Generalized and degenerate Whittaker models for representations of reductive groups over local fields

Dmitry Gourevitch
http://www.wisdom.weizmann.ac.il/~dimagur

Sphericity workshop, Kloster Reute, Germany

February 2016

Whittaker models

- $\mathbb{F}$: local field of char.0, G: reductive group over $\mathbb{F}$.

Whittaker models

- $\mathbb{F}$: local field of char.0, G: reductive group over $\mathbb{F}$.
- $\mathcal{M}(G)$: smooth admissible f.g. representations (of moderate growth).

Whittaker models

- $\mathbb{F}$: local field of char.0, G: reductive group over $\mathbb{F}$.
- $\mathcal{M}(G)$: smooth admissible f.g. representations (of moderate growth).
- Just for this slide assume G is quasisplit and fix a Borel $B = HU$,

Whittaker models

- $\mathbb{F}$: local field of char.0, G: reductive group over $\mathbb{F}$.
- $\mathcal{M}(G)$: smooth admissible f.g. representations (of moderate growth).
- Just for this slide assume G is quasisplit and fix a Borel $B = HU$,
- $\Psi^{\times} :=$ non-degenerate unitary characters of U.

Whittaker models

- $\mathbb{F}$: local field of char.0, G: reductive group over $\mathbb{F}$.
- $\mathcal{M}(G)$: smooth admissible f.g. representations (of moderate growth).
- Just for this slide assume G is quasisplit and fix a Borel $B = HU$,
- $\Psi^{\times} :=$ non-degenerate unitary characters of U.
- For $\psi \in \Psi^{\times}$ define $\mathcal{W}_{\psi} := \operatorname{ind}_U^G(\psi)$.

Whittaker models

- $\mathbb{F}$: local field of char.0, G: reductive group over $\mathbb{F}$.
- $\mathcal{M}(G)$: smooth admissible f.g. representations (of moderate growth).
- Just for this slide assume G is quasisplit and fix a Borel $B = HU$,
- $\Psi^{\times} :=$ non-degenerate unitary characters of U.
- For $\psi \in \Psi^{\times}$ define $\mathcal{W}_{\psi} := \mathrm{ind}_U^G(\psi)$.
- For $\pi \in \mathcal{M}(G)$ define

$$\mathcal{W}_{\psi}(\pi) := \mathrm{Hom}_G(\mathcal{W}_{\psi}, \pi^*) = \mathrm{Hom}_U(\pi, \psi).$$

Whittaker models

- $\mathbb{F}$: local field of char.0, G: reductive group over $\mathbb{F}$.
- $\mathcal{M}(G)$: smooth admissible f.g. representations (of moderate growth).
- Just for this slide assume G is quasisplit and fix a Borel $B = HU$,
- $\Psi^\times :=$ non-degenerate unitary characters of U.
- For $\psi \in \Psi^\times$ define $\mathcal{W}_\psi := \mathrm{ind}_U^G(\psi)$.
- For $\pi \in \mathcal{M}(G)$ define

$$\mathcal{W}_\psi(\pi) := \mathrm{Hom}_G(\mathcal{W}_\psi, \pi^*) = \mathrm{Hom}_U(\pi, \psi).$$

Whittaker models

- $\mathbb{F}$: local field of char.0, G: reductive group over $\mathbb{F}$.
- $\mathcal{M}(G)$: smooth admissible f.g. representations (of moderate growth).
- Just for this slide assume G is quasisplit and fix a Borel $B = HU$,
- $\Psi^{\times} :=$ non-degenerate unitary characters of U.
- For $\psi \in \Psi^{\times}$ define $\mathcal{W}_{\psi} := \mathrm{ind}_U^G(\psi)$.
- For $\pi \in \mathcal{M}(G)$ define

$$\mathcal{W}_{\psi}(\pi) := \mathrm{Hom}_G(\mathcal{W}_{\psi}, \pi^*) = \mathrm{Hom}_U(\pi, \psi).$$

Theorem (Jacquet,Gelfand-Kazhdan, Shalika, Rodier, Kostant, Vogan, Casselman-Hecht-Milicic, 70s-90s)

- *For any* $\psi \in \Psi^{\times}$, $\quad \pi \mapsto \mathcal{W}_{\psi}(\pi)$ *is an exact functor.*

Whittaker models

- $\mathbb{F}$: local field of char.0, G: reductive group over $\mathbb{F}$.
- $\mathcal{M}(G)$: smooth admissible f.g. representations (of moderate growth).
- Just for this slide assume G is quasisplit and fix a Borel $B = HU$,
- $\Psi^{\times}$:= non-degenerate unitary characters of U.
- For $\psi \in \Psi^{\times}$ define $\mathcal{W}_\psi := \mathrm{ind}_U^G(\psi)$.
- For $\pi \in \mathcal{M}(G)$ define

$$\mathcal{W}_\psi(\pi) := \mathrm{Hom}_G(\mathcal{W}_\psi, \pi^*) = \mathrm{Hom}_U(\pi, \psi).$$

Theorem (Jacquet,Gelfand-Kazhdan, Shalika, Rodier, Kostant, Vogan, Casselman-Hecht-Milicic, 70s-90s)

- *For any* $\psi \in \Psi^{\times}$, $\quad \pi \mapsto \mathcal{W}_\psi(\pi)$ *is an exact functor.*
- *For any* $\pi \in Irr(G)$, $\psi \in \Psi^{\times}$ *we have* $\dim \mathcal{W}_\psi(\pi) \leq 1$.

