Wave-front sets of distinguished representations

Dmitry Gourevitch
Weizmann Institute of Science, Israel
http://www.wisdom.weizmann.ac.il/~dimagur

CIRM workshop Relative Aspects of the Langlands Program, L-Functions and Beyond Endoscopy

j.w. Eitan Sayag

following Harris-Weich, Prasad, Sakellaridis, Vogan

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- F: local field, $\mathbf{G}$: reductive group over F, $\mathbf{H} \subset \mathbf{G}$ algebraic subgroup, $G := \mathbf{G}(F)$, $H := \mathbf{H}(F)$, $\mathfrak{g} := Lie(\mathbf{G})$, $\mathfrak{g}_0 := Lie(G)$, $\mathcal{N}(\mathfrak{g}^*)$:=nilpotent cone, $\mathfrak{h} := Lie(\mathbf{H})$, $\mathfrak{h}_0 := Lie(H)$.
- $\pi \in \mathrm{Irr}(G)$ is called *H-distinguished* if $(\pi^*)^H \neq 0$.
- $\mathrm{Irr}_H(G)$:=all smooth admissible H-distinguished irreps of G.
- $WF^o(\pi) \subset \mathcal{N}(\mathfrak{g}_0^*)$:= union of top G-orbits in the wave-front set of π.

Conjecture

For any $\pi \in \mathrm{Irr}_H(G)$, $WF^o(\pi)|_{\mathfrak{h}_0} \ni 0$.

Theorem (G.-Sayag 2020)

1. *If F is Archimedean and $\mathbf{H} \subset \mathbf{G}$ is spherical then*
$$\mathbf{G}(WF^0(\pi))|_{\mathfrak{h}} \ni 0.$$
2. *If F is p-adic then $WF^o(\pi) \subset \overline{G\mathfrak{h}_0^{\perp}} \subset \mathfrak{g}_0^*$*

$\mathbf{H}$ is called spherical if it has an open orbit on the flag variety $\mathbf{G}/\mathbf{B}$.

Uniform applications

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1. *If F is Archimedean and $\mathbf{H} \subset \mathbf{G}$ is spherical then*
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2. *If F is p-adic then $WF^o(\pi) \subset \overline{G\mathfrak{h}_0^\perp}$*

Assume char $F = 0$.

Corollary (following Prasad - Sakellaridis)

$\forall$ unimodular spherical $H \subset G$,

$$\max_{\pi \in \mathrm{Irr}_H(G)} \dim \mathrm{WF}^o(\pi) = \dim \mathcal{N}(\mathfrak{g}_0^*) \cap G\mathfrak{h}_0^\perp \quad \textit{and}$$

1. *$\exists$ inf. dim. $\pi \in \mathrm{Irr}_H(G) \iff \exists$ unipotent $u \in G - H$.*
2. *for quasisplit G: $\exists$ generic $\pi \in \mathrm{Irr}_H(G) \iff H \cap U = \{1\}$ for some maximal unipotent subgroup $U \subset G$.*

Archimedean setting & applications to symmetric pairs

From now on $F := \mathbb{R}$.

Theorem (Rossmann 1995, using Borho-Brylinski + Joseph)

$\forall \pi \in \mathrm{Irr}(G)$, $\mathrm{WF}^o(\pi)$ *lies in a unique* $\mathbf{G}$*-orbit* $\mathcal{O}(\pi) \subset \mathcal{N}(\mathfrak{g}^*)$.

Let $\mathbf{H} \subset \mathbf{G}$ be symmetric sbgrp, and $G_H^{\mathbb{R}}$ the corresponding real form of $\mathbf{G}$.
Kostant-Sekiguchi: $\mathbf{H}$-orbits on $\mathfrak{h}^{\perp} \cong (\mathfrak{g}/\mathfrak{h})^* \leftrightarrow$ real $G_H^{\mathbb{R}}$-orbits on $\mathfrak{g}^*$.
Prasad philosophy: $G_H^{\mathbb{R}}$ describes H-distinguished $\mathrm{rep}^{\mathrm{ns}}$ of G.

