Relations between Fourier coefficients of automorphic forms with applications to vanishing and Eulerianity

Dmitry Gourevitch
Weizmann Institute of Science, Israel
http://www.wisdom.weizmann.ac.il/~dimagur
Number Theory / Representation Theory Seminar, University of Wisconsin - Madison
j.w. R. Gomez, H. P. A. Gustafsson, A. Kleinschmidt, D. Persson, and S. Sahi
Following Piatetski-Shapiro–Shalika, Jian-Shu Li, Ginzburg–Rallis–Soudry, Moeglin-Waldspurger, Jiang–Liu–Savin, Ahlen, Hundley–Sayag, Shen, Green-Miller-Vanhove, Kazhdan–Polishchuk, Bossard–Pioline

Definitions

- $\mathbb{K}$: number field, $\mathbb{A} := \mathbb{A}_{\mathbb{K}}$, $\mathbf{G}$: reductive group over $\mathbb{K}$, $\Gamma := \mathbf{G}(\mathbb{K})$, $G := \mathbf{G}(\mathbb{A})$, $\mathfrak{g} := Lie(\Gamma)$.

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- Let $[N] := (\Gamma \cap N)\backslash N$. For automorphic form η on G, define *Fourier coefficient*

$$\mathcal{F}_{H,f}[\eta](g) := \int_{[N]} \eta(ng)\chi_f(n)^{-1}dn.$$

Two central cases of Fourier coefficients

$[H, f] = -2f$, $\mathfrak{n} = (\mathfrak{g}_1 \cap \mathfrak{g}^f) \oplus \bigoplus_{i>1} \mathfrak{g}_i$, $N = \mathrm{Exp}(\mathfrak{n})(\mathbb{A})$,

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Coefficients that are both neutral and Whittaker are Eulerian by local uniqueness of Whittaker models.

Examples of Fourier coefficients

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Lemma (Root exchange, cf. Ginzburg-Rallis-Soudry)

1. $\mathcal{F}_{H,f}[\eta](g) = \sum_{\gamma \in (U/I^\perp)(\mathbb{K})} \mathcal{F}^I_{H,f}[\eta](\gamma g)$
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Cf. θ, Stone-von-Neumann thm, Poisson summation formula.

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Theorem

Let $(H, f) \succ (S, f)$. Then

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2. *If* $\Gamma f \in \mathrm{WO}^{\max}(\eta)$ *and* $\mathfrak{g}^H_1 = \mathfrak{g}^S_1 = 0$ *let* $\mathfrak{v} := \mathfrak{g}^H_{>1} \cap \mathfrak{g}^S_{<1}$. *Then*

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Corollary

1. *If* η *is cuspidal then any* $\mathcal{O} \in \mathrm{WO}^{\max}(\eta)$ *is* $\mathbb{K}$*-distinguished.*
   In particular, $\mathcal{O}$ *is totally even for* $G = \mathrm{Sp}_{2n}$, *totally odd for* $G = SO(V)$, *not minimal for* $\mathrm{rk} G > 1$,
   and not next-to-minimal for $\mathrm{rk} G > 2$, $G \neq F_4$.
2. *Lower bounds for partitions of* $\mathcal{O} \in \mathrm{WO}^{\max}(\eta)$ *with cuspidal* η*:*
   2^n *for* Sp_{2n}, $3^n 1^n$ *for* $SO(2n,2n)$, $53^{n-1}1^n$ *for* $SO(2n+1,2n+1)$,
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Proof of (i).

Let $\mathfrak{l} \subset \mathfrak{g}$ be Levi subalgebra intersecting $\mathcal{O}$. Let $(e, h, f) \in \mathfrak{l}$ be an $\mathfrak{sl}_2$-triple with $f \in \mathcal{O}$. Let $Z \in \mathfrak{g}$ be a (rational) semi-simple element s.t. $\mathfrak{l} = \mathfrak{g}^Z$. Let $T >> 0 \in \mathbb{Z}$ and let $H := h + TZ$.
Then $\mathcal{F}_{H,f}(\eta) = \mathcal{F}_{H,f}(c_L(\eta))$, where $c_L(\eta)$ denotes the constant term.
Since $\mathcal{F}_{H,f}(\eta) \neq 0$ by the theorem and η is cuspidal, $L = G$. □

Example for the proof of the Theorem

$G := \mathrm{GL}(4, \mathbb{A})$, $f := E_{21} + E_{43}$, $H := \mathrm{diag}(3, 1, -1, -3)$,
$h = \mathrm{diag}(1, -1, 1, -1)$, $Z = H - h = \mathrm{diag}(2, 2, -2, -2)$, $H_t := h + tZ$.

