Fourier coefficients of automorphic forms & applications to minimal and next-to-minimal forms.

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Automorphic Structures in String Theory, SCGP, Stony Brooke, NY
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Following Piatetski-Shapiro–Shalika, Ginzburg–Rallis–Soudry, Moeglin-Waldspurger, Jiang–Liu–Savin, Gomez, Ahlen, Hundley–Sayag, Green-Miller-Vanhove, Kazhdan–Polishchuk, Bossard–Pioline

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Definitions

- $\mathbb{K}$: number field, $\mathbb{A} := \mathbb{A}_{\mathbb{K}}$, $\mathbf{G}$: reductive group over $\mathbb{K}$, $\Gamma := \mathbf{G}(\mathbb{K})$, $G := \mathbf{G}(\mathbb{A})$, $\mathfrak{g} := Lie(\Gamma)$.

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- $C^\infty(\Gamma\backslash G) :=$ functions on $\Gamma\backslash G$ smooth on G_∞ and finite under K_{fin}.
- Let $[N] := (\Gamma \cap N)\backslash N$. For $\eta \in C^\infty(\Gamma\backslash G)$, define *Fourier coefficient*

$$\mathcal{F}_{H,f}[\eta](g) := \int_{[N]} \eta(ng)\chi_f(n)^{-1}dn.$$

Two central cases of Fourier coefficients

$$[H, f] = -2f,\ \mathfrak{n} = (\mathfrak{g}_1 \cap \mathfrak{g}^f) \oplus \bigoplus_{i>1} \mathfrak{g}_i,\ N = \mathrm{Exp}(\mathfrak{n})(\mathbb{A}),\ \eta \in C^\infty(\Gamma \backslash G),$$

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- Neutral Fourier coefficient, coming from $\mathfrak{sl}_2$-triple (e,H,f), *e.g.*:

$$H=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\ f=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}\ \mathfrak{n}=\begin{pmatrix}0&\underline{*}&0&*\\0&0&0&0\\0&*&0&\underline{*}\\0&0&0&0\end{pmatrix}$$

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Coefficients that are both neutral and Whittaker are Eulerian by local uniqueness of Whittaker models.

Examples of Fourier coefficients

$[H, f] = -2f$, $\mathfrak{n} = (\mathfrak{g}_1 \cap \mathfrak{g}^f) \oplus \bigoplus_{i>1} \mathfrak{g}_i$, $N = \mathrm{Exp}(\mathfrak{n})(\mathbb{A})$, $\eta \in C^\infty(\Gamma \backslash G)$,

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Comparison for $G = \mathrm{GL}_3(\mathbb{K})$:

- Neutral Fourier coefficient:

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Lemma

(i) $\mathcal{F}_{H,f}[\eta](g) = \sum_{\gamma \in (U/I^{\perp})(\mathbb{K})} \mathcal{F}^I_{H,f}[\eta](\gamma g)$

(ii) *For any isotropic subspace* $\mathfrak{j} \subset \mathfrak{u}$ *with* $\dim \mathfrak{j} = \dim \mathfrak{i}$ *and* $\mathfrak{j} \cap \mathfrak{i}^{\perp} = \{0\}$,

$$\mathcal{F}^J_{H,f}[\eta](g) = \int_{J(\mathbb{A})} \mathcal{F}^I_{H,f}[\eta](ug)\, du$$

For $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$: $\begin{pmatrix} 0 & \mathfrak{i} & \underline{\mathfrak{n}} \\ 0 & 0 & \mathfrak{j} \\ 0 & 0 & 0 \end{pmatrix}$

Relating different coefficients

- $\mathrm{WO}(\eta) := \{\mathcal{O} \in \mathcal{N}(\mathfrak{g}) \,|\, \forall$ neutral (h, f) with $f \in \mathcal{O}$, $\mathcal{F}_{h,f}(\eta) \not\equiv 0\}$.