Corollary (of our thm)

$\forall \pi \in \mathrm{Irr}_H(G)$, $\mathcal{O}(\pi)$ *includes a real* $G_H^{\mathbb{R}}$*-orbit.*

Real orbits are classified by Ohta and Djokovic.

Conjecture (Prasad)

For split G, H*:* $\exists$ *tempered* $\pi \in \mathrm{Irr}_H(G) \Rightarrow \exists$ *generic* $\pi \in \mathrm{Irr}_H(G)$.

Our theorem + result of Harris $\Rightarrow$ conjecture holds for most pairs.

Applications to strongly spherical pairs

Corollary

Assume that $\mathbf{H}$ *is reductive and* $\mathbf{\Delta H} \subset \mathbf{G} \times \mathbf{H}$ *spherical.*
Let $\pi \in \operatorname{Irr}(G)$, $\tau \in \operatorname{Irr}(H)$ *with* $\operatorname{Hom}_H(\pi|_H, \tau) \neq 0$*. Then*

$$\mathcal{O}(\tau) \subset \mathcal{O}(\pi)|_{\mathfrak{h}}.$$

For classical G, $\mathcal{O}(\pi)$ is described by its Jordan partition $\lambda(\pi)$.

Corollary

Let $\mathbf{G}$ *be* GL_{n+1}, O_{n+1} *or* U_{n+1} *and* $\mathbf{H}$ *be* GL_n, O_n *or* U_n *respectively. Let* $\pi \in \operatorname{Irr}(G)$, $\tau \in \operatorname{Irr}(H)$ *with* $\operatorname{Hom}_H(\pi|_H, \tau) \neq 0$*. Then for any index* i*:*

$$|\lambda(\pi)_i^t - \lambda(\tau)_i^t| \leq 1.$$

Applications to Jacquet quotients

Corollary

Let $P = LU \subset G$ be parabolic subgroup, $\pi \in \mathrm{Irr}(G)$, and $\tau \in \mathrm{Irr}(L)$ be a quotient of the Jacquet module $r_U(\pi)$. Then $\mathcal{O}(\tau) \subset \mathcal{O}(\pi)|_{\mathfrak{p}^*}$.

$$\mathfrak{l} = \mathfrak{p}/\mathfrak{u}, \quad \mathcal{O}(\tau) \subset \mathfrak{l}^* \subset \mathfrak{p}^* \twoheadleftarrow \mathfrak{g}^* \supset \mathcal{O}(\pi)$$

Proof.

Let $H := \{(p, pU) \in G \times L\} \subset G \times L$. Then $\mathrm{Hom}_H(\pi \hat{\otimes} \tilde{\tau}, \mathbb{C}) \neq 0$. $\square$

For a maximal parabolic of GL_n:

$$\text{if } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{O}(\tau) \text{ then } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{O}(\pi) \text{ for some } C.$$

Proposition (Zhuohui Zhang 2020)

$\exists C$ *s. t.* $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ *has given Jordan partition* $\mu \iff c^{\mu}_{\lambda(A),\lambda(B)} \neq 0$

(Littlewood-Richardson coefficient)

Twisted induction of spherical pairs

Let $P = LU \subset G$ parabolic subgroup, $\mathbf{S} \subset \mathbf{L}$ be a spherical subgroup. Let $H := SU$. Fix algebraic character $\psi : U \to \mathbb{R}$, and let $\chi : H \to \mathbb{C}^\times$ be a character s.t. $\chi(u) := \exp(i\psi(u))\,\forall u \in U$. Consider $d\psi \in \mathfrak{u}^* \subset \mathfrak{h}^*$.

$$\mathrm{Irr}_{H,\chi}(G) := \{\pi \in \mathrm{Irr}(G)\ : \mathrm{Hom}_H(\pi|_H, \chi) \neq 0\}$$

