Degenerate Whittaker functionals for real reductive groups

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Whittaker functionals

- $\mathbb{F}$: local field of char.0, G: reductive group over $\mathbb{F}$.
- $\mathcal{M}(G)$: smooth admissible representations (of moderate growth).
- Assume G **quasisplit:** fix Borel $B = HN$, $\mathfrak{n} := Lie_{\mathbb{F}}(N)$.

$$\text{Define} \quad \mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}], \mathfrak{v} = \mathfrak{n}/\mathfrak{n}', \Psi = \mathfrak{v}^* \subset \mathfrak{n}^*$$

- $\Psi \longleftrightarrow$ Lie algebra characters of $\mathfrak{n} \longleftrightarrow$ unitary group characters of N
- $\Psi \supset \Psi^\times :=$ non-degenerate characters.
- For $\psi \in \Psi$, $\pi \in \mathcal{M}(G)$ define

$$Wh^*_\psi(\pi) := \mathrm{Hom}^{cts}_N(\pi, \psi), \Psi(\pi) := \{\psi \in \Psi : Wh^*_\psi(\pi) \neq 0\}$$

- (Casselman) For any $\psi \in \Psi^\times$, $\quad \pi \mapsto Wh^*_\psi(\pi)$ is an exact functor.

Theorem (Gelfand-Kazhdan, Shalika)

For $\pi \in Irr(G)$, $\psi \in \Psi^\times$, $\dim Wh^*_\psi(\pi) \leq 1$.

Kostant's theorem

- We say π is *generic* if $\exists \psi \in \Psi^\times$ s.t. $Wh_\psi(\pi) \neq 0$.

Theorem (Kostant, Rodier)

π is generic iff it is large.

Theorem (Harish-Chandra, Howe)

Near $e \in G$, the character distribution (asymptotically) equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

$$\chi_\pi \approx \sum a_\mathcal{O} \mathcal{F}(\mu_\mathcal{O})$$

- Define $\mathrm{WF}(\pi) = \cup\{\overline{\mathcal{O}} \mid a_\mathcal{O} \neq 0\} \subset \mathcal{N}$, where $\mathcal{N} \subset \mathfrak{g}^*$ denotes the nilpotent cone.
- π is called *large* if $\mathrm{WF}(\pi) = \mathcal{N}$.

Case of non-generic representations

- In the p-adic case, Moeglin and Waldspurger give a very general definition of degenerate Whittaker models and give a precise connection between their existence and the wave-front set $\mathrm{WF}(\pi)$. In the real case there is no full analog currently.
- Several authors (Matumoto, Yamashita, ...) consider *generalized* Whittaker functionals $\sim$ generic characters for *smaller* nilradicals
- We consider *degenerate* functionals $\sim$ *arbitrary* characters of $\mathfrak{n}$.

Main results

- From now on, let $\mathbb{F} = \mathbb{R}$.
- The finite group $F_G = Norm_{G_{\mathbb{C}}}(G) / (Z_{G_{\mathbb{C}}} \cdot G)$ acts on $\mathcal{M}(G)$.
- For $\pi \in \mathcal{M}(G)$, define $\tilde{\pi} = \bigoplus \{\pi^a : a \in F_G\}$.

Theorem (1)

For $\pi \in \mathcal{M}(G)$ we have

$$\Psi(\pi) \subset \mathrm{WF}(\pi) \cap \Psi \subset \Psi(\widetilde{\pi}) \tag{1}$$

Moreover if $G = GL_n(\mathbb{R})$ or if G is a complex group then $\widetilde{\pi} = \pi$ and

$$\Psi(\pi) = \mathrm{WF}(\pi) \cap \Psi \tag{2}$$

Main results

Theorem (2)

The sets $\Psi(\pi)$ *and* $\mathrm{WF}(\pi)$ *determine one another if*

1. $G = GL_n(\mathbb{R})$ *or* $GL_n(\mathbb{C})$ *or* $SL_n(\mathbb{C})$ *and* $\pi \in \mathcal{M}(G)$
2. $G = Sp_{2n}(\mathbb{C})$ *or* $O_n(\mathbb{C})$ *or* $SO_n(\mathbb{C})$ *and* π *is irreducible*

Key observation for the second statement:

Theorem (3)

Let $\mathcal{O}$ *be a nilpotent orbit for a complex classical Lie algebra then* $\mathcal{O}$ *is uniquely determined by* $\overline{\mathcal{O}} \cap \Psi$.