Whittaker models

- $\mathbb{F}$: local field of char.0, G: reductive group over $\mathbb{F}$.
- $\mathcal{M}(G)$: smooth admissible f.g. representations (of moderate growth).
- Just for this slide assume G is quasisplit and fix a Borel $B = HU$,
- $\Psi^{\times} :=$ non-degenerate unitary characters of U.
- For $\psi \in \Psi^{\times}$ define $\mathcal{W}_{\psi} := \mathrm{ind}_U^G(\psi)$.
- For $\pi \in \mathcal{M}(G)$ define

$$\mathcal{W}_{\psi}(\pi) := \mathrm{Hom}_G(\mathcal{W}_{\psi}, \pi^*) = \mathrm{Hom}_U(\pi, \psi).$$

Theorem (Jacquet,Gelfand-Kazhdan, Shalika, Rodier, Kostant, Vogan, Casselman-Hecht-Milicic, 70s-90s)

- *For any* $\psi \in \Psi^{\times}$, $\quad \pi \mapsto \mathcal{W}_{\psi}(\pi)$ *is an exact functor.*
- *For any* $\pi \in Irr(G)$, $\psi \in \Psi^{\times}$ *we have* $\dim \mathcal{W}_{\psi}(\pi) \leq 1$.
- *For any* $\pi \in \mathcal{M}(G)$,

$$(\exists \psi \in \Psi^{\times} \text{ with } \mathcal{W}_{\psi}(\pi) \neq 0) \Leftrightarrow \pi \text{ is large.}$$

Wave front set and wave-front cycle

Theorem (Howe, Harish-Chandra, Barbasch-Vogan, 70s)

Near $e \in G$, the character distribution (asymptotically) equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

$$\exp^*(\chi_\pi) \approx \sum c_{\mathcal{O}} \mathcal{F}(\mu_{\mathcal{O}})$$

Wave front set and wave-front cycle

Theorem (Howe, Harish-Chandra, Barbasch-Vogan, 70s)

Near $e \in G$, the character distribution (asymptotically) equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

$$\exp^*(\chi_\pi) \approx \sum c_{\mathcal{O}} \mathcal{F}(\mu_{\mathcal{O}})$$

- Let $\mathcal{N} \subset \mathfrak{g}^*$ denote the nilpotent cone.

Wave front set and wave-front cycle

Theorem (Howe, Harish-Chandra, Barbasch-Vogan, 70s)

Near $e \in G$, the character distribution (asymptotically) equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

$$\exp^*(\chi_\pi) \approx \sum c_{\mathcal{O}} \mathcal{F}(\mu_{\mathcal{O}})$$

- Let $\mathcal{N} \subset \mathfrak{g}^*$ denote the nilpotent cone.
- Define $\mathrm{WF}(\pi) = \cup\{\overline{\mathcal{O}} \mid c_{\mathcal{O}} \neq 0\} \subset \mathcal{N}$.

Wave front set and wave-front cycle

Theorem (Howe, Harish-Chandra, Barbasch-Vogan, 70s)

Near $e \in G$, the character distribution (asymptotically) equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

$$\exp^*(\chi_\pi) \approx \sum c_{\mathcal{O}} \mathcal{F}(\mu_{\mathcal{O}})$$

- Let $\mathcal{N} \subset \mathfrak{g}^*$ denote the nilpotent cone.
- Define $\mathrm{WF}(\pi) = \cup\{\overline{\mathcal{O}} \mid c_{\mathcal{O}} \neq 0\} \subset \mathcal{N}$.
- π is called *large* if $\dim \mathrm{WF}(\pi) = \dim \mathcal{N}$.

The result of Moeglin and Waldspurger in the p-adic case

- Fix a semisimple $H \in \mathfrak{g}$, and let $\mathfrak{g}_i$ denote the eigenspaces of $ad(H)$. Assume that all the eigenvalues i lie in $\mathbb{Q}$.

The result of Moeglin and Waldspurger in the p-adic case

- Fix a semisimple $H \in \mathfrak{g}$, and let $\mathfrak{g}_i$ denote the eigenspaces of $ad(H)$. Assume that all the eigenvalues i lie in $\mathbb{Q}$.
- Let $\varphi \in \mathfrak{g}^*_{-2}$ and let $\mathfrak{l} \subset \mathfrak{g}_1$ be a maximal isotropic subspace w.r.t $\omega(X, Y) := \varphi([X, Y])$.

The result of Moeglin and Waldspurger in the p-adic case

- Fix a semisimple $H \in \mathfrak{g}$, and let $\mathfrak{g}_i$ denote the eigenspaces of $ad(H)$. Assume that all the eigenvalues i lie in $\mathbb{Q}$.
- Let $\varphi \in \mathfrak{g}^*_{-2}$ and let $\mathfrak{l} \subset \mathfrak{g}_1$ be a maximal isotropic subspace w.r.t $\omega(X, Y) := \varphi([X, Y])$.
- Define $\mathfrak{v} := \bigoplus_{i>1} \mathfrak{g}_i$, $\mathfrak{n} := \mathfrak{v} \oplus \mathfrak{l}$, $N := \mathrm{Exp}(\mathfrak{n})$

The result of Moeglin and Waldspurger in the p-adic case

- Fix a semisimple $H \in \mathfrak{g}$, and let $\mathfrak{g}_i$ denote the eigenspaces of $ad(H)$. Assume that all the eigenvalues i lie in $\mathbb{Q}$.
- Let $\varphi \in \mathfrak{g}^*_{-2}$ and let $\mathfrak{l} \subset \mathfrak{g}_1$ be a maximal isotropic subspace w.r.t $\omega(X, Y) := \varphi([X, Y])$.
- Define $\mathfrak{v} := \bigoplus_{i>1} \mathfrak{g}_i$, $\mathfrak{n} := \mathfrak{v} \oplus \mathfrak{l}$, $N := \mathrm{Exp}(\mathfrak{n})$
- Define $\mathcal{W}_{H,\varphi} := \mathrm{ind}_N^G \varphi$, and for $\pi \in \mathcal{M}(G)$ let

$$\mathcal{W}_{H,\varphi}(\pi) := \mathrm{Hom}_G(\mathcal{W}_{H,\varphi}, \pi^*) = \mathrm{Hom}_N(\pi, \varphi).$$

