called spherical if it has an open orbit on the flag variety $\mathbf{G}/\mathbf{B}$.

Uniform applications

- $\mathrm{Irr}_H(G)$:=all smooth admissible H-distinguished irreps.
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Assume char $F = 0$.

Corollary (following Prasad - Sakellaridis)

$\forall$ *unimodular spherical* $H \subset G$,

$$\max_{\pi \in \mathrm{Irr}_H(G)} \dim \mathrm{WF}^o(\pi) = \dim \mathcal{N}(\mathfrak{g}_0^*) \cap G\mathfrak{h}_0^{\perp} \quad \textit{and}$$

1. $\exists$ *inf. dim.* $\pi \in \mathrm{Irr}_H(G) \iff \exists$ *unipotent* $u \in G - H$.
2. *for quasisplit G:* $\exists$ *generic* $\pi \in \mathrm{Irr}_H(G) \iff H \cap U = \{1\}$ *for some maximal unipotent subgroup* $U \subset G$.

From now on $F := \mathbb{R}$.

Theorem (Rossmann 1995, using Borho-Brylinski + Joseph)

$\forall \pi \in \mathrm{Irr}(G)$, $\mathrm{WF}^o(\pi)$ *lies in a unique* $\mathbf{G}$*-orbit* $\mathcal{O}(\pi) \subset \mathcal{N}(\mathfrak{g}^*)$.

Archimedean setting & applications to symmetric pairs

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Let $\mathbf{H} \subset \mathbf{G}$ be symmetric sbgrp, and $G_H^{\mathbb{R}}$ the corresponding real form of $\mathbf{G}$.
Kostant-Sekiguchi: $\mathbf{H}$-orbits on $\mathfrak{h}^{\perp} \cong (\mathfrak{g}/\mathfrak{h})^* \leftrightarrow$ real $G_H^{\mathbb{R}}$-orbits on $\mathfrak{g}^*$.
Prasad philosophy: $G_H^{\mathbb{R}}$ describes H-distinguished $\mathrm{rep}^{\mathrm{ns}}$ of G.

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Corollary (of our thm)

$\forall \pi \in \mathrm{Irr}_H(G)$, $\mathcal{O}(\pi)$ *includes a real* $G_H^{\mathbb{R}}$*-orbit.*

Real orbits are classified by Ohta and Djokovic.

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Our theorem + result of Harris $\Rightarrow$ conjecture holds for most pairs.

Applications to strongly spherical pairs

Corollary

Assume that $\mathbf{H}$ *is reductive and* $\mathbf{\Delta H} \subset \mathbf{G} \times \mathbf{H}$ *spherical.*
Let $\pi \in \mathrm{Irr}(G)$, $\tau \in \mathrm{Irr}(H)$ *with* $\mathrm{Hom}_H(\pi|_H, \tau) \neq 0$. *Then*

$$\mathcal{O}(\tau) \subset \mathcal{O}(\pi)|_{\mathfrak{h}}.$$

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For classical G, $\mathcal{O}(\pi)$ is described by its Jordan partition $\lambda(\pi)$.

Corollary

Let $\mathbf{G}$ *be* GL_{n+1}, O_{n+1} *or* U_{n+1} *and* $\mathbf{H}$ *be* GL_n, O_n *or* U_n *respectively. Let* $\pi \in \operatorname{Irr}(G)$, $\tau \in \operatorname{Irr}(H)$ *with* $\operatorname{Hom}_H(\pi|_H, \tau) \neq 0$. *Then for any index* i*:*

$$|\lambda(\pi)_i^t - \lambda(\tau)_i^t| \leq 1.$$

Applications to Jacquet quotients

Corollary

Let $P = LU \subset G$ be parabolic subgroup, $\pi \in \mathrm{Irr}(G)$, and $\tau \in \mathrm{Irr}(L)$ be a quotient of the Jacquet module $r_U(\pi)$. Then $\mathcal{O}(\tau) \subset \mathcal{O}(\pi)|_{\mathfrak{p}^*}$.

$$\mathfrak{l} = \mathfrak{p}/\mathfrak{u}, \quad \mathcal{O}(\tau) \subset \mathfrak{l}^* \subset \mathfrak{p}^* \twoheadleftarrow \mathfrak{g}^* \supset \mathcal{O}(\pi)$$

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Proof.