Theorem (G.-Sayag 2020)

For all $\pi \in \mathrm{Irr}(G)_{H,\chi}, \quad d\psi \in \mathcal{O}(\pi)|_{\mathfrak{h}}$

Shalika models: $\mathbf{G} = \mathrm{GL}_{2n}, \mathbf{L} = \mathrm{GL}_n \times \mathrm{GL}_n$, $\mathbf{S} = \Delta\,\mathrm{GL}_n$, $\mathfrak{u} = Mat_{n\times n}$,

$$H = \left\{ \begin{pmatrix} g & A \\ 0 & g \end{pmatrix} \right\}, \psi(A) = \mathrm{tr}(A).$$

Corollary

For all $\pi \in \mathrm{Irr}_{H,\chi}(G)$, $\lambda(\pi)$ *consists of even parts.*

$P = LU \subset G$ parabolic , $\mathbf{S} \subset \mathbf{L}$ spherical, $H := SU, \psi : U \to \mathbb{R}$,
$\chi : H \to \mathbb{C}^\times$ character s.t. $\chi(u) := \exp(i\psi(u)) \, \forall u \in U$.

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  Cor. For all $\pi \in \mathrm{Irr}_{H,\chi}(G)$, $\lambda(\pi)$ consists of even parts.
- Klyachko models: $\mathbf{G} = \mathrm{GL}_{n+k}$, $\mathbf{L} = \mathrm{GL}_n \times (\mathrm{GL}_1)^k$, $\mathbf{S} = \mathrm{Sp}_n$,

$$H = \left\{ \begin{pmatrix} g & A \\ 0 & B \end{pmatrix} : g \in \mathrm{Sp}_n,\ B \text{ upper-uni-triangular} \right\}, \psi(B) = \sum B_{i,i+1}$$

Corollary

$\forall \pi \in \mathrm{Irr}_{H,\chi}(G)$, $\#\{i : \lambda(\pi)^t_i$ *is odd*$\} = k$.

G.-Offen-Sahi-Sayag: for unitary π,

$$\pi \in \mathrm{Irr}_{H,\chi}(G) \iff \#\{i : \lambda(\pi)^t_i \text{ is odd}\} = k.$$

Twisted induction of strongly spherical pairs

$P = LU \subset G$ parabolic, $\mathbf{S} \subset \mathbf{L}$ with $\Delta\mathbf{S} \subset \mathbf{S} \times \mathbf{L}$ spherical,
$H := SU, \psi : U \to \mathbb{R}, \chi : H \to \mathbb{C}^\times$ char s.t. $\chi(u) := \exp(i\psi(u)) \, \forall u \in U$.

Corollary (G.-Sayag 2020)

Let $\pi \in \mathrm{Irr}(G), \tau \in \mathrm{Irr}(S)$ *with* $\mathrm{Hom}_H(\pi|_H, \tau \otimes \chi) \neq 0$. *Then*

$$\mathcal{O}(\tau) + d\psi \subset \mathcal{O}(\pi)|_{\mathfrak{h}}$$

Rankin-Selberg models: $\mathbf{G} = \mathrm{GL}_{n+k}$, $\mathbf{L} = \mathrm{GL}_{n+1} \times (\mathrm{GL}_1)^{k-1}$, $\mathbf{S} = \mathrm{GL}_n$,

$$H = \left\{ \begin{pmatrix} g & 0 & * & \dots & * \\ 0 & 1 & \underline{*} & \dots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \underline{*} \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} \right\}, \quad \psi = \text{ sum of underlined entries}$$

Corollary

If $\mathrm{Hom}_H(\pi|_H, \tau \otimes \chi) \neq 0$ *then* $|\lambda(\pi)^t_i - \lambda(\tau)^t_i| \leq 1$ *for any index* i.