The result of Moeglin and Waldspurger in the p-adic case

- Fix a semisimple $H \in \mathfrak{g}$, and let $\mathfrak{g}_i$ denote the eigenspaces of $ad(H)$. Assume that all the eigenvalues i lie in $\mathbb{Q}$.
- Let $\varphi \in \mathfrak{g}^*_{-2}$ and let $\mathfrak{l} \subset \mathfrak{g}_1$ be a maximal isotropic subspace w.r.t $\omega(X, Y) := \varphi([X, Y])$.
- Define $\mathfrak{v} := \bigoplus_{i>1} \mathfrak{g}_i$, $\mathfrak{n} := \mathfrak{v} \oplus \mathfrak{l}$, $N := \mathrm{Exp}(\mathfrak{n})$
- Define $\mathcal{W}_{H,\varphi} := \mathrm{ind}_N^G \varphi$, and for $\pi \in \mathcal{M}(G)$ let

$$\mathcal{W}_{H,\varphi}(\pi) := \mathrm{Hom}_G(\mathcal{W}_{H,\varphi}, \pi^*) = \mathrm{Hom}_N(\pi, \varphi).$$

Theorem (Moeglin-Waldspurger, 87')

Assume that $\mathbb{F}$ is non-archimedean, and H is tangent to a 1-parameter subgroup. Let $\pi \in \mathcal{M}(G)$. Then

The result of Moeglin and Waldspurger in the p-adic case

- Fix a semisimple $H \in \mathfrak{g}$, and let $\mathfrak{g}_i$ denote the eigenspaces of $ad(H)$. Assume that all the eigenvalues i lie in $\mathbb{Q}$.
- Let $\varphi \in \mathfrak{g}^*_{-2}$ and let $\mathfrak{l} \subset \mathfrak{g}_1$ be a maximal isotropic subspace w.r.t $\omega(X, Y) := \varphi([X, Y])$.
- Define $\mathfrak{v} := \bigoplus_{i>1} \mathfrak{g}_i$, $\mathfrak{n} := \mathfrak{v} \oplus \mathfrak{l}$, $N := \mathrm{Exp}(\mathfrak{n})$
- Define $\mathcal{W}_{H,\varphi} := \mathrm{ind}_N^G \varphi$, and for $\pi \in \mathcal{M}(G)$ let

$$\mathcal{W}_{H,\varphi}(\pi) := \mathrm{Hom}_G(\mathcal{W}_{H,\varphi}, \pi^*) = \mathrm{Hom}_N(\pi, \varphi).$$

Theorem (Moeglin-Waldspurger, 87')

Assume that $\mathbb{F}$ is non-archimedean, and H is tangent to a 1-parameter subgroup. Let $\pi \in \mathcal{M}(G)$. Then

- *If $\mathcal{W}_{H,\varphi}(\pi) \neq 0$ then $\varphi \in \mathrm{WF}(\pi)$.*

The result of Moeglin and Waldspurger in the p-adic case

- Fix a semisimple $H \in \mathfrak{g}$, and let $\mathfrak{g}_i$ denote the eigenspaces of $ad(H)$. Assume that all the eigenvalues i lie in $\mathbb{Q}$.
- Let $\varphi \in \mathfrak{g}^*_{-2}$ and let $\mathfrak{l} \subset \mathfrak{g}_1$ be a maximal isotropic subspace w.r.t $\omega(X, Y) := \varphi([X, Y])$.
- Define $\mathfrak{v} := \bigoplus_{i>1} \mathfrak{g}_i$, $\mathfrak{n} := \mathfrak{v} \oplus \mathfrak{l}$, $N := \mathrm{Exp}(\mathfrak{n})$
- Define $\mathcal{W}_{H,\varphi} := \mathrm{ind}_N^G \varphi$, and for $\pi \in \mathcal{M}(G)$ let

$$\mathcal{W}_{H,\varphi}(\pi) := \mathrm{Hom}_G(\mathcal{W}_{H,\varphi}, \pi^*) = \mathrm{Hom}_N(\pi, \varphi).$$

Theorem (Moeglin-Waldspurger, 87')

Assume that $\mathbb{F}$ is non-archimedean, and H is tangent to a 1-parameter subgroup. Let $\pi \in \mathcal{M}(G)$. Then

- *If $\mathcal{W}_{H,\varphi}(\pi) \neq 0$ then $\varphi \in \mathrm{WF}(\pi)$.*
- *For any (H, φ) with $G\varphi$ open in $\mathrm{WF}(\pi)$,*

$$\dim \mathcal{W}_{H,\varphi}(\pi) = c_\varphi.$$

Examples

- We call (H, φ) a *Whittaker pair*, and $\mathcal{W}_{H,\varphi}$ a *degenerate Whittaker model*. We call them *generalized* if (H, φ) can be completed to an $\mathfrak{sl}_2$-triple, and *principal degenerate* if they come from a regular Whittaker pair of a Levi subgroup. Some examples for $G = \mathrm{GL}_4(\mathbb{F})$:

