Generalized and degenerate Whittaker models for representations of reductive groups

Dmitry Gourevitch
Weizmann Institute of Science, Israel
http://www.wisdom.weizmann.ac.il/~dimagur

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Degenerate Whittaker space

- $\mathbb{F}$: local field of char.0, G: reductive group over $\mathbb{F}$, $\mathfrak{g} := Lie(G)$.
- $\mathcal{M}(G)$: smooth (Frechet, moderate growth) representations.
- Fix a semisimple $H \in \mathfrak{g}$, and let $\mathfrak{g}_i$ denote the eigenspaces of $ad(H)$. Assume that all the eigenvalues i lie in $\mathbb{Q}$.
- Let $\varphi \in \mathfrak{g}^*_{-2}$ and let $\mathfrak{l} \subset \mathfrak{g}_1$ be a maximal isotropic subspace w.r.t $\omega(X, Y) := \varphi([X, Y])$.
- Define $\mathfrak{v} := \bigoplus_{i>1} \mathfrak{g}_i$, $\mathfrak{n} := \mathfrak{v} \oplus \mathfrak{l}$, $N := \mathrm{Exp}(\mathfrak{n})$,
- Fix a non-trivial unitary additive character $\psi : \mathbb{F} \to \mathbb{C}$ and define $\chi_\varphi : N \to \mathbb{C}$ by $\chi_\varphi(\mathrm{Exp}\, X) := \psi(\varphi(X))$.
- For $\pi \in \mathcal{M}(G)$ let $\mathcal{W}_{H,\varphi}(\pi) = \mathrm{Hom}_N(\pi, \chi_\varphi)$.
- We call (H, φ) a *Whittaker pair*, and $\mathcal{W}_{H,\varphi}(\pi)$ a *degenerate Whittaker space*. We call them *generalized* or *neutral* if (H, φ) can be completed to an $\mathfrak{sl}_2$-triple. The generalized Whittaker space depends only on the coadjoint orbit of φ.
- $\mathrm{WS}(\pi)$:=maximal orbits with $\mathcal{W}_{\mathcal{O}}(\pi) \neq 0$ (w.r.t. the closure order)

Examples

Some examples for $G = \mathrm{GL}_4(\mathbb{F})$:

- $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathfrak{n} = \begin{pmatrix} 0 & \underline{*} & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & \underline{*} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- $$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathfrak{n} = \begin{pmatrix} 0 & \underline{*} & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In both cases $\mathfrak{n} = \mathfrak{v}$.

Theorem (Gomez-G.-Sahi, 2016 following Moeglin-Waldspurger)

Let (H, φ) be a Whittaker pair, and $\pi \in \mathcal{M}(G)$.

- *We have a natural functorial embedding $\nu : \mathcal{W}_{H,\Phi}(\pi) \hookrightarrow \mathcal{W}_{\varphi}(\pi)$.*
- *If $G\varphi \in \mathrm{WS}(\pi)$ then $\nu \neq 0$.*
- *If $G\varphi \in \mathrm{WS}(\pi)$ and F is p-adic then ν is an isomorphism.*

We also have a global (adelic) analogue.

Corollary

Let F be p-adic, π be cuspidal, $\mathcal{O} \in \mathrm{WS}(\pi)$ and $L \subset G$ a Levi subgroup with $\mathcal{O} \cap \mathfrak{l}^* \neq \emptyset$. Then $L = G$.

Proof.

Let $(e, h, f) \in \mathfrak{l}$ be an $\mathfrak{sl}_2$-triple such that $f \in \mathcal{O}$. Let $Z \in \mathfrak{g}$ be a (rational) semi-simple element s.t. $\mathfrak{l} = \mathfrak{g}^Z$. Let $T >> 0 \in \mathbb{Z}$ and let $H := h + TZ$. Then $\mathcal{W}_{H,\varphi}(\pi)$ embeds into the dual of the Jacquet module $r_L(\pi)$. Since $\mathcal{W}_{H,\varphi}(\pi) \neq 0$ and π is cuspidal, $L = G$. □

Example for our theorem

Let $G := \mathrm{GL}(4, \mathbb{F})$ and define φ by $\varphi(X) := \mathrm{tr}(X(E_{21} + E_{43}))$.
Let $\Phi := \varphi$, $H := \mathrm{diag}(3, 1, -1, -3)$, $h = \mathrm{diag}(1, -1, 1, -1)$,
$Z = H - h = \mathrm{diag}(2, 2, -2, -2)$, $H_t := h + tZ$.
Then $\mathfrak{n}_0 \subset \mathfrak{n}_{1/4} \sim \mathfrak{n}'_{1/4} \subset \mathfrak{n}_{3/4} = \mathfrak{n}_1$:

$$\begin{pmatrix} 0 & - & 0 & - \\ 0 & 0 & 0 & 0 \\ 0 & - & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & - & a & - \\ 0 & 0 & 0 & a \\ 0 & * & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & - & * & - \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\subset \begin{pmatrix} 0 & - & - & - \\ 0 & 0 & * & - \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & - & - & - \\ 0 & 0 & - & - \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Both $*$ and $-$ denote arbitrary elements. $-$ denotes the entries in $\mathfrak{v}_t$ and $*$ those in $\mathfrak{l}_t$. a denotes equal elements.