Let $H := \{(p, pU) \in G \times L\} \subset G \times L$. Then $\mathrm{Hom}_H(\pi \hat{\otimes} \tilde{\tau}, \mathbb{C}) \neq 0$. □

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For a maximal parabolic of GL_n:

$$\text{if } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{O}(\tau) \text{ then } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{O}(\pi) \text{ for some } C.$$

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Proposition (Zhuohui Zhang 2020)

$\exists C$ *s. t.* $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ *has given Jordan partition* $\mu \iff c^{\mu}_{\lambda(A),\lambda(B)} \neq 0$

(Littlewood-Richardson coefficient)

Twisted induction of spherical pairs

Let $P = LU \subset G$ parabolic subgroup, $\mathbf{S} \subset \mathbf{L}$ be a spherical subgroup. Let $H := SU$. Fix algebraic character $\psi : U \to \mathbb{R}$, and let $\chi : H \to \mathbb{C}^\times$ be a character s.t. $\chi(u) := \exp(i\psi(u))\, \forall u \in U$. Consider $d\psi \in \mathfrak{u}^* \subset \mathfrak{h}^*$.

$$\mathrm{Irr}_{H,\chi}(G) := \{\pi \in \mathrm{Irr}(G) \;: \mathrm{Hom}_H(\pi|_H, \chi) \neq 0\}$$

Theorem (G.-Sayag 2020)

For all $\pi \in \mathrm{Irr}(G)_{H,\chi}, \quad d\psi \in \mathcal{O}(\pi)|_{\mathfrak{h}}$

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Shalika models: $\mathbf{G} = \mathrm{GL}_{2n}, \mathbf{L} = \mathrm{GL}_n \times \mathrm{GL}_n$, $\mathbf{S} = \Delta\,\mathrm{GL}_n$, $\mathfrak{u} = Mat_{n\times n}$,

$$H = \left\{ \begin{pmatrix} g & A \\ 0 & g \end{pmatrix} \right\}, \ \psi(A) = \mathrm{tr}(A).$$

Corollary

For all $\pi \in \mathrm{Irr}_{H,\chi}(G)$, $\lambda(\pi)$ *consists of even parts.*

$P = LU \subset G$ parabolic , $\mathbf{S} \subset \mathbf{L}$ spherical, $H := SU, \psi : U \to \mathbb{R}$, $\chi : H \to \mathbb{C}^\times$ character s.t. $\chi(u) := \exp(i\psi(u)) \, \forall u \in U$.

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Cor. For all $\pi \in \mathrm{Irr}_{H,\chi}(G)$, $\lambda(\pi)$ consists of even parts.
- Klyachko models: $\mathbf{G} = \mathrm{GL}_{n+k}$, $\mathbf{L} = \mathrm{GL}_n \times (\mathrm{GL}_1)^k$, $\mathbf{S} = \mathrm{Sp}_n$,
$$H = \left\{ \begin{pmatrix} g & A \\ 0 & B \end{pmatrix} : g \in \mathrm{Sp}_n,\ B \text{ upper-uni-triangular} \right\}, \psi(B) = \sum B_{i,i+1}$$

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Corollary

$\forall \pi \in \mathrm{Irr}_{H,\chi}(G)$, $\#\{i : \lambda(\pi)^t_i$ *is odd*$\} = k$.

G.-Offen-Sahi-Sayag: for unitary π,

$$\pi \in \mathrm{Irr}_{H,\chi}(G) \iff \#\{i : \lambda(\pi)^t_i \text{ is odd}\} = k.$$

Twisted induction of strongly spherical pairs

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If $\mathrm{Hom}_H(\pi|_H, \tau \otimes \chi) \neq 0$ *then* $|\lambda(\pi)^t_i - \lambda(\tau)^t_i| \leq 1$ *for any index* i.

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- Bessel models: analogues for $O(p,q), U(p,q), O_n(\mathbb{C})$.

Corollary (holds for both kinds of models: Rankin-Selberg and Bessel)

If $\mathrm{Hom}_H(\pi|_H, \tau \otimes \chi) \neq 0$ *then* $|\lambda(\pi)^t_i - \lambda(\tau)^t_i| \leq 1$ *for any index* i.

Related to the non-tempered Gan-Gross-Prasad conjecture.

Proof ingredients

Theorem (Wen-Wei Li 2019)

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- For a $\mathfrak{g}$-module M, $Ann(M)\subset\mathcal{U}(\mathfrak{g})$ - annihilator, and $\mathrm{An}\mathcal{V}(M)\subset\mathfrak{g}^*$ - annihilator variety:= zero set of symbols of $Ann(M)$

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Twisted version uses Kazhdan filtration + other ideas from W-algebras.

Beyond spherical subgroups

The only place where we used sphericity in the proof is to have $2\dim\mathcal{O}\cap\mathfrak{h}^{\perp}\leq\dim\mathcal{O}$ for any $\mathcal{O}\subset\overline{\mathcal{O}(\pi)}$. Thus, we can just assume this in the theorem, instead sphericity.