Examples

- We call (H,φ) a *Whittaker pair*, and $\mathcal{W}_{H,\varphi}$ a *degenerate Whittaker model*. We call them *generalized* if (H,φ) can be completed to an $\mathfrak{sl}_2$-triple, and *principal degenerate* if they come from a regular Whittaker pair of a Levi subgroup. Some examples for $G = \mathrm{GL}_4(\mathbb{F})$:
- $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathfrak{n} = \begin{pmatrix} 0 & \underline{*} & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & \underline{*} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Examples

- We call (H,φ) a *Whittaker pair*, and $\mathcal{W}_{H,\varphi}$ a *degenerate Whittaker model*. We call them *generalized* if (H,φ) can be completed to an $\mathfrak{sl}_2$-triple, and *principal degenerate* if they come from a regular Whittaker pair of a Levi subgroup. Some examples for $G = \mathrm{GL}_4(\mathbb{F})$:
- $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathfrak{n} = \begin{pmatrix} 0 & \underline{*} & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & \underline{*} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
- $$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathfrak{n} = \begin{pmatrix} 0 & \underline{*} & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In both cases $\mathfrak{n} = \mathfrak{v}$.

Some results in the real case

In this slide $\mathbb{F} = \mathbb{R}$.

Theorem (Matumoto 87',92')

- *If* $\mathcal{W}_{H,\varphi}(\pi) \neq 0$ *then* $\varphi \in G_{\mathbb{C}} \cdot \mathrm{WF}(\pi)$.
- *If* φ *is regular nilpotent then* $\mathcal{W}_{H,\varphi}(\pi) \neq 0 \Leftrightarrow \varphi \in \mathrm{WF}(\pi)$.

Some results in the real case

In this slide $\mathbb{F} = \mathbb{R}$.

Theorem (Matumoto 87',92')

- *If* $\mathcal{W}_{H,\varphi}(\pi) \neq 0$ *then* $\varphi \in G_{\mathbb{C}} \cdot \mathrm{WF}(\pi)$.
- *If* φ *is regular nilpotent then* $\mathcal{W}_{H,\varphi}(\pi) \neq 0 \Leftrightarrow \varphi \in \mathrm{WF}(\pi)$.

Theorem (G.-Sahi, 2013)

Let G be quasi-split and (H, φ) be **principal** *degenerate Whittaker pair.*

$$\mathcal{W}_{H,\varphi}(\pi) \neq 0 \Rightarrow \varphi \in \mathrm{WF}(\pi) \Rightarrow \exists g \in F_G \text{ s.t. } \mathcal{W}_{H,g\cdot\varphi}(\pi) \neq 0,$$

where $F_G = \{1\}$ if $G = GL_n(\mathbb{R})$ or if G is a complex group,
and $F_G =$ finite abelian group $Norm_{G_{\mathbb{C}}}(G) / (Z_{G_{\mathbb{C}}} \cdot G)$ otherwise.

Some results in the real case

In this slide $\mathbb{F} = \mathbb{R}$.

Theorem (Matumoto 87',92')

- *If* $\mathcal{W}_{H,\varphi}(\pi) \neq 0$ *then* $\varphi \in G_{\mathbb{C}} \cdot \mathrm{WF}(\pi)$.
- *If* φ *is regular nilpotent then* $\mathcal{W}_{H,\varphi}(\pi) \neq 0 \Leftrightarrow \varphi \in \mathrm{WF}(\pi)$.

Theorem (G.-Sahi, 2013)

Let G *be quasi-split and* (H, φ) *be* **principal** *degenerate Whittaker pair.*

$$\mathcal{W}_{H,\varphi}(\pi) \neq 0 \Rightarrow \varphi \in \mathrm{WF}(\pi) \Rightarrow \exists g \in F_G \text{ s.t. } \mathcal{W}_{H,g\cdot\varphi}(\pi) \neq 0,$$

where $F_G = \{1\}$ *if* $G = GL_n(\mathbb{R})$ *or if* G *is a complex group, and* $F_G =$ *finite abelian group* $Norm_{G_{\mathbb{C}}}(G) / (Z_{G_{\mathbb{C}}} \cdot G)$ *otherwise. Moreover, if* G *is complex classical and* $\pi \in Irr(G)$ *then the set of principal degenerate Whittaker pairs* (H, φ) *with* $\mathcal{W}_{H,\varphi}(\pi) \neq 0$ *determines* $\mathrm{WF}(\pi)$.

Our uniform results

Theorem (Gomez-G.-Sahi, 2015)

- *Let* (H, Φ) *be a Whittaker pair, let* $\varphi \in \overline{G_H}\Phi$. *Then* $\exists \mathcal{W}_{\varphi}^{gen} \twoheadrightarrow \mathcal{W}_{H,\Phi}$.

Our uniform results

Theorem (Gomez-G.-Sahi, 2015)

- *Let* (H, Φ) *be a Whittaker pair, let* $\varphi \in \overline{G_H}\Phi$. *Then* $\exists \mathcal{W}_{\varphi}^{gen} \twoheadrightarrow \mathcal{W}_{H,\Phi}$.
- *For* $G = \mathrm{GL}_n(\mathbb{F}), \pi \in \mathcal{M}(G)$, $\mathcal{W}_{\varphi}^{gen}(\pi) \neq 0 \Leftrightarrow \varphi \in \mathrm{WF}(\pi)$.

Our uniform results

Theorem (Gomez-G.-Sahi, 2015)

- *Let* (H,Φ) *be a Whittaker pair, let* $\varphi \in \overline{G_H}\Phi$*. Then* $\exists \mathcal{W}_{\varphi}^{gen} \twoheadrightarrow \mathcal{W}_{H,\Phi}$.
- *For* $G = \mathrm{GL}_n(\mathbb{F}), \pi \in \mathcal{M}(G)$, $\mathcal{W}_{\varphi}^{gen}(\pi) \neq 0 \Leftrightarrow \varphi \in \mathrm{WF}(\pi)$.

