Finite multiplicities beyond spherical pairs

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Basic Functions, Orbital Integrals, and Beyond Endoscopy
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Theorem (Kobayashi-Oshima, 2013)

Let $\mathbf{X} = \mathbf{G}/\mathbf{H}$. *Then*

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Theorem (Casselman, 1978)

$0 < m_\sigma(\mathcal{S}(G/U)) < \infty \quad \forall \sigma \in \operatorname{Irr}(G)$, *where* U=*maximal unipotent.*

Ξ-spherical spaces

$\forall x \in \mathbf{X}$, have action map $\mathbf{G} \to \mathbf{X}$, thus $\mathfrak{g} \to T_x\mathbf{X}$, and $T_x^*\mathbf{X} \to \mathfrak{g}^*$.
This gives the moment map $\mu : T^*\mathbf{X} \to \mathfrak{g}^*$.
For $\mathbf{X} = \mathbf{G}/\mathbf{H}$: $T^*\mathbf{X} \cong \mathbf{G} \times_H \mathfrak{h}^\perp$ and $\mu(g, \alpha) = g \cdot \alpha$

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Definition

- *For a nilpotent orbit* $\mathbf{O} \subset \mathcal{N}(\mathfrak{g}^*)$, *say* $\mathbf{X}$ *is* $\mathbf{O}$*-spherical if*

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Corollary (following Wen-Wei Li)

- *$\mathbf{X}$ is $\mathcal{N}(\mathfrak{g}^*)$-spherical $\iff$ $\mathbf{X}$ is spherical*
- *$\mathbf{X}$ is $\{0\}$-spherical $\iff$ $\mathbf{G}$ has finitely many orbits on $\mathbf{X}$.*

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Theorem 2 (Aizenbud - G. 2021)

Let $\Xi \subset \mathcal{N}(\mathfrak{g}^*)$ closed **G***-invariant. Let* **X** *be Ξ-spherical* **G***-manifold, and let $\sigma \in \mathcal{M}_\Xi(G)$. Then* $\dim \operatorname{Hom}(\mathcal{S}(X), \sigma) < \infty$

Applications to branching problems

Corollary

Let $\mathbf{H}\subset\mathbf{G}$ *be reductive subgroup. Let* $\mathbf{P}\subset\mathbf{G}$ *and* $\mathbf{Q}\subset\mathbf{H}$ *be parabolic subgroups with* $|\mathbf{P}\backslash\mathbf{G}/\mathbf{Q}|<\infty$*. Then* $\forall\pi\in\mathcal{M}_{\overline{\mathbf{O_P}}}(G)$ *and* $\tau\in\mathcal{M}_{\overline{\mathbf{O_Q}}}(H)$*,*

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1. *Let* $\mathbf{P} \subset \mathbf{G}$ *be a parabolic subgroup s.t.* $\mathbf{G}/\mathbf{P}$ *is a spherical* $\mathbf{H}$*-variety. Then* $\forall \pi \in \mathcal{M}_{\overline{\mathbf{O_P}}}(G)$, $\pi|_H$ *has finite multiplicities.*
2. *Let* $\mathbf{Q} \subset \mathbf{H}$ *be a parabolic subgroup that is spherical as a subgroup of* $\mathbf{G}$. *Then for any* $\tau \in \mathcal{M}_{\overline{\mathbf{O_Q}}}(H)$, $\operatorname{ind}_H^G \tau$ *has finite multiplicities.*

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For simple $\mathbf{G}$ and symmetric $\mathbf{H}\subset\mathbf{G}$, all $\mathbf{P}\subset\mathbf{G}$ satisfying (i), and all $\mathbf{Q}\subset\mathbf{H}$ satisfying (ii) are classified by He, Nishiyama, Ochiai, Oshima. For classical $\mathbf{G}$, all $\mathbf{H}$: Avdeev-Petukhov. They also have a strategy $\forall\mathbf{G}$.