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Then $\mathfrak{n}_0 \subset \mathfrak{n}_{1/4} \oplus \mathfrak{i} \sim \mathfrak{n}_{1/4} \oplus \mathfrak{j} \subset \mathfrak{n}_{3/4} = \mathfrak{n}_1$:

$$\begin{pmatrix} 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & * & a & * \\ 0 & 0 & 0 & a \\ 0 & - & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & * & - & * \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\subset \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & - & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Both $*$ and $-$ denote arbitrary elements. $*$ denotes the entries in $\mathfrak{g}^{H_t}_{>1}$ and $-$ those in $\mathfrak{g}^{H_t}_1$. a denotes equal elements in $\mathfrak{g}^{H_t}_1 \cap \mathfrak{g}^f$.

Corollary (Hidden symmetry)

Let η be an automorphic form on G, and let (H, f) be a Whittaker pair with $\Gamma f \in \mathrm{WO}^{\max}(\eta)$. Then any unipotent element u of the centralizer of the pair (H, f) in G acts trivially on the Fourier coefficient $\mathcal{F}_{H,f}[\eta]$ using the left regular action.

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Example: $G = GL_4(\mathbb{A}), f = E_{31} + E_{42}, H = \mathrm{diag}(1, 1, -1, -1), u = Id + E_{12} + E_{34}, Z = \mathrm{diag}(1, -1, 1, -1)$.

$$\mathfrak{n}_{H,f} = \begin{pmatrix} 0 & b & \underline{*} & * \\ 0 & 0 & * & \underline{*} \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}; \mathfrak{n}_{H+Z,f} = \begin{pmatrix} 0 & * & \underline{*} & * \\ 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Corollary: if $\mathrm{WO}^{\max}(\eta) = \{2^n\}$ then η has almost-Shalika model.

Applications of hidden symmetry

Corollary (Hidden symmetry)

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Corollary

If $G \in \{SO(n,n), SO(n+1,n)\}$ and $\mathrm{WO}^{\max}(\eta) = \{31\ldots1\}$ then $\mathcal{F}_{H,f}[\eta]$ is Eulerian.

Follows from uniqueness of Bessel models.

Fourier-Jacobi periods and the Weil representation

For a symplectic space V over $\mathbb{K}$, let $\mathcal{H}(V) := V \oplus \mathbb{K}$ be the Heisenberg group and $\widetilde{J(V)} := \widetilde{\mathrm{Sp}(V(\mathbb{A}))} \ltimes \mathcal{H}(V(\mathbb{A}))$ be the double cover Jacobi group. It has unique irreducible unitarizable representation ϖ_V with central character χ. It has automorphic realization given by theta functions:

$$\theta_f(g) = \sum_{a \in \mathcal{E}(K)} \omega_\chi(g) f(a), \text{ where } g \in \widetilde{J(V)}, f \in \mathcal{S}(\mathcal{E}(\mathbb{A})), \mathcal{E} \subset V \text{ Lagrangi}$$

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$$f \otimes \eta \mapsto \int_{U(K) \backslash U(\mathbb{A})} f(u\tilde{g}) \theta_\eta(\ell(u, \tilde{g})) du$$

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If $\Gamma \cdot f \in \mathrm{WO}^{\max}(\pi)$ then $\widetilde{M}$ acts on the image of FJ by ± 1.

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Corollary

If $\Gamma \cdot f \in \mathrm{WO}^{\max}(\pi)$ and G is classical then the orbit of f is special.

Eulerianity

Lemma

Let (S, f) *and* (H, f') *be two Whittaker pairs such that* $\Gamma f = \Gamma f' \in \mathrm{WO}^{\max}(\eta)$. *Let* I, I' *be maximal isotropic. Suppose that* $\mathcal{F}^{I}_{S,f}[\eta]$ *is Eulerian. Then* $\mathcal{F}^{I'}_{H,\psi}[\eta]$ *is also Eulerian.*

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Verified for:

1. Minimal representations of most split simply-laced groups
2. Next-to-minimal Eisenstein series of most split simply-laced groups
3. Discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$.