- For p-adic $\mathbb{F}$, and φ in the interior of $\mathrm{WF}(\pi)$ we obtain a functorial isomorphism $\mathcal{W}_{H,\phi}(\pi) \simeq \mathcal{W}_{H',\phi}(\pi)$ for any H, H' tangent to 1-parameter subgroups.

Our uniform results

Theorem (Gomez-G.-Sahi, 2015)

- *Let (H, Φ) be a Whittaker pair, let $\varphi \in \overline{G_H}\Phi$. Then $\exists \mathcal{W}_{\varphi}^{gen} \twoheadrightarrow \mathcal{W}_{H,\Phi}$.*
- *For $G = \mathrm{GL}_n(\mathbb{F}), \pi \in \mathcal{M}(G)$, $\mathcal{W}_{\varphi}^{gen}(\pi) \neq 0 \Leftrightarrow \varphi \in \mathrm{WF}(\pi)$.*

- For p-adic $\mathbb{F}$, and φ in the interior of $\mathrm{WF}(\pi)$ we obtain a functorial isomorphism $\mathcal{W}_{H,\phi}(\pi) \simeq \mathcal{W}_{H',\phi}(\pi)$ for any H, H' tangent to 1-parameter subgroups.
- For $\mathrm{GL}_n(\mathbb{F})$ we also describe $\mathcal{W}_{\varphi}^{gen}(\pi)$ in terms of an analog of Bernstein-Zelevinsky derivatives. This enables us to extend to $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ the results of Moeglin-Waldspurger on the dimension of $\mathcal{W}_{\varphi}^{gen}(\pi)$ and on the exactness of the generalized Whittaker functor.

Our uniform results

Theorem (Gomez-G.-Sahi, 2015)

- *Let* (H,Φ) *be a Whittaker pair, let* $\varphi \in \overline{G_H}\Phi$*. Then* $\exists \mathcal{W}_{\varphi}^{gen} \twoheadrightarrow \mathcal{W}_{H,\Phi}$*.*
- *For* $G = \mathrm{GL}_n(\mathbb{F}), \pi \in \mathcal{M}(G)$, $\mathcal{W}_{\varphi}^{gen}(\pi) \neq 0 \Leftrightarrow \varphi \in \mathrm{WF}(\pi)$.

- For p-adic $\mathbb{F}$, and φ in the interior of $\mathrm{WF}(\pi)$ we obtain a functorial isomorphism $\mathcal{W}_{H,\phi}(\pi) \simeq \mathcal{W}_{H',\phi}(\pi)$ for any H, H' tangent to 1-parameter subgroups.
- For $\mathrm{GL}_n(\mathbb{F})$ we also describe $\mathcal{W}_{\varphi}^{gen}(\pi)$ in terms of an analog of Bernstein-Zelevinsky derivatives. This enables us to extend to $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ the results of Moeglin-Waldspurger on the dimension of $\mathcal{W}_{\varphi}^{gen}(\pi)$ and on the exactness of the generalized Whittaker functor.
- We also have a global (adelic) analogue.

Basic lemma

Define anti-symmetric form ω on $\mathfrak{g}$ by $\omega(X, Y) := \varphi([X, Y])$.

Lemma (Following Ginzburg-Soudry-Rallis, Jiang-Liu, Lapid-Mao)

Let $\mathfrak{n}, \mathfrak{m} \subset \mathfrak{g}$ *be nilpotent subalgebras such that* $[\mathfrak{n}, \mathfrak{m}] \subset \mathfrak{n} \cap \mathfrak{m}$, $\omega|_{\mathfrak{n}} = 0$, $\omega|_{\mathfrak{m}} = 0$ *and the radical of* $\omega|_{\mathfrak{n}+\mathfrak{m}}$ *is* $\mathfrak{n} \cap \mathfrak{m}$. *Then* $\mathfrak{n} + \mathfrak{m}$ *is a nilpotent Lie algebra and*

$$\operatorname{ind}_{\operatorname{Exp}(\mathfrak{n})}^{\operatorname{Exp}(\mathfrak{n}+\mathfrak{m})} \varphi \simeq \operatorname{ind}_{\operatorname{Exp}(\mathfrak{m})}^{\operatorname{Exp}(\mathfrak{n}+\mathfrak{m})} \varphi.$$

Basic lemma

Define anti-symmetric form ω on $\mathfrak{g}$ by $\omega(X, Y) := \varphi([X, Y])$.

Lemma (Following Ginzburg-Soudry-Rallis, Jiang-Liu, Lapid-Mao)

Let $\mathfrak{n}, \mathfrak{m} \subset \mathfrak{g}$ *be nilpotent subalgebras such that* $[\mathfrak{n}, \mathfrak{m}] \subset \mathfrak{n} \cap \mathfrak{m}$, $\omega|_{\mathfrak{n}} = 0$, $\omega|_{\mathfrak{m}} = 0$ *and the radical of* $\omega|_{\mathfrak{n}+\mathfrak{m}}$ *is* $\mathfrak{n} \cap \mathfrak{m}$. *Then* $\mathfrak{n} + \mathfrak{m}$ *is a nilpotent Lie algebra and*

$$\operatorname{ind}_{\operatorname{Exp}(\mathfrak{n})}^{\operatorname{Exp}(\mathfrak{n}+\mathfrak{m})} \varphi \simeq \operatorname{ind}_{\operatorname{Exp}(\mathfrak{m})}^{\operatorname{Exp}(\mathfrak{n}+\mathfrak{m})} \varphi.$$

Proof.