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Let $\mathbf{H}$ *be a reductive group, and* $\mathbf{P}, \mathbf{Q} \subset \mathbf{H}$ *be parabolic subgroups s.t.* $\mathbf{H}/\mathbf{P} \times \mathbf{H}/\mathbf{Q}$ *is a spherical* $\mathbf{H}$*-variety, under the diagonal action.*
Then $\forall \pi \in \mathcal{M}_{\overline{\mathbf{O_P}}}(H)$, *and* $\tau \in \mathcal{M}_{\overline{\mathbf{O_Q}}}(H)$, $\pi \otimes \tau$ *has finite multiplicities.*

All such triples $(\mathbf{H}, \mathbf{P}, \mathbf{Q})$ were classified by Stembridge.
Example: $\mathbf{H} = \mathrm{GL}_n$, $\tau \in \mathcal{M}_{\overline{\mathbf{O}_{\min}}}(H)$, or classical $\mathbf{H}$ and $\pi, \tau \in \mathcal{M}_{\overline{\mathbf{O}_{2^n}}}(H)$.

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Example (Generalized Shalika model)

Let $\mathbf{G} = \mathrm{GL}_{2n}$, $\mathbf{R} = \mathbf{LU} \subset \mathbf{G}$ *with* $\mathbf{L} = \mathrm{GL}_n \times \mathrm{GL}_n$ *and* $\mathbf{U} = \mathrm{Mat}_{\mathbf{n}\times\mathbf{n}}$,
$\mathbf{M} = \Delta\,\mathrm{GL}_n \subset \mathbf{L}$, $\mathbf{H} := \mathbf{MU}$.
Let $\mathfrak{m}^* \supset \mathbf{O}_{\min} :=$ *minimal nilpotent orbit, and* $\pi \in \mathcal{M}_{\overline{\mathbf{O}_{\min}}}(M)$.
Let ψ *be a unitary character of* H.
Then $\mathrm{ind}_H^G(\pi \otimes \psi)$ *has finite multiplicities.*

Similar case: $\mathbf{G} = O_{4n}$, $\mathbf{L} = \mathrm{GL}_{2n}$, $\mathbf{M} = \mathrm{Sp}_{2n}$, $\mathbf{O}_{\mathrm{ntm}} \subset \mathfrak{m}^*$.

Some necessary conditions for finite multiplicities

Theorem (Tauchi)

Let $P \subset G$ be a parabolic subgroup. If all degenerate principal series representations of the form $\mathrm{Ind}_P^G \rho$, with $\dim \rho < \infty$, have finite H-multiplicities, then H has finitely many orientable orbits on G/P.

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Corollary

Let $\mathbf{P} \subset \mathbf{G}$ be a parabolic subgroup defined over $\mathbb{R}$. Suppose that for all but finitely many orbits of $\mathbf{H}$ on $\mathbf{G}/\mathbf{P}$, the set of real points is non-empty and orientable. Then the following are equivalent.

1. $\mathbf{H}$ *is* $\overline{\mathbf{O_P}}$*-spherical.*
2. *Every* $\pi \in \mathcal{M}_{\overline{\mathbf{O_P}}}(G)$ *has finite multiplicities in* $\mathcal{S}(G/H)$.
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The assumption of the corollary holds if H and G are complex reductive groups.

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Let $\mathbf{P} \subset \mathbf{G}$ *be a parabolic subgroup defined over* $\mathbb{R}$*. Suppose that for all but finitely many orbits of* $\mathbf{H}$ *on* $\mathbf{G}/\mathbf{P}$*, the set of real points is non-empty and orientable. Then the following are equivalent.*

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Branching multiplicities for degenerate principal series were computed in various cases by Frahm-Orsted-Oshima, and Kobayashi.
Kobayashi: Conditions for bounded multiplicities in terms of distinction w.r. to symmetric $G'\subset G$.

Further Examples

Example (I. Karshon, related to Howe correspondance in type II)

$\mathbf{G} := \mathrm{Sp}(V \otimes W \oplus V^* \otimes W^*)$, $\mathbf{H} := \mathrm{GL}(V) \times \mathrm{GL}(W) \hookrightarrow G$.
Then $\mathbf{G}/\mathbf{B_H}$ *is* $\overline{\mathbf{O}_{\min}}$*-spherical.*

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Example (D. Panyushev, strict inequality)

$\mathbf{G} := \mathrm{Sp}_{\mathbf{2n}}$, $\mathbf{P} = \mathbf{LU} \subset \mathbf{G}$- *maximal parabolic subgroup with* $U \cong$ *Heisenberg group,* $\mathbf{O} := O_{\min}$. *Then* $\dim \mathbf{O} = 2n$, *while* $\dim \mathbf{O} \cap \mathfrak{p}^{\perp} = 1$.
Thus $\dim \mu^{-1}_{\mathbf{G}/\mathbf{P}}(\mathbf{O}) < \dim \mathbf{G}/\mathbf{P} + \dim \mathbf{O}/2$.

