Generalized and degenerate Whittaker models for representations of reductive groups

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- We call (H, φ) a *Whittaker pair*, and $\mathcal{W}_{H,\varphi}(\pi)$ a *degenerate Whittaker space*. We call them *generalized* or *neutral* if (H, φ) can be completed to an $\mathfrak{sl}_2$-triple. The generalized Whittaker space depends only on the coadjoint orbit of φ.

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- $\operatorname{WS}(\pi)$:=maximal orbits with $\mathcal{W}_{\mathcal{O}}(\pi) \neq 0$ (w.r.t. the closure order)

Examples

Some examples for $G = \mathrm{GL}_4(\mathbb{F})$:

- $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathfrak{n} = \begin{pmatrix} 0 & \underline{*} & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & \underline{*} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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- $$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathfrak{n} = \begin{pmatrix} 0 & \underline{*} & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In both cases $\mathfrak{n} = \mathfrak{v}$.

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Corollary

Let F be p-adic, π be cuspidal, $\mathcal{O} \in \mathrm{WS}(\pi)$ and $L \subset G$ a Levi subgroup with $\mathcal{O} \cap \mathfrak{l}^* \neq \emptyset$. Then $L = G$.

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Proof.

Let $(e, h, f) \in \mathfrak{l}$ be an $\mathfrak{sl}_2$-triple such that $f \in \mathcal{O}$. Let $Z \in \mathfrak{g}$ be a (rational) semi-simple element s.t. $\mathfrak{l} = \mathfrak{g}^Z$. Let $T >> 0 \in \mathbb{Z}$ and let $H := h + TZ$. Then $\mathcal{W}_{H,\varphi}(\pi)$ embeds into the dual of the Jacquet module $r_L(\pi)$. Since $\mathcal{W}_{H,\varphi}(\pi) \neq 0$ and π is cuspidal, $L = G$. □

Example for our theorem

Let $G := \mathrm{GL}(4, \mathbb{F})$ and define φ by $\varphi(X) := \mathrm{tr}(X(E_{21} + E_{43}))$.
Let $\Phi := \varphi$, $H := \mathrm{diag}(3, 1, -1, -3)$, $h = \mathrm{diag}(1, -1, 1, -1)$,
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Then $\mathfrak{n}_0 \subset \mathfrak{n}_{1/4} \sim \mathfrak{n}'_{1/4} \subset \mathfrak{n}_{3/4} = \mathfrak{n}_1$:

$$\begin{pmatrix} 0 & - & 0 & - \\ 0 & 0 & 0 & 0 \\ 0 & - & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & - & a & - \\ 0 & 0 & 0 & a \\ 0 & * & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & - & * & - \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Both $*$ and $-$ denote arbitrary elements. $-$ denotes the entries in $\mathfrak{v}_t$ and $*$ those in $\mathfrak{l}_t$. a denotes equal elements.

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Theorem

(i) *$\mathcal{WF}_{H,\varphi}(f)$ is obtained from $\mathcal{WF}_\varphi(f)$ by an integral transform.*

(ii) *If $\varphi \in \mathrm{WS}(f)$ then $\mathcal{WF}_\varphi$ is obtained from $\mathcal{WF}_{H,\varphi}$ by another integral transform.*

Wave front set and wave-front cycle

Let $\pi \in \mathcal{M}(G)$ be admissible and finitely generated.

Theorem (Howe, Harish-Chandra, Barbasch-Vogan, 70s)

Near $e \in G$, the character distribution (asymptotically) equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

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Some extensions of the Moeglin-Waldspurger theorem

Theorem (Matumoto 87',92')

- *If* $\mathcal{W}_{H,\varphi}(\pi) \neq 0$ *then* φ *lies in the Zariski closure of* $\mathrm{WF}(\pi)$.
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In particular, $\mathrm{WS}(\pi)$ *consists of the Richardson orbits of* $\mathfrak{p}$.