$P = LU \subset G$ parabolic, $\mathbf{S} \subset \mathbf{L}$ with $\Delta\mathbf{S} \subset \mathbf{S} \times \mathbf{L}$ spherical,
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- Bessel models: analogues for $O(p, q), U(p, q), O_n(\mathbb{C})$.

Corollary (holds for both kinds of models: Rankin-Selberg and Bessel)

If $\mathrm{Hom}_H(\pi|_H, \tau \otimes \chi) \neq 0$ *then* $|\lambda(\pi)^t_i - \lambda(\tau)^t_i| \leq 1$ *for any index* i.

Related to the non-tempered Gan-Gross-Prasad conjecture.

Proof ingredients

Theorem (Wen-Wei Li 2019)

For any nilpotent orbit $\mathcal{O}$, $2\dim \mathcal{O}\cap \mathfrak{h}^{\perp} \leq \dim \mathcal{O}$.

- $\mathcal{U}_n(\mathfrak{g})$ - PBW filtration on universal enveloping algebra.
- $\mathrm{gr}\,\mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g}) \cong \mathrm{Pol}(\mathfrak{g}^*)$.
- For a $\mathfrak{g}$-module M, $Ann(M) \subset \mathcal{U}(\mathfrak{g})$ - annihilator, and $\mathrm{An}\mathcal{V}(M) \subset \mathfrak{g}^*$ - annihilator variety:= zero set of symbols of $Ann(M)$

Theorem (Rossmann 1995)

For all $\pi \in \mathrm{Irr}(G)$, $\mathrm{An}\mathcal{V}(\pi) = \overline{\mathcal{O}(\pi)}$.

For finitely-generated M, define $\mathrm{As}\mathcal{V}(M)$ to be the set of common zeros of symbols of annihilators of the generators.

Theorem (Gabber-Joseph)

$2\dim \mathrm{As}\mathcal{V}(M) \geq \dim \mathrm{An}\mathcal{V}(M)$.

- For any nilpotent orbit $\mathcal{O}$, $2\dim \mathcal{O}\cap\mathfrak{h}^\perp \leq \dim\mathcal{O}$.
- $\mathrm{An}\mathcal{V}(\pi)=\overline{\mathcal{O}(\pi)}$
- For f.g. M, $2\dim \mathrm{As}\mathcal{V}(M)\geq \dim \mathrm{An}\mathcal{V}(M)$.
- If M is generated by an $\mathfrak{h}$-finite vector then $\mathrm{As}\mathcal{V}(M)\subset \mathrm{An}\mathcal{V}(M)\cap\mathfrak{h}^\perp$.

Theorem (G. - Sayag 2020)

For any $\pi\in \mathrm{Irr}(G)_H$, $\mathcal{O}(\pi)\cap\mathfrak{h}^\perp\neq\emptyset$.

Proof.

Let $\eta\neq 0\in(\pi^*)^H$ and $M:=\mathcal{U}(\mathfrak{g})\eta\subset\pi^*$. M is non-degenerately paired with π, thus $Ann(M)=Ann(\pi)$ and $\mathrm{An}\mathcal{V}(M)=\mathrm{An}\mathcal{V}(\pi)$. From

$$2\dim \mathrm{As}\mathcal{V}(M)\geq \dim \mathrm{An}\mathcal{V}(M)=\dim \mathrm{An}\mathcal{V}(\pi)=\dim\mathcal{O}(\pi), \tag{1}$$

$$\mathrm{As}\mathcal{V}(M)\subset \mathrm{An}\mathcal{V}(M)\cap\mathfrak{h}^\perp, \text{ and} \tag{2}$$

$$\text{intersection with } \mathfrak{h}^\perp \text{ reduces dimension times 2,} \tag{3}$$

we get that $\mathcal{O}(\pi)$ intersects $\mathfrak{h}^\perp$. $\square$

Twisted version uses Kazhdan filtration + other ideas from W-algebras.

