Degenerate Whittaker functionals for real reductive groups

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Theorem (Gelfand-Kazhdan, Shalika)

For $\pi \in Irr(G)$, $\psi \in \Psi^{\times}$, $\dim Wh^*_{\psi}(\pi) \leq 1$.

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- π is called *large* if $\mathrm{WF}(\pi) = \mathcal{N}$.

Case of non-generic representations

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- Several authors (Matumoto, Yamashita, ...) consider *generalized* Whittaker functionals $\sim$ generic characters for *smaller* nilradicals
- We consider *degenerate* functionals $\sim$ *arbitrary* characters of $\mathfrak{n}$.

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Theorem (1)

For $\pi \in \mathcal{M}(G)$ we have

$$\Psi(\pi) \subset \mathrm{WF}(\pi) \cap \Psi \subset \Psi(\widetilde{\pi}) \quad (1)$$

Moreover if $G = GL_n(\mathbb{R})$ or if G is a complex group then $\widetilde{\pi} = \pi$ and

$$\Psi(\pi) = \mathrm{WF}(\pi) \cap \Psi \quad (2)$$

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Theorem (2)

The sets $\Psi(\pi)$ *and* $\mathrm{WF}(\pi)$ *determine one another if*

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Key observation for the second statement:

Theorem (3)

Let $\mathcal{O}$ be a nilpotent orbit for a complex classical Lie algebra then $\mathcal{O}$ is uniquely determined by $\overline{\mathcal{O}} \cap \Psi$.

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Let $\pi \in \hat{G}$ be an irreducible unitary representation. Let λ be such that $WF(\pi) = \overline{\mathcal{O}_\lambda}$. Then $\dim Wh_{\psi_\lambda}(\pi) = 1$.

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- In the p-adic case the analogous theorem was proven by Zelevinsky without the assumption that π is unitary. The proofs in both cases use "derivatives".

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- Kostant showed that π is generic iff $\pi^{K-\text{finite}}$ is generic, though dimensions of Whittaker spaces differ considerably.

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Theorem (0)

For $M \in \mathcal{HC}$ we have $\Psi(M) = pr_{\mathfrak{n}^*}(\mathrm{As}\mathcal{V}(M)) \cap \Psi$.

Idea of the proof

- Since $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ is commutative, from Nakayama's lemma we have $\Psi(M) = \mathrm{Supp}(M/[\mathfrak{n},\mathfrak{n}]M)$. Now, restriction to $\mathfrak{n}$ corresponds to projection on $\mathfrak{n}^*$ and quotient by $[\mathfrak{n},\mathfrak{n}]$ corresponds to intersection with $\Psi = [\mathfrak{n},\mathfrak{n}]^\perp$.

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- However, in non-commutative situation one could even have $V = [\mathfrak{n},\mathfrak{n}]V$. For example, let $G = GL(3,\mathbb{R})$ and consider the identification of $\mathfrak{n}$ with the Heisenberg Lie algebra $\langle x, \frac{d}{dx}, 1\rangle$ acting on $V = \mathbb{C}[x]$.

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- Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ be the Borel subalgebra of $\mathfrak{g}$, let V be a $\mathfrak{b}$-module. We define the n-adic completion and Jacquet module as follows:
$$\widehat{V} = \widehat{V}_{\mathfrak{n}} = \varprojlim V/\mathfrak{n}^i V, \quad J(V) = J_{\mathfrak{b}}(V) = \left(\widehat{V}_{\mathfrak{n}}\right)^{\mathfrak{h}\text{-finite}}$$

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- (Casselman-Osborne+Gabber) $\mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) = pr_{\mathfrak{n}^*}(\mathrm{As}\mathcal{V}_{\mathfrak{g}}(M))$.
- Thus $\Psi(M) \supset pr_{\mathfrak{n}^*}(\mathrm{As}\mathcal{V}_{\mathfrak{g}}(M)) \cap \Psi$; other inclusion is easy.

Proof of Theorem 3

Proof.

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- Orbits for $Sp_{2n}(\mathbb{C})$ or $O_n(\mathbb{C}) \sim$ partitions satisfying certain conditions
- An orbit meets Ψ iff it has at most one part ≥ 2 with odd multiplicity
- For each partition λ and each k there is a partition $\mu \leq \lambda$, which meets Ψ and satisfies $\mu_1 + \cdots + \mu_k = \lambda_1 + \cdots + \lambda_k$

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- Result for $SO_n(\mathbb{C})$ requires slight additional argument.

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  5. $\underline{E_6(a_1)}$ and $\underline{E_7(a_4)}$
  6. $\underline{E_8(a_7)}$, $E_7(a_5)$, $E_6(a_3) + A_1$, and $D_6(a_2)$.