Uniqueness

- An analog of Shalika's result would be uniqueness of "minimally degenerate" Whittaker models. So far it is known only for GL_n, both in real and p-adic cases.
- Let G be $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$. For a partition λ of n let $\mathcal{O}_\lambda$ denote the corresponding nilpotent orbit and ψ_λ denote the corresponding character of N.

Theorem (Aizenbud-G-Sahi)

Let $\pi \in \hat{G}$ be an irreducible unitary representation. Let λ be such that $WF(\pi) = \overline{\mathcal{O}_\lambda}$. Then $\dim Wh_{\psi_\lambda}(\pi) = 1$.

- In the p-adic case the analogous theorem was proven by Zelevinsky without the assumption that π is unitary. The proofs in both cases use "derivatives".

Algebraic setting

- From now on, we let $\mathfrak{n}, \mathfrak{g}$, etc. denote *complexified* Lie algebras.
- Let $K \subset G$ be maximal compact subgroup. A $(\mathfrak{g}, K)$-module is a complex vector space with compatible actions of $\mathfrak{g}$ and K such that every vector is K-finite.
- Let $\mathcal{HC}(G)$ denote the category of $(\mathfrak{g}, K)$-modules of finite length

Theorem (Casselman-Wallach)

The functor $\pi \mapsto \pi^{K-finite}$ is an equivalence of categories $\mathcal{M}(G) \cong \mathcal{HC}(G)$

- For $M \in \mathcal{HC}(G)$ and $\psi \in \Psi_{\mathbb{C}}$ we define

$$Wh'_{\psi}(M) := \mathrm{Hom}_{\mathfrak{n}}(M, \psi), \quad \Psi(M) := \{\psi \in \Psi_{\mathbb{C}} \mid Wh'_{\psi}(M) \neq 0\}$$

- Kostant showed that π is generic iff $\pi^{K-\text{finite}}$ is generic, though dimensions of Whittaker spaces differ considerably.

Associated varieties and our algebraic theorem

- Using PBW filtration, $\operatorname{gr}\mathcal{U}(\mathfrak{g}) = \mathsf{Sym}(\mathfrak{g}) = \mathsf{Pol}(\mathfrak{g}^*)$
- Using this, one can define

$$\mathrm{As}\mathcal{V}(M) \subset \mathrm{An}\mathcal{V}(M) \subset \mathcal{N}$$

- Schmid and Vilonen proved that $\mathsf{WF}(\pi)$ and $\mathrm{As}\mathcal{V}(\pi^{K-\text{finite}})$ determine each other.
- Let $pr_{\mathfrak{n}^*} : \mathfrak{g}^* \to \mathfrak{n}^*$ denote the natural projection (restriction to $\mathfrak{n}$).

Theorem (0)

For $M \in \mathcal{HC}$ we have $\Psi(M) = pr_{\mathfrak{n}^*}(\mathrm{As}\mathcal{V}(M)) \cap \Psi$.

Idea of the proof

- Since $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ is commutative, from Nakayama's lemma we have $\Psi(M) = \mathrm{Supp}(M/[\mathfrak{n},\mathfrak{n}]M)$. Now, restriction to $\mathfrak{n}$ corresponds to projection on $\mathfrak{n}^*$ and quotient by $[\mathfrak{n},\mathfrak{n}]$ corresponds to intersection with $\Psi = [\mathfrak{n},\mathfrak{n}]^{\perp}$.
- However, in non-commutative situation one could even have $V = [\mathfrak{n},\mathfrak{n}]V$. For example, let $G = GL(3,\mathbb{R})$ and consider the identification of $\mathfrak{n}$ with the Heisenberg Lie algebra $\langle x, \frac{d}{dx}, 1\rangle$ acting on $V = \mathbb{C}[x]$.
- Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ be the Borel subalgebra of $\mathfrak{g}$, let V be a $\mathfrak{b}$-module. We define the *n*-adic completion and Jacquet module as follows: $\widehat{V} = \widehat{V}_{\mathfrak{n}} = \varprojlim V/\mathfrak{n}^i V, \quad J(V) = J_{\mathfrak{b}}(V) = \left(\widehat{V}_{\mathfrak{n}}\right)^{\mathfrak{h}\text{-finite}}$