Adelic setting

- K - number field, $\mathbf{G}$ defined over K, $G := \mathbf{G}(\mathbb{A}_K)$, $\Gamma := \mathbf{G}(K)$
- $(H, \varphi) \in \mathfrak{g}(K) \times \mathfrak{g}^*(K)$ - Whittaker pair, $N \subset G$ defined as before.
- For any automorphic function $f \in C^\infty(G/\Gamma)$, let a new function $\mathcal{WF}_{H,\varphi}(f)$ be

$$\mathcal{WF}_{H,\varphi}(f)(x) := \int_{N(\mathbb{A})/N(K)} \chi_\varphi(n)^{-1} f(xn)dn.$$

Theorem

(i) *$\mathcal{WF}_{H,\varphi}(f)$ is obtained from $\mathcal{WF}_\varphi(f)$ by an integral transform.*

(ii) *If $\varphi \in \mathrm{WS}(f)$ then $\mathcal{WF}_\varphi$ is obtained from $\mathcal{WF}_{H,\varphi}$ by another integral transform.*

Wave front set and wave-front cycle

Let $\pi \in \mathcal{M}(G)$ be admissible and finitely generated.

Theorem (Howe, Harish-Chandra, Barbasch-Vogan, 70s)

Near $e \in G$, the character distribution (asymptotically) equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

$$\exp^*(\chi_\pi) \approx \sum c_{\mathcal{O}} \mathcal{F}(\mu_{\mathcal{O}})$$

- Let $\mathcal{N} \subset \mathfrak{g}^*$ denote the nilpotent cone.
- $\mathrm{WF}(\pi) := \cup\{\overline{\mathcal{O}} \mid c_{\mathcal{O}} \neq 0\} \subset \mathcal{N}$.
- $\mathrm{WF}^{\max}(\pi) :=$ union of maximal orbits in $\mathrm{WF}(\pi)$.

Theorem (Moeglin-Waldspurger, 87')

Let $\mathbb{F}$ be p-adic and let (H, φ) be a Whittaker pair.

- *If $\mathcal{W}_{H,\varphi}(\pi) \neq 0$ then $\varphi \in \mathrm{WF}(\pi)$.*
- *If $\varphi \in \mathrm{WF}^{\max}(\pi)$ then $\dim \mathcal{W}_{H,\varphi}(\pi) = c_\varphi$.*

Some extensions of the Moeglin-Waldspurger theorem

Theorem (Matumoto 87',92')

- *If* $\mathcal{W}_{H,\varphi}(\pi) \neq 0$ *then* φ *lies in the Zariski closure of* $\mathrm{WF}(\pi)$.
- *If* φ *is regular nilpotent then* $\mathcal{W}_{H,\varphi}(\pi) \neq 0 \Leftrightarrow \varphi \in \mathrm{WF}(\pi)$.

Theorem (Gomez-G.-Sahi 2016)

Let $G = \mathrm{GL}(n, \mathbb{F})$. *Then*

(i) $\varphi \in \mathrm{WF}(\pi) \Leftrightarrow \mathcal{W}_{\varphi}(\pi) \neq 0$

(ii) *If* $\varphi \in \mathrm{WF}^{max}(\pi)$ *and* π *is irreducible & unitary then* $\dim \mathcal{W}_{H,\varphi}(\pi) = 1$.

Theorem (G.-Sahi-Sayag 2016)

Let $P \subset G$ *be a parabolic subgroup and* $\mathfrak{p}$ *be its Lie algebra. Let* (H, φ) *be a Whittaker pair with* $\varphi \in \mathfrak{p}^{\perp}$. *Let* $\sigma \in \mathcal{M}(P)$ *with* $\dim \sigma < \infty$ *and let* $\pi := Ind_P^G(\sigma)$. *Then* $\mathcal{W}_{H,\varphi}(\pi) \neq 0$.
In particular, $\mathrm{WS}(\pi)$ *consists of the Richardson orbits of* $\mathfrak{p}$.