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The smallest non-zero orbit is called "minimal". For GL_n and for Sp_n, $\mathcal{O}_{\min}$ consists of rank one matrices $v \otimes \varphi$ (with $\varphi = v^t$ for Sp_n).

Example

Let $\mathbf{G} := \mathrm{GL}_n \times \mathrm{GL}_n \times \mathrm{GL}_n$ *and* $H := \Delta\, \mathrm{GL}_n$*. Let* $\mathcal{O}_1, \mathcal{O}_2 \subset \mathfrak{gl}_n^*$ *be arbitrary orbits, and* $\mathcal{O} := \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_{\min}$*. Then*

$$2\dim \mathcal{O} \cap \mathfrak{h}^{\perp} = \dim \mathcal{O}$$

Application: restrictions on annihilator varieties of quotients of $\pi \otimes \tau$ if $\mathcal{O}(\tau)$ is minimal.

Applications to local theta correspondence in type II

- $\iota : \mathrm{GL}(U) \times \mathrm{GL}(W) \hookrightarrow \mathrm{Sp}(\widetilde{V \oplus V^*})$
- ϖ- Weil representation
- π in $\mathrm{Irr}(\mathrm{GL}(V))$, $\varpi|_{\mathrm{GL}(V)} \twoheadrightarrow \pi \otimes \Theta(\pi)$
- $\theta(\pi) \in \mathrm{Irr}(\mathrm{GL}(W))$ - unique irr. quotient of $\Theta(\Pi)$.

Let $G := \mathrm{GL}(U) \times \mathrm{GL}(W) \times \mathrm{Sp}(\widetilde{V \oplus V^*})$ and $H = \mathrm{Graph}(\iota) \subset G$. Then $\pi \boxtimes \theta(\pi) \boxtimes \varpi \in \mathrm{Irr}(G)$ is H-distinguished.

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Theorem (Ido Karshon)

Let $\mathcal{O}_1 \subset \mathcal{N}(\mathfrak{gl}(U)^*)$ and $\mathcal{O}_2 \subset \mathcal{N}(\mathfrak{gl}(W)^*)$ be nilpotent orbits. Then

1. $2 \dim \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_{\min} \cap \mathfrak{h}^\perp \leq \dim \mathcal{O}_1 + \dim \mathcal{O}_2 + \dim \mathcal{O}_{\min}$
2. $2 \dim \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_{\min} \cap \mathfrak{h}^\perp \neq \emptyset$ *if and only if* $\mathcal{O}_1 \times \mathcal{O}_2$ *intersects the image of the moment map*

$$V = \mathrm{Hom}(U, W) \to \mathfrak{gl}(U) \times \mathfrak{gl}(W) \cong \mathfrak{gl}(U)^* \times \mathfrak{gl}(W)^*$$

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Corollary

$\mathcal{O}(\pi) \times \mathcal{O}(\theta(\pi))$ *intersects the image of the moment map.*

Definition of the wave front set

Assume $\mathrm{char}\, F = 0$.

Theorem (Howe, Harish-Chandra, Barbasch-Vogan, 70s)

Near $e \in G$, the character distribution (asymptotically) equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

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- Define $\mathrm{WF}(\pi) = \{\mathcal{O} : c_{\mathcal{O}} \neq 0\}$.
- Say $\mathcal{O} < \mathcal{O}'$ if $\mathcal{O} \subset \overline{\mathcal{O}'}$.
- $\mathrm{WF}^o(\pi) :=$ union of maximal elements of $\mathrm{WF}(\pi)$.

Evidence for the p-adic case, for $\mathbf{G} = \mathrm{GL}_n$

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p-adic theorem

$\forall$ nilpotent $\varphi \in \mathfrak{g}^*$, choose sl_2-triple $f, h, e \in \mathfrak{g}$ using Killing form + Jacobson-Morozov; let $\mathfrak{v}_\varphi := \mathfrak{g}^h_{\geq 2} \subset \mathfrak{g}$, $V_\varphi := \mathrm{Exp}(\mathfrak{v}_\varphi) \subset G$.

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(ii) $\mathcal{S}(G/H)_{V_\varphi,\varphi} \neq 0 \Rightarrow \varphi(\mathfrak{v}_\varphi \cap g(\mathfrak{h})) = \{0\} \Rightarrow \varphi \in (\mathfrak{g}^*)^h_{>-2} + g(\mathfrak{h}^\perp)$.
Kazhdan action preserves φ and $G\mathfrak{h}^\perp$, contracts $(\mathfrak{g}^*)^h_{>-2}$ thus $\varphi \in \overline{G\mathfrak{h}^\perp}$ □