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Theorem

Let $(H, f) \succ (S, f)$. Then

(i) *$\mathcal{F}_{H,f}[\eta]$ linearly determines $\mathcal{F}_{S,f}[\eta]$.*

(ii) *If $\Gamma f \in \mathrm{WO}^{\max}(\eta)$ and $\mathfrak{g}^H_1 = \mathfrak{g}^S_1 = 0$ let $\mathfrak{v} := \mathfrak{g}^H_{>1} \cap \mathfrak{g}^S_{<1}$. Then*

$$\mathcal{F}_{H,f}[\eta](g) = \int_{V(\mathbb{A})} \mathcal{F}_{S,f}[\eta](vg)\, dv$$

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Corollary

(i) *If* η *is cuspidal then any* $\mathcal{O} \in \mathrm{WO}^{\max}(\eta)$ *is* $\mathbb{K}$*-distinguished.*
In particular, $\mathcal{O}$ *is totally even for* $G = \mathrm{Sp}_{2n}$, *totally odd for* $G = SO(V)$, *not minimal for* $\mathrm{rk} G > 1$,
and not next-to-minimal for $\mathrm{rk} G > 2$, $G \neq F_4$.

(ii) *If* $f \notin \mathrm{WO}(\eta)$ *then* $\mathcal{F}_{H,f}(\eta) = 0$ *for any* H.

(iii) *Let* G *be simply-laced,* H *define a maximal parabolic,* $f \in \mathrm{WO}^{\max}(\eta)$:
f *is minimal* $\Rightarrow \mathcal{F}_{H,f}(\eta) = \mathcal{W}_f(\eta)$
f *is next-to-minimal* $\Rightarrow \mathcal{F}_{H,f}(\eta) = \int_{V(\mathbb{A})} \mathcal{W}_f(\eta)$.
The RHS is frequently Eulerian.

Example

$G := \mathrm{GL}(4, \mathbb{A})$, $f := E_{21} + E_{43}$, $H := \mathrm{diag}(3, 1, -1, -3)$,
$h = \mathrm{diag}(1, -1, 1, -1)$, $Z = H - h = \mathrm{diag}(2, 2, -2, -2)$, $H_t := h + tZ$.

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Then $\mathfrak{n}_0 \subset \mathfrak{n}_{1/4} \oplus \mathfrak{i} \sim \mathfrak{n}_{1/4} \oplus \mathfrak{j} \subset \mathfrak{n}_{3/4} = \mathfrak{n}_1$:

$$\begin{pmatrix} 0 & - & 0 & - \\ 0 & 0 & 0 & 0 \\ 0 & - & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & - & a & - \\ 0 & 0 & 0 & a \\ 0 & * & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & - & * & - \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Both $*$ and $-$ denote arbitrary elements. $-$ denotes the entries in $\mathfrak{g}^{H_t}_{>0}$ and $*$ those in $\mathfrak{g}^{H_t}_1$. a denotes equal elements in $\mathfrak{g}^{H_t}_1 \cap \mathfrak{g}^f$.

Expressing forms through their Whittaker coefficients

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(i) *Any* $\eta \in C^{\infty}(\Gamma \backslash G)$ *is linearly determined by all its Levi-distinguished Fourier coefficients.*

(ii) *If all* $\mathcal{O} \in \mathrm{WO}(\eta)$ *admit Whittaker coefficients then* η *is linearly determined by its Whittaker coefficients.*

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(ii) *If all $\mathcal{O} \in \mathrm{WO}(\eta)$ admit Whittaker coefficients then η is linearly determined by its Whittaker coefficients.*

(iii) *If G is split and simply-laced, and η is minimal or next-to-minimal then all Fourier coefficients of η are linearly determined by Whittaker coefficients of η.*

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Explanation for GL_n (PS-Shalika, Ahlen–Gustafsson-Liu-Kleinschmidt-Persson)

Let $\eta \in C^\infty(\Gamma \backslash \mathrm{GL}_n(\mathbb{A}))$. Restrict to the last column and decompose to Fourier series. All non-trivial characters are conjugate by $\mathrm{GL}_{n-1}(\mathbb{K})$.

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