Beyond spherical subgroups

The only place where we used sphericity in the proof is to have $2\dim \mathcal{O}\cap \mathfrak{h}^{\perp} \leq \dim \mathcal{O}$ for any $\mathcal{O}\subset \overline{\mathcal{O}(\pi)}$. Thus, we can just assume this in the theorem, instead sphericity.

The smallest non-zero orbit is called "minimal". For GL_n and for Sp_n, $\mathcal{O}_{\min}$ consists of rank one matrices $v\otimes\varphi$ (with $\varphi = v^t$ for Sp_n).

Example

Let $\mathbf{G} := \mathrm{GL}_n\times \mathrm{GL}_n\times \mathrm{GL}_n$ *and* $H := \Delta\,\mathrm{GL}_n$. *Let* $\mathcal{O}_1,\mathcal{O}_2\subset \mathfrak{gl}_n^*$ *be arbitrary orbits, and* $\mathcal{O} := \mathcal{O}_1\times\mathcal{O}_2\times\mathcal{O}_{\min}$. *Then*

$$2\dim \mathcal{O}\cap \mathfrak{h}^{\perp} = \dim \mathcal{O}$$

Application: restrictions on annihilator varieties of quotients of $\pi\otimes\tau$ if $\mathcal{O}(\tau)$ is minimal.

Applications to local theta correspondence in type II

- $\iota : \mathrm{GL}(U) \times \mathrm{GL}(W) \hookrightarrow \mathrm{Sp}(\widetilde{V \oplus V^*})$
- ϖ- Weil representation
- π in $\mathrm{Irr}(\mathrm{GL}(V))$, $\varpi|_{\mathrm{GL}(V)} \twoheadrightarrow \pi \otimes \Theta(\pi)$
- $\theta(\pi) \in \mathrm{Irr}(\mathrm{GL}(W))$ - unique irr. quotient of $\Theta(\Pi)$.

Let $G := \mathrm{GL}(U) \times \mathrm{GL}(W) \times \mathrm{Sp}(\widetilde{V \oplus V^*})$ and $H = \mathrm{Graph}(\iota) \subset G$. Then $\pi \boxtimes \theta(\pi) \boxtimes \varpi \in \mathrm{Irr}(G)$ is H-distinguished.

Theorem (Ido Karshon)

Let $\mathcal{O}_1 \subset \mathcal{N}(\mathfrak{gl}(U)^*)$ and $\mathcal{O}_2 \subset \mathcal{N}(\mathfrak{gl}(W)^*)$ be nilpotent orbits. Then

1. $2 \dim \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_{\min} \cap \mathfrak{h}^\perp \leq \dim \mathcal{O}_1 + \dim \mathcal{O}_2 + \dim \mathcal{O}_{\min}$
2. $2 \dim \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_{\min} \cap \mathfrak{h}^\perp \neq \emptyset$ *if and only if* $\mathcal{O}_1 \times \mathcal{O}_2$ *intersects the image of the moment map*

$$V = \mathrm{Hom}(U, W) \to \mathfrak{gl}(U) \times \mathfrak{gl}(W) \cong \mathfrak{gl}(U)^* \times \mathfrak{gl}(W)^*$$

given by $A \mapsto (A^*A, AA^*)$

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Corollary

$\mathcal{O}(\pi) \times \mathcal{O}(\theta(\pi))$ *intersects the image of the moment map.*

Definition of the wave front set

Assume $\operatorname{char} F = 0$.

Theorem (Howe, Harish-Chandra, Barbasch-Vogan, 70s)

Near $e \in G$, the character distribution (asymptotically) equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

$$\exp^*(\chi_\pi) \approx \sum c_{\mathcal{O}} \mathcal{F}(\mu_{\mathcal{O}})$$

- Define $\mathrm{WF}(\pi) = \{\mathcal{O} : c_{\mathcal{O}} \neq 0\}$.
- Say $\mathcal{O} < \mathcal{O}'$ if $\mathcal{O} \subset \overline{\mathcal{O}'}$.
- $\mathrm{WF}^o(\pi) :=$ union of maximal elements of $\mathrm{WF}(\pi)$.

