Wave-front sets of distinguished representations

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following Harris-Weich, Prasad, Sakellaridis, Vogan
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Conjecture

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Theorem (G.-Sayag 2020)

1. *If F is Archimedean and $\mathbf{H} \subset \mathbf{G}$ is spherical then*

$$\mathbf{G}(WF^0(\pi))|_{\mathfrak{h}} \ni 0.$$

2. *If F is p-adic then $WF^o(\pi) \subset \overline{G\mathfrak{h}_0^{\perp}} \subset \mathfrak{g}_0^*$*

$\mathbf{H}$ is called spherical if it has an open orbit on the flag variety $\mathbf{G}/\mathbf{B}$.

Uniform applications

- $\mathrm{Irr}_H(G)$:=all smooth admissible H-distinguished irreps.
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(ii) *If F is p-adic then $WF^o(\pi) \subset \overline{G\mathfrak{h}_0^{\perp}}$*

Assume char $F = 0$.

Corollary (following Prasad - Sakellaridis)

$\forall$ *unimodular spherical $H \subset G$,*

$$\max_{\pi \in \mathrm{Irr}_H(G)} \dim \mathrm{WF}^o(\pi) = \dim \mathcal{N}(\mathfrak{g}_0^*) \cap G\mathfrak{h}_0^{\perp} \quad \textit{and}$$

(i) $\exists$ *inf. dim.* $\pi \in \mathrm{Irr}_H(G) \iff \exists$ *unipotent $u \in G - H$.*

(ii) *for quasisplit G: $\exists$ generic $\pi \in \mathrm{Irr}_H(G) \iff H \cap U = \{1\}$ for some maximal unipotent subgroup $U \subset G$.*

Archimedean setting & applications to symmetric pairs

From now on $F := \mathbb{R}$.

Theorem (Rossmann 1995, using Borho-Brylinski + Joseph)

$\forall \pi \in \mathrm{Irr}(G)$, $\mathrm{WF}^o(\pi)$ *lies in a unique* $\mathbf{G}$*-orbit* $\mathcal{O}(\pi) \subset \mathcal{N}(\mathfrak{g}^*)$.

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Let $\mathbf{H} \subset \mathbf{G}$ be symmetric sbgrp, and $G_H^{\mathbb{R}}$ the corresponding real form of $\mathbf{G}$.
Kostant-Sekiguchi: $\mathbf{H}$-orbits on $\mathfrak{h}^{\perp} \cong (\mathfrak{g}/\mathfrak{h})^* \leftrightarrow$ real $G_H^{\mathbb{R}}$-orbits on $\mathfrak{g}^*$.
Prasad philosophy: $G_H^{\mathbb{R}}$ describes H-distinguished $\mathrm{rep}^{\mathrm{ns}}$ of G.

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Corollary (of our thm)

$\forall \pi \in \mathrm{Irr}_H(G)$, $\mathcal{O}(\pi)$ *includes a real* $G_H^{\mathbb{R}}$*-orbit.*

Real orbits are classified by Ohta and Djokovic.

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For split G, H*:* $\exists$ *tempered* $\pi \in \mathrm{Irr}_H(G) \Rightarrow \exists$ *generic* $\pi \in \mathrm{Irr}_H(G)$.

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Our theorem + result of Harris $\Rightarrow$ conjecture holds for most pairs.

Applications to strongly spherical pairs

Corollary

Assume that $\mathbf{H}$ *is reductive and* $\boldsymbol{\Delta}\mathbf{H} \subset \mathbf{G} \times \mathbf{H}$ *spherical.*
Let $\pi \in \mathrm{Irr}(G)$, $\tau \in \mathrm{Irr}(H)$ *with* $\mathrm{Hom}_H(\pi|_H, \tau) \neq 0$*. Then*

$$\mathcal{O}(\tau) \subset \mathcal{O}(\pi)|_{\mathfrak{h}}.$$

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For classical G, $\mathcal{O}(\pi)$ is described by its Jordan partition $\lambda(\pi)$.

Corollary

Let $\mathbf{G}$ *be* $\mathrm{GL}_{n+1}, O_{n+1}$ *or* U_{n+1} *and* $\mathbf{H}$ *be* GL_n, O_n *or* U_n *respectively. Let* $\pi \in \mathrm{Irr}(G)$, $\tau \in \mathrm{Irr}(H)$ *with* $\mathrm{Hom}_H(\pi|_H, \tau) \neq 0$. *Then for any index* i*:*

$$|\lambda(\pi)^t_i - \lambda(\tau)^t_i| \leq 1.$$

Applications to Jacquet quotients

Corollary

Let $P = LU \subset G$ be parabolic subgroup, $\pi \in \mathrm{Irr}(G)$, and $\tau \in \mathrm{Irr}(L)$ be a quotient of the Jacquet module $r_U(\pi)$. Then $\mathcal{O}(\tau) \subset \mathcal{O}(\pi)|_{\mathfrak{p}^*}$.

$$\mathfrak{l} = \mathfrak{p}/\mathfrak{u}, \quad \mathcal{O}(\tau) \subset \mathfrak{l}^* \subset \mathfrak{p}^* \twoheadleftarrow \mathfrak{g}^* \supset \mathcal{O}(\pi)$$

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Proof.