Sketch of the proof

- Define $n' = [\mathfrak{n}, \mathfrak{n}]$ and $CV = H_0(\mathfrak{n}', V) = V/\mathfrak{n}'V$.
- (Nakayama) $\Psi(M) = \mathrm{Supp}_{\mathfrak{v}}(CM) = \mathrm{An}\mathcal{V}_{\mathfrak{v}}(CM)$
- (Joseph+Gabber) $\mathrm{An}\mathcal{V}_{\mathfrak{v}}(CM) = \mathrm{An}\mathcal{V}_{\mathfrak{v}}\left(\widehat{CM}\right) = \mathrm{An}\mathcal{V}_{\mathfrak{v}}(J(CM))$
- (Easy) $J(CM) \approx C(JM)$ as $\mathfrak{b}$-modules.
- (Bernstein+Joseph)
  $\mathrm{An}\mathcal{V}_{\mathfrak{v}}(J(CM)) = \mathrm{As}\mathcal{V}_{\mathfrak{v}}(C(JM)) = \mathrm{As}\mathcal{V}_{\mathfrak{n}}(JM) \cap \Psi$.
- (Ginzburg+ENV) $\mathrm{As}\mathcal{V}_{\mathfrak{n}}(JM) \supset \mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) \cap \Psi$.
- (Casselman-Osborne+Gabber) $\mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) = pr_{\mathfrak{n}^*}(\mathrm{As}\mathcal{V}_{\mathfrak{g}}(M))$.
- Thus $\Psi(M) \supset pr_{\mathfrak{n}^*}(\mathrm{As}\mathcal{V}_{\mathfrak{g}}(M)) \cap \Psi$; other inclusion is easy.

Proof of Theorem 3

Proof.

For $GL(n, \mathbb{R})$ and $SL(n, \mathbb{C}) \sim$ Jordan form

- Orbits for $Sp_{2n}(\mathbb{C})$ or $O_n(\mathbb{C}) \sim$ partitions satisfying certain conditions
- An orbit meets Ψ iff it has at most one part ≥ 2 with odd multiplicity
- For each partition λ and each k there is a partition $\mu \leq \lambda$, which meets Ψ and satisfies $\mu_1 + \cdots + \mu_k = \lambda_1 + \cdots + \lambda_k$
- Result for $SO_n(\mathbb{C})$ requires slight additional argument.

□

Counterexamples for exceptional groups

Fact

Theorem 3 is false for every exceptional group.

- We list all orbits whose closures have the same intersection with Ψ.
- We follow Bala-Carter notation and we have underlined the special orbits.
- For $G = G_2$: $\underline{G_2(a_1)}$ and $\widetilde{A_1}$
- For $G = F_4$:
  1. $\underline{F_4(a_1)}$ and $\underline{F_4(a_2)}$
  2. $\underline{F_4(a_3)}$ and $C_3(a_1)$
- For $G = E_6$:
  1. $\underline{E_6(a_1)}$ and $\underline{D_5}$
  2. $\underline{D_4(a_1)}$ and $A_3 + A_1$
- For $G = E_7$:
  1. $\underline{E_7(a_1)}$ and $\underline{E_7(a_2)}$
  2. $\underline{E_7(a_3)}$ and D_6
  3. $\underline{E_6(a_1)}$ and $\underline{E_7(a_4)}$.

- For $G = E_8$:
  1. $\underline{E_8(a_1)}$, $\underline{E_8(a_2)}$, and $\underline{E_8(a_3)}$
  2. $\underline{E_8(a_4)}$, $\underline{E_8(b_4)}$ and $\underline{E_8(a_5)}$
  3. $\underline{E_7(a_1)}$, $\underline{E_8(b_5)}$ and $E_7(a_2)$
  4. $\underline{E_8(a_6)}$ and $\underline{D_7(a_1)}$
  5. $\underline{E_6(a_1)}$ and $\underline{E_7(a_4)}$
  6. $\underline{E_8(a_7)}$, $E_7(a_5)$, $E_6(a_3) + A_1$, and $D_6(a_2)$.