$\mathfrak{k} := \mathfrak{n} \cap \mathfrak{m} \cap \operatorname{Ker}(\varphi)$. Then $\operatorname{Exp}(\mathfrak{n} + \mathfrak{m}) / \operatorname{Exp}(\mathfrak{k})$ is Heisenberg group corresponding to $(\mathfrak{n} + \mathfrak{m}) / (\mathfrak{n} \cap \mathfrak{m})$. The subspaces $\mathfrak{m} / (\mathfrak{n} \cap \mathfrak{m})$, $\mathfrak{n} / (\mathfrak{n} \cap \mathfrak{m})$ are Lagrangian, thus

$$\operatorname{ind}_{\operatorname{Exp}(\mathfrak{n}) / \operatorname{Exp}(\mathfrak{k})}^{\operatorname{Exp}(\mathfrak{n}+\mathfrak{m}) / \operatorname{Exp}(\mathfrak{k})} \varphi \simeq \operatorname{ind}_{\operatorname{Exp}(\mathfrak{m}) / \operatorname{Exp}(\mathfrak{k})}^{\operatorname{Exp}(\mathfrak{n}+\mathfrak{m}) / \operatorname{Exp}(\mathfrak{k})} \varphi,$$

since both $\simeq$ oscillator representation of $\operatorname{Exp}(\mathfrak{n} + \mathfrak{m}) / \operatorname{Exp}(\mathfrak{k})$ with central character φ. □

Example 1

Let $G := \mathrm{GL}(4, \mathbb{F})$ and define φ by $\varphi(X) := \mathrm{tr}(X(E_{21} + E_{43}))$.
Let $\Phi := \varphi$, $H := \mathrm{diag}(3, 1, -1, -3)$, $h = \mathrm{diag}(1, -1, 1, -1)$,
$Z = H - h = \mathrm{diag}(2, 2, -2, -2), H_t := h + tZ$.

Example 1

Let $G := \mathrm{GL}(4, \mathbb{F})$ and define φ by $\varphi(X) := \mathrm{tr}(X(E_{21} + E_{43}))$.
Let $\Phi := \varphi$, $H := \mathrm{diag}(3, 1, -1, -3)$, $h = \mathrm{diag}(1, -1, 1, -1)$,
$Z = H - h = \mathrm{diag}(2, 2, -2, -2), H_t := h + tZ$.
Then $\mathfrak{n}_0 \subset \mathfrak{n}_{1/4} \sim \mathfrak{n}'_{1/4} \subset \mathfrak{n}_{3/4} = \mathfrak{n}_1$:

$$\begin{pmatrix} 0 & - & 0 & - \\ 0 & 0 & 0 & 0 \\ 0 & - & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & - & a & - \\ 0 & 0 & 0 & a \\ 0 & * & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & - & * & - \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\subset \begin{pmatrix} 0 & - & - & - \\ 0 & 0 & * & - \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & - & - & - \\ 0 & 0 & - & - \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Both $*$ and $-$ denote arbitrary elements. $-$ denotes the entries in $\mathfrak{v}_t$ and $*$ those in $\mathfrak{l}_t$. a denotes equal elements.

Example 2

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Let $H := \mathrm{diag}(0, -2, 2, 0)$, $h := (1, -1, 1, -1)$, $Z = H - h = \mathrm{diag}(-1, -1, 1, 1), H_t := h + tZ$. Then $\mathfrak{n}_0 = \mathfrak{n}_{1/2} \sim \mathfrak{n}'_{1/2} \subset \mathfrak{n}_1$

Example 2

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Let $H := \operatorname{diag}(0, -2, 2, 0)$, $h := (1, -1, 1, -1)$, $Z = H - h = \operatorname{diag}(-1, -1, 1, 1)$, $H_t := h + tZ$. Then $\mathfrak{n}_0 = \mathfrak{n}_{1/2} \sim \mathfrak{n}'_{1/2} \subset \mathfrak{n}_1$

$$\begin{pmatrix} 0 & * & 0 & - \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & * & 0 & * \\ 0 & -a & 0 & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & 0 & * \\ 0 & * & 0 & 0 \end{pmatrix}.$$

Prederivatives

Definition

$$P_n = \left\{ \begin{pmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\} \subset G_n := GL_n(F)$$

- Over p-adic $\mathbb{F}$, the category $\mathrm{Rep}^{\infty}(P_n)$ of smooth P_n-rep-ns is equivalent to the category of G_{n-1}-equivariant sheaves on $V_n^* := (F^{n-1})^*$.

Prederivatives

Definition

$$P_n = \left\{ \begin{pmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\} \subset G_n := GL_n(F)$$

- Over p-adic $\mathbb{F}$, the category $\mathrm{Rep}^\infty(P_n)$ of smooth P_n-rep-ns is equivalent to the category of G_{n-1}-equivariant sheaves on $V_n^* := (F^{n-1})^*$.
- Define $\Phi : \mathrm{Rep}^\infty(P_n) \to \mathrm{Rep}^\infty(P_{n-1})$ by $\Phi(\pi) := \pi_{V_n,\psi}$, and $E^k : \mathcal{M}(G_n) \to \mathrm{Rep}^\infty(G_{n-k})$ by $E^k(\pi) := \Phi^{k-1}(\pi|_{P_n})|_{G_{n-k}}$.