Let $H := \{(p, pU) \in G \times L\} \subset G \times L$. Then $\mathrm{Hom}_H(\pi \hat{\otimes} \tilde{\tau}, \mathbb{C}) \neq 0$. □

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For a maximal parabolic of GL_n:

$$\text{if } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{O}(\tau) \text{ then } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{O}(\pi) \text{ for some } C.$$

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Proposition (Zhang, work in progress)

$\exists C$ s. t. $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ has given Jordan partition $\mu \iff c^{\mu}_{\lambda(A),\lambda(B)} \neq 0$

(Littlewood-Richardson coefficient)

Twisted induction of spherical pairs

Let $P = LU \subset G$ parabolic subgroup, $\mathbf{S} \subset \mathbf{L}$ be a spherical subgroup. Let $H := SU$. Fix algebraic character $\psi : U \to \mathbb{R}$, and let $\chi : H \to \mathbb{C}^\times$ be a character s.t. $\chi(u) := \exp(i\psi(u))\, \forall u \in U$. Consider $d\psi \in \mathfrak{u}^* \subset \mathfrak{h}^*$.

$$\mathrm{Irr}_{H,\chi}(G) := \{\pi \in \mathrm{Irr}(G)\ : \mathrm{Hom}_H(\pi|_H, \chi) \neq 0\}$$

Theorem (G.-Sayag 2020)

For all $\pi \in \mathrm{Irr}(G)_{H,\chi}, \quad d\psi \in \mathcal{O}(\pi)|_{\mathfrak{h}}$

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Shalika models: $\mathbf{G} = \mathrm{GL}_{2n}, \mathbf{L} = \mathrm{GL}_n \times \mathrm{GL}_n$, $\mathbf{S} = \Delta\, \mathrm{GL}_n$, $\mathfrak{u} = Mat_{n\times n}$,

$$H = \left\{ \begin{pmatrix} g & A \\ 0 & g \end{pmatrix} \right\}, \ \psi(A) = \mathrm{tr}(A).$$

Corollary

For all $\pi \in \mathrm{Irr}_{H,\chi}(G)$, $\lambda(\pi)$ *consists of even parts.*

$P = LU \subset G$ parabolic , $\mathbf{S} \subset \mathbf{L}$ spherical, $H := SU, \psi : U \to \mathbb{R}$,
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- Klyachko models: $\mathbf{G} = \mathrm{GL}_{n+k}$, $\mathbf{L} = \mathrm{GL}_n \times (\mathrm{GL}_1)^k$, $\mathbf{S} = \mathrm{Sp}_n$,

$$H = \left\{ \begin{pmatrix} g & A \\ 0 & B \end{pmatrix} : g \in \mathrm{Sp}_n,\ B \text{ upper-uni-triangular} \right\}, \psi(B) = \sum B_{i,i+1}$$

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Corollary

$\forall \pi \in \mathrm{Irr}_{H,\chi}(G)$, $\#\{i : \lambda(\pi)^t_i \text{ is odd}\} = k$.

G.-Offen-Sahi-Sayag: for unitary π,

$$\pi \in \mathrm{Irr}_{H,\chi}(G) \iff \#\{i : \lambda(\pi)^t_i \text{ is odd}\} = k.$$

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Let $\pi \in \mathrm{Irr}(G), \tau \in \mathrm{Irr}(S)$ *with* $\mathrm{Hom}_H(\pi|_H, \tau \otimes \chi) \neq 0$. *Then*

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Corollary

If $\mathrm{Hom}_H(\pi|_H, \tau \otimes \chi) \neq 0$ *then* $|\lambda(\pi)^t_i - \lambda(\tau)^t_i| \leq 1$ *for any index* i.

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- Bessel models: analogues for $O(p,q), U(p,q), O_n(\mathbb{C})$.

Corollary (holds for both kinds of models: Rankin-Selberg and Bessel)

If $\mathrm{Hom}_H(\pi|_H, \tau \otimes \chi) \neq 0$ *then* $|\lambda(\pi)_i^t - \lambda(\tau)_i^t| \leq 1$ *for any index* i.

Related to the non-tempered Gan-Gross-Prasad conjecture.

Proof ingredients

Theorem (Wen-Wei Li 2019)

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Twisted version uses Kazhdan filtration + other ideas from W-algebras.

Definition of the wave front set

Assume $\mathrm{char}\, F = 0$.

Theorem (Howe, Harish-Chandra, Barbasch-Vogan, 70s)

Near $e \in G$, the character distribution (asymptotically) equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

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- Say $\mathcal{O} < \mathcal{O}'$ if $\mathcal{O} \subset \overline{\mathcal{O}'}$.
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$\forall$ nilpotent $\varphi \in \mathfrak{g}^*$, choose sl_2-triple $f, h, e \in \mathfrak{g}$ using Killing form + Jacobson-Morozov; let $\mathfrak{v}_\varphi := \mathfrak{g}^h_{\geq 2} \subset \mathfrak{g}$, $V_\varphi := \mathrm{Exp}(\mathfrak{v}_\varphi) \subset G$.

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Kazhdan action preserves φ and $G\mathfrak{h}^\perp$, contracts $(\mathfrak{g}^*)^h_{>-2}$ thus $\varphi \in \overline{G\mathfrak{h}^\perp}$ $\square$