Step 1 of the proof: Reduction to distributions

Theorem 3 (Aizenbud - G. 2021)

Let $I \subset \mathcal{U}(\mathfrak{g})$ *be a two-sided ideal such that* $\mathcal{V}(I) \subset \mathcal{N}(\mathfrak{g}^*)$. *Let* $\mathbf{X}, \mathbf{Y}$ *be* $\mathcal{V}(I)$*-spherical* $\mathbf{G}$*-manifolds. Let* $\mathcal{S}^*(X \times Y)^{\Delta G, I}$ *denote the space of* ΔG*-invariant tempered distributions on* $X \times Y$ *annihilated by* I. *Then*

$$\dim \mathcal{S}^*(X \times Y)^{\Delta G, I} < \infty$$

Step 1 of the proof: Reduction to distributions

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Proof of Theorem 2.

$\Xi \subset \mathcal{N}(\mathfrak{g}^*)$, $\mathbf{X}$ is Ξ-spherical, $\sigma \in \mathcal{M}_\Xi$. Need: $\dim \operatorname{Hom}_G(\mathcal{S}(X), \sigma) < \infty$. Let $\mathcal{E}$ be a bundle on $Y := G/K$ s.t. $\sigma \hookrightarrow \mathcal{S}^*(Y, \mathcal{E})$. Let $I := \operatorname{Ann}(\sigma)$. Then $\mathcal{V}(I) \subset \Xi$, and

$$\operatorname{Hom}_G(\mathcal{S}(X), \sigma) \hookrightarrow \operatorname{Hom}_G(\mathcal{S}(X), \mathcal{S}^*(Y, \mathcal{E}))^I \hookrightarrow \mathcal{S}^*(X \times Y, \mathbb{C} \boxtimes \mathcal{E})^{\Delta G, I}$$

□

Main technique: D-modules

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- $D_{\mathbf{X}}$:=sheaf of algebraic differential operators. $\operatorname{Gr} D_{\mathbf{X}} \cong \mathcal{O}(T^*\mathbf{X})$.
- For a fin.gen. sheaf M of $D_{\mathbf{X}}$-modules, $\operatorname{SingS}(M) := \operatorname{Supp} \operatorname{Gr}(M) \subset T^*\mathbf{X}$.
- Bernstein: if $M \neq 0$ then $\dim \operatorname{SingS}(M) \geq \dim \mathbf{X}$.
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Theorem (Bernstein-Kashiwara)

For any holonomic M, $\dim \operatorname{Hom}_{D_{\mathbf{X}}}(M, \mathcal{S}^*(X)) < \infty$.

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Let $\Xi \subset \mathcal{N}(\mathfrak{g}^*)$ and let $\mathbf{X}, \mathbf{Y}$ be Ξ-spherical $\mathbf{G}$-manifolds. Then

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Proof of Theorem 3.

$M := D_{\mathbf{X} \times \mathbf{Y}}$-module with $\mathcal{S}^*(X \times Y)^{\Delta G, I} \hookrightarrow \operatorname{Hom}(M, \mathcal{S}^*(X, Y))$.
By the lemma, M is holonomic.

Proof of the geometric lemma

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Proof.

$\forall$ orbit $\mathbf{O} \subset \Xi$ we have

$$\dim \mu_{\mathbf{X}\times\mathbf{Y}}^{-1}((\mathbf{O} \times \mathbf{O}) \cap (\Delta\mathfrak{g})^{\perp}) = \dim \mu_{\mathbf{X}}^{-1}(\mathbf{O}) + \dim \mu_{\mathbf{Y}}^{-1}(\mathbf{O}) - \dim \mathbf{O} \leq$$
$$\dim \mathbf{X} + \dim \mathbf{O}/2 + \dim \mathbf{Y} + \dim \mathbf{O}/2 - \dim \mathbf{O} = \dim \mathbf{X} + \dim \mathbf{Y}$$

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Happy Birthday, Bill!