Prederivatives

Definition

$$P_n = \left\{ \begin{pmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\} \subset G_n := GL_n(F)$$

- Over p-adic $\mathbb{F}$, the category $\text{Rep}^\infty(P_n)$ of smooth P_n-rep-ns is equivalent to the category of G_{n-1}-equivariant sheaves on $V_n^* := (F^{n-1})^*$.
- Define $\Phi : \text{Rep}^\infty(P_n) \to \text{Rep}^\infty(P_{n-1})$ by $\Phi(\pi) := \pi_{V_n,\psi}$, and $E^k : \mathcal{M}(G_n) \to \text{Rep}^\infty(G_{n-k})$ by $E^k(\pi) := \Phi^{k-1}(\pi|_{P_n})|_{G_{n-k}}$.
- For $\pi \in \mathcal{M}(G_n)$, $\text{depth}(\pi) :=$ size of max. Jordan block in $\text{WF}(\pi)$.

Theorem (Aizenbud - G. - Sahi, 2012)

Let $\mathcal{M}^d(G_n) \subset \mathcal{M}(G_n)$ *be the subcategory of rep-s of depth* $\leq d$. *Then*

- E^d *defines a non-zero exact functor* $\mathcal{M}^d(G_n) \to \mathcal{M}(G_{n-d})$.

Theorem (Aizenbud - G. - Sahi, 2012)

Let $\mathcal{M}^d(G_n) \subset \mathcal{M}(G_n)$ *be the subcategory of rep-s of depth* $\leq d$. *Then*

- E^d *defines a non-zero exact functor* $\mathcal{M}^d(G_n) \to \mathcal{M}(G_{n-d})$.
- $E^k|_{\mathcal{M}^d(G_n)} = 0$ *for any* $k > d$.

Theorem (Aizenbud - G. - Sahi, 2012)

Let $\mathcal{M}^d(G_n) \subset \mathcal{M}(G_n)$ be the subcategory of rep-s of depth $\leq d$. Then

- *E^d defines a non-zero exact functor $\mathcal{M}^d(G_n) \to \mathcal{M}(G_n - d)$.*
- *$E^k|_{\mathcal{M}^d(G_n)} = 0$ for any $k > d$.*
- *If* $\mathrm{WF}(\pi) = \overline{G_n J_{(d,d_2,\dots,d_k)}}$ *then* $\mathrm{WF}(E^d(\pi)) = \overline{G_n J_{(d_2,\dots,d_k)}}$

Theorem (Aizenbud - G. - Sahi, 2012)

Let $\mathcal{M}^d(G_n) \subset \mathcal{M}(G_n)$ *be the subcategory of rep-s of depth* $\leq d$. *Then*

- E^d *defines a non-zero exact functor* $\mathcal{M}^d(G_n) \to \mathcal{M}(G_{n-d})$.
- $E^k|_{\mathcal{M}^d(G_n)} = 0$ *for any* $k > d$.
- *If* $\mathrm{WF}(\pi) = \overline{G_n J_{(d,d_2,\ldots,d_k)}}$ *then* $\mathrm{WF}(E^d(\pi)) = \overline{G_n J_{(d_2,\ldots,d_k)}}$
- *Let* $n = n_1 + \ldots + n_d$ *and let* χ_i *be characters of* G_{n_i}. *Let* $\pi = \chi_1 \times \ldots \times \chi_d \in \mathcal{M}^d(G_n)$ *denote the corresponding degenerate principal series representation. Then* $\mathrm{depth}(\pi) = d$ *and* $E^d(\pi) \cong (\chi_1)|_{G_{n_1-1}} \times \ldots \times (\chi_d)|_{G_{n_d-1}}$.

Theorem (Aizenbud - G. - Sahi, 2012)

Let $\mathcal{M}^d(G_n) \subset \mathcal{M}(G_n)$ *be the subcategory of rep-s of depth* $\leq d$*. Then*

- E^d *defines a non-zero exact functor* $\mathcal{M}^d(G_n) \to \mathcal{M}(G_{n-d})$*.*
- $E^k|_{\mathcal{M}^d(G_n)} = 0$ *for any* $k > d$*.*
- *If* $\mathrm{WF}(\pi) = \overline{G_n J_{(d,d_2,\ldots,d_k)}}$ *then* $\mathrm{WF}(E^d(\pi)) = \overline{G_n J_{(d_2,\ldots,d_k)}}$
- *Let* $n = n_1 + \ldots + n_d$ *and let* χ_i *be characters of* G_{n_i}*. Let* $\pi = \chi_1 \times \ldots \times \chi_d \in \mathcal{M}^d(G_n)$ *denote the corresponding degenerate principal series representation. Then* $depth(\pi) = d$ *and* $E^d(\pi) \cong (\chi_1)|_{G_{n_1 - 1}} \times \ldots \times (\chi_d)|_{G_{n_d - 1}}$*.*
- *For a unitarizable irrep* π *of depth* d*,* $E^d(\pi) = A(\pi)$*.*

Theorem (Aizenbud - G. - Sahi, 2012)

Let $\mathcal{M}^d(G_n) \subset \mathcal{M}(G_n)$ *be the subcategory of rep-s of depth* $\leq d$. *Then*

- E^d *defines a non-zero exact functor* $\mathcal{M}^d(G_n) \to \mathcal{M}(G_{n-d})$.
- $E^k|_{\mathcal{M}^d(G_n)} = 0$ *for any* $k > d$.
- *If* $\mathrm{WF}(\pi) = \overline{G_n J_{(d,d_2,\ldots,d_k)}}$ *then* $\mathrm{WF}(E^d(\pi)) = \overline{G_n J_{(d_2,\ldots,d_k)}}$
- *Let* $n = n_1 + \ldots + n_d$ *and let* χ_i *be characters of* G_{n_i}. *Let* $\pi = \chi_1 \times \ldots \times \chi_d \in \mathcal{M}^d(G_n)$ *denote the corresponding degenerate principal series representation. Then depth*$(\pi) = d$ *and* $E^d(\pi) \cong (\chi_1)|_{G_{n_1-1}} \times \ldots \times (\chi_d)|_{G_{n_d-1}}$.
- *For a unitarizable irrep* π *of depth* d, $E^d(\pi) = A(\pi)$.