Expressing forms through their Whittaker coefficients

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Any $\mathcal{F}_{H,f}$ can be expressed through all Levi-distinguished Fourier coefficients $\mathcal{F}_{S,F}$ with $\Gamma F \geq \Gamma f$.

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1. *Any η can be expressed through all its Levi-distinguished Fourier coefficients.*
2. *If all $\mathcal{O} \in \mathrm{WO}(\eta)$ admit Whittaker coefficients then η can be expressed through its Whittaker coefficients.*
3. *For split simply-laced G, we obtained expressions for all minimal or next-to-minimal η, and all their Fourier coefficients in terms of Whittaker coefficients.*

Explanation for GL_n (PS-Shalika, Ahlen–Gustafsson-Liu-Kleinschmidt-Persson)

Let $\eta \in C^\infty(\Gamma\backslash \mathrm{GL}_n(\mathbb{A}))$. Restrict to the last column and decompose to Fourier series. All non-trivial characters are conjugate by $\mathrm{GL}_{n-1}(\mathbb{K})$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Example: Sp(4)

$$\mathfrak{sp}_4 = \left\{ \begin{pmatrix} A & B = B^t \\ C = C^t & -A^t \end{pmatrix} \right\}.$$

Let $\mathfrak{n}$ be the Borel nilradical, and $\mathfrak{u} \subset \mathfrak{n}$ be the Siegel nilradical, spanned by B. Characters given by $\bar{\mathfrak{u}} \cong \mathrm{Sym}^2(\mathbb{K}^2)$. Restricting η to B and decomposing into Fourier series we obtain $\eta = \sum_{f \in \bar{\mathfrak{u}}} \mathcal{F}_{\mathfrak{u},f}[\eta]$.

1. Constant term $\mathcal{F}_{\mathfrak{u},0}[\eta]$: Restrict to the Siegel Levi $L \cong \mathrm{GL}_2(\mathbb{A})$, and decompose to Fourier series on the abelian group $N \cap L$:

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3. Split non-degenerate forms are conjugate to $f_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   Using Weyl group conjugation (24) and root exchange, we express $\mathcal{F}_{\mathfrak{u},f_2}$ through $\mathcal{F}_{\mathfrak{u}',e_{21}}$, where $\mathfrak{u}' = Span(e_{12} - e_{43}, e_{13}, e_{24}) \subset \mathfrak{n}$. Fourier expansion by the remaining coordinate of $e_{14} + e_{23} \in \mathfrak{n}$:

$$\mathcal{F}_{\mathfrak{u},f}[\eta](g) = \int_{x \in \mathbb{A}} \mathcal{W}_{1,a}[\eta]((Id + xe_{24})wg).$$

X:=set of anisotropic 2×2 forms. For $f\in X$, we cannot simplify $\mathcal{F}_{\mathfrak{u},f}[\eta]$.
Summarizing, for any η on $G=\mathrm{Sp}_4(\mathbb{A})$ we have

$$\eta(g)=\sum_{f\in X}\mathcal{F}_{\mathfrak{u},f}[\eta](g)+\sum_{a\in\mathbb{K}}\Big(\sum_{\gamma\in L/O(1,1)}\int_{x\in\mathbb{A}}\mathcal{W}_{1,a}[\eta](v_x w\gamma g)+\sum_{\gamma\in L/(N\cap L)}\mathcal{W}_{a,1}[\eta](\gamma g)+\mathcal{W}_{a,0}[\eta](g)\Big)$$

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X:=set of anisotropic 2×2 forms. For $f\in X$, we cannot simplify $\mathcal{F}_{\mathfrak{u},f}[\eta]$. Summarizing, for any η on $G=\mathrm{Sp}_4(\mathbb{A})$ we have

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If η is cuspidal then $\mathcal{W}_{0,a}[\eta]=\mathcal{W}_{a,0}[\eta]=0$. If η is non-generic, then $\mathcal{W}_{1,a}[\eta]=\mathcal{W}_{a,1}[\eta]=0$, unless $a=0$. Thus

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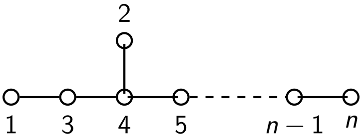

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- All the theorems above have local analogues with similar proofs.

Wave front set and wave-front cycle

Let π be smooth, admissible and finitely generated.

Theorem (Howe, Harish-Chandra, Barbasch-Vogan 70s)

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