Evidence for the p-adic case, for $\mathbf{G} = \mathrm{GL}_n$

- Klyachko models: Offen-Sayag
- Jacquet quotients of ladder representations: Lapid - Minguez
- Restriction from GL_{n+1} to GL_n for representations of Arthur type: Maxim Gurevich, Kei Yuen Chan
- Local theta correspondence: Gomez-Zhu, Gan

p-adic theorem

$\forall$ nilpotent $\varphi \in \mathfrak{g}^*$, choose sl_2-triple $f, h, e \in \mathfrak{g}$ using Killing form + Jacobson-Morozov; let $\mathfrak{v}_\varphi := \mathfrak{g}^h_{\geq 2} \subset \mathfrak{g}$, $V_\varphi := \mathrm{Exp}(\mathfrak{v}_\varphi) \subset G$.

Definition

For any rep-n Π, let $\mathrm{WO}(\Pi) := \{\varphi \in \mathfrak{g}^* : \Pi_{V_\varphi,\varphi} \neq 0\}$.

Theorem (Moeglin-Waldspurger 1987)

If F is p-adic then $\forall \pi \in \mathrm{Irr}(G)$, $\overline{\mathrm{WO}(\pi)} = \overline{\mathrm{WF}^o(\pi)}$.

Theorem (G.-Sayag 2020)

$G\mathfrak{h}^\perp \cap \mathcal{N} \subset \mathrm{WO}(\mathcal{S}(G/H)) \subset \overline{G\mathfrak{h}^\perp} \cap \mathcal{N}$.

Ingredients and sketch of proof

$\forall$ nilpotent $\varphi \in \mathfrak{g}^*$, choose $\mathfrak{sl}_2$-triple $f, h, e \in \mathfrak{g}$ using Killing form + Jacobson-Morozov; let $\mathfrak{v}_\varphi := \mathfrak{g}^h_{\geq 2} \subset \mathfrak{g}$, $V_\varphi := \mathrm{Exp}(\mathfrak{v}_\varphi) \subset G$.

Definition

For any rep-n Π, let $\mathrm{WO}(\Pi) := \{\varphi \in \mathfrak{g}^* \ : \Pi_{V_\varphi,\varphi} \neq 0\}$.

- Kazhdan action of $F^\times$ on $\mathfrak{g}^*$: $t \cdot \psi := t^2 \exp(th)\psi$. Fixes φ.
- $\mathcal{S}(G/H)_{V_\varphi,\varphi} \neq 0 \iff \varphi(\mathfrak{v}_\varphi \cap g(\mathfrak{h})) = \{0\}$ for some $g \in G$.

Theorem (G.-Sayag 2020)

$G\mathfrak{h}^\perp \cap \mathcal{N} \subset \mathrm{WO}(\mathcal{S}(G/H)) \subset \overline{G\mathfrak{h}^\perp} \cap \mathcal{N}$.

Proof.

(i) $\varphi \in g(\mathfrak{h}^\perp) \Rightarrow \varphi(\mathfrak{v}_\varphi \cap g(\mathfrak{h})) = \{0\} \Rightarrow \mathcal{S}(G/H)_{V_\varphi,\varphi} \neq 0$.

(ii) $\mathcal{S}(G/H)_{V_\varphi,\varphi} \neq 0 \Rightarrow \varphi(\mathfrak{v}_\varphi \cap g(\mathfrak{h})) = \{0\} \Rightarrow \varphi \in (\mathfrak{g}^*)^h_{>-2} + g(\mathfrak{h}^\perp)$.

Kazhdan action preserves φ and $G\mathfrak{h}^\perp$, contracts $(\mathfrak{g}^*)^h_{>-2}$ thus $\varphi \in \overline{G\mathfrak{h}^\perp}$ $\square$