Theorem (Aizenbud - G. - Sahi, 2012)

Let $\mathcal{M}^d(G_n) \subset \mathcal{M}(G_n)$ be the subcategory of rep-s of depth $\leq d$. Then

- *E^d defines a non-zero exact functor $\mathcal{M}^d(G_n) \to \mathcal{M}(G_{n-d})$.*
- *$E^k|_{\mathcal{M}^d(G_n)} = 0$ for any $k > d$.*
- *If $\mathrm{WF}(\pi) = \overline{G_n J_{(d,d_2,\ldots,d_k)}}$ then $\mathrm{WF}(E^d(\pi)) = \overline{G_n J_{(d_2,\ldots,d_k)}}$*
- *Let $n = n_1 + \ldots + n_d$ and let χ_i be characters of G_{n_i}. Let $\pi = \chi_1 \times \ldots \times \chi_d \in \mathcal{M}^d(G_n)$ denote the corresponding degenerate principal series representation. Then depth$(\pi) = d$ and $E^d(\pi) \cong (\chi_1)|_{G_{n_1-1}} \times \ldots \times (\chi_d)|_{G_{n_d-1}}$.*
- *For a unitarizable irrep π of depth d, $E^d(\pi) = A(\pi)$.*

Theorem (Gomez - G. - Sahi, 2015)

Let $\lambda = (n_1, \ldots, n_k)$ be a partition of n and J_λ be the corresponding nilpotent Jordan matrix. Then $\mathcal{W}^{gen}_{J_\lambda}(\pi) \cong E^{n_k}(\cdots(E^{n_1}(\pi))\cdots)^*$.

Example for $\lambda = (3, 2, 1)$

$$\left\{ \begin{pmatrix} 0 & 0 & * & 0 & * & * \\ 0 & 0 & \underline{*} & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & \underline{*} & * \\ 0 & 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

Adelic setting

- K - number field, $\mathbf{G}$ defined over K, $G := \mathbf{G}(\mathbb{A}_K)$.

Adelic setting

- K - number field, $\mathbf{G}$ defined over K, $G := \mathbf{G}(\mathbb{A}_K)$.
- (H, Φ) - Whittaker pair, N defined as before.

Adelic setting

- K - number field, $\mathbf{G}$ defined over K, $G := \mathbf{G}(\mathbb{A}_K)$.
- (H, Φ) - Whittaker pair, N defined as before.
- For any automorphic function f, let a new function $\mathcal{WF}_{H,\varphi}(f)$ be

$$\mathcal{WF}_{H,\varphi}(f)(x) := \int_{N(\mathbb{A})/N(K)} \Phi(n)^{-1} f(xn) dn.$$

Adelic setting

- K - number field, $\mathbf{G}$ defined over K, $G := \mathbf{G}(\mathbb{A}_K)$.
- (H, Φ) - Whittaker pair, N defined as before.
- For any automorphic function f, let a new function $\mathcal{WF}_{H,\varphi}(f)$ be

$$\mathcal{WF}_{H,\varphi}(f)(x) := \int_{N(\mathbb{A})/N(K)} \Phi(n)^{-1} f(xn) dn.$$

Adelic setting

- K - number field, $\mathbf{G}$ defined over K, $G := \mathbf{G}(\mathbb{A}_K)$.
- (H, Φ) - Whittaker pair, N defined as before.
- For any automorphic function f, let a new function $\mathcal{WF}_{H,\varphi}(f)$ be

$$\mathcal{WF}_{H,\varphi}(f)(x) := \int_{N(\mathbb{A})/N(K)} \Phi(n)^{-1} f(xn)dn.$$

Theorem

Let $\varphi \in \overline{G\Phi}$ and suppose that either $G = \mathrm{GL}_n(\mathbb{A})$ or $\varphi \in \overline{G_H\Phi}$. Then $\mathcal{WF}_{H,\Phi}$ is obtained from $\mathcal{WF}_{\varphi}^{gen}$ by an integral transform. In particular, for any automorphic representation π we have

$$\mathcal{WF}_{(H,\Phi)}(\pi) \neq 0 \Rightarrow \mathcal{WF}_{\varphi}^{gen}(\pi) \neq 0.$$

Adelic setting

- K - number field, $\mathbf{G}$ defined over K, $G := \mathbf{G}(\mathbb{A}_K)$.
- (H, Φ) - Whittaker pair, N defined as before.
- For any automorphic function f, let a new function $\mathcal{WF}_{H,\varphi}(f)$ be

$$\mathcal{WF}_{H,\varphi}(f)(x) := \int_{N(\mathbb{A})/N(K)} \Phi(n)^{-1} f(xn) dn.$$

Theorem

Let $\varphi \in \overline{G\Phi}$ and suppose that either $G = \mathrm{GL}_n(\mathbb{A})$ or $\varphi \in \overline{G_H\Phi}$. Then $\mathcal{WF}_{H,\Phi}$ is obtained from $\mathcal{WF}_{\varphi}^{gen}$ by an integral transform. In particular, for any automorphic representation π we have

$$\mathcal{WF}_{(H,\Phi)}(\pi) \neq 0 \Rightarrow \mathcal{WF}_{\varphi}^{gen}(\pi) \neq 0.$$

Thank you for your attention!