ANALYTIC CONTINUATION OF EQUIVARIANT DISTRIBUTIONS

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Abstract. We establish a method for constructing equivariant distributions on smooth real algebraic varieties from equivariant distributions on Zariski open subsets. This is based on Bernstein's theory of analytic continuation of holonomic distributions. We use this to construct H-equivariant functionals on principal series representations of G, where G is a real reductive group and H is an algebraic subgroup. We also deduce the existence of generalized Whittaker models for degenerate principal series representations. As a special case, this gives short proofs of existence of Whittaker models on principal series representations, and of analytic continuation of standard intertwining operators. Finally, we extend our constructions to the p-adic case using a recent result of Hong and Sun.

1. Introduction

The utility of the theory of distributions in representation theory and harmonic analysis is well established since the foundational works of Bruhat and Harish-Chandra. In particular, invariant distributions provide a basic tool to study linear functionals on induced representations, maps between induced representations, and characters of infinite dimensional representations. In many of these applications, the representation theoretic question is translated to a question of existence of certain equivariant distributions on a homogeneous space, usually obtained as points over a local field of an algebraic variety. In some cases one can guess the restriction of the equivariant distribution to an open subset and would like to extend it to the entire space. Although such an extension problem is too general to decide we have found that in many interesting cases, the mere existence of an equivariant distribution on a subvariety implies the existence of such an equivariant distribution on the entire space.

To illustrate our idea we let a linear algebraic group $\mathbf{S}$ act on a smooth affine algebraic variety $\mathbf{X}$, both defined over $\mathbb{R}$. Let q be a polynomial on $\mathbf{X}$ transforming under the action of $\mathbf{S}$ by some character ψ. Define a polynomial $\bar{q}$ by $\bar{q}(x) := \overline{q(\bar{x})}$. Let $p := q\bar{q}$ and let $\mathbf{Y} := \mathbf{X}_p$ be the basic open subset defined by p. Let X, Y and S denote the real points of $\mathbf{X}, \mathbf{Y}$ and $\mathbf{S}$. Let ξ be an (S, χ)-equivariant tempered distribution on Y, *i.e.* a continuous functional on the space of Schwartz functions on Y (see §2.4 below). Since p is positive on Y, we can consider the product $p^\lambda \xi$ as a tempered distribution on Y, for any $\lambda \in \mathbb{C}$. Since p vanishes on the complement of Y, for $\operatorname{Re}\lambda$ large enough the new distributions $p^\lambda \xi$ naturally extend to the entire space X. If ξ generates a holonomic $D(\mathbf{X})$-module (see §3 below for this notion) then [Ber72] implies that this family of distributions on X has a meromorphic continuation to the entire complex plane. The obtained family η_λ is $(S, \chi|\psi|^{2\lambda})$-equivariant. This implies that the leading coefficient of the family η_λ at 0 is (S, χ)-equivariant, and the constant term η_0 is generalized (S, χ)-equivariant. In addition we have $\eta_0|_Y = \xi$. For the detailed proof see Lemma 4.1 below.

Let us provide a class of examples in which for given $\mathbf{S}, \mathbf{X}$ and $\mathbf{Y}$, an equivariant polynomial q describing $\mathbf{Y}$ always exists. Namely, we require $\mathbf{S}$ to be solvable and assume that it has an open orbit $\mathbf{O}$ on $\mathbf{X}$. Then the Lie-Kolchin theorem implies that the ideal of polynomials vanishing on the complement to $\mathbf{Y} := \mathbf{O}$ has a non-zero $\mathbf{S}$-equivariant element q. Note that in this case, the equivariant distribution ξ on $\mathbf{O}$ is necessarily a measure.

We show that it suffices to assume the existence of a measure on an arbitrary orbit, not necessary open. Since many invariant distributions arising in representation theory are measures on orbits, we obtain numerous applications, recasting many well known results under one roof.

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Theorem A. *Let a solvable linear algebraic group* $\mathbf{S}$ *act on a smooth affine algebraic variety* $\mathbf{X}$*, both defined over* $\mathbb{R}$*. Let* S *and* X *denote the real points of* $\mathbf{S}$ *and* $\mathbf{X}$*, and let* χ *be a (continuous) character of* S*. Suppose that some* S*-orbit* $O \subset X$ *admits a non-zero* (S, χ)*-equivariant tempered measure.*

Then there exists a non-zero (S, χ)*-equivariant tempered distribution on* X*.*

Here, a measure is called tempered if it defines a tempered distribution. Note that an (S, χ)-equivariant measure on O is unique up to a multiplicative constant, and it is tempered if and only if the restriction of χ to the unipotent radical of S is unitary. See §2.4 below for more details.

Example 1.1. *Let* $S := \mathbb{R}^\times$*,* $X := \mathbb{R}$ *and* $O := \mathbb{R}^\times$*. Then* O *carries an* S*-invariant measure* $\mu = dx/x$*, that extends to a generalized-invariant distribution* ξ *on* X *given by*

$$\xi(f) := \int_{x=-\infty}^{-1} f(x)x^{-1}dx + \int_{-1}^{1} (f(x) - f(0))x^{-1}dx + \int_{x=1}^{\infty} f(x)x^{-1}dx.$$

However, ξ *is not* S*-invariant. To obtain an* S*-invariant distribution on* X*, we derive* ξ *and get the delta-distribution* δ_0*. This shows that we can either obtain a generalized-invariant extension of the original distribution, or an invariant distribution, but (in general) not an invariant extension.*

To demonstrate the power of Theorem A, let G be a quasi-split real reductive group, B be a Borel subgroup, N be the unipotent radical of B, χ_1 be a character of B and χ_2 be a non-degenerate unitary character of N. Take $S := B \times N$, consider the two-sided action of S on G, let $X := G$ and $\chi := \chi_1 \otimes \chi_2$. Then Theorem A implies the existence of Whittaker models for the principal series representations of G.

We prove a generalization of Theorem A in §4 below. This generalization allows S to be an extension of a compact group by a solvable one. An example of such S is any algebraic subgroup of a minimal parabolic subgroup of a real reductive group. We also extend the result to p-adic local fields, using a recent work [HS]. Our method also works in the case of quasi-projective $\mathbf{X}$, provided that the action of $\mathbf{S}$ extends to the ambient projective space.

Another natural way to generalize Theorem A is to consider also generalized sections of bundles on X. We do that for the case when $\mathbf{X}$ is a transitive space of a group $\mathbf{G}$ that includes $\mathbf{S}$. More specifically, let F be a local field of characteristic zero, and let $\mathbf{G}$ be a linear algebraic F-group. Let $\mathbf{P}_0 \subset \mathbf{G}$ be a minimal parabolic F-subgroup. Let $\mathbf{H} \subset \mathbf{G}$ and $\mathbf{S} \subset \mathbf{P}_0$ be F-subgroups and let $\mathbf{N}$ be the unipotent radical of $\mathbf{S} \times \mathbf{H}$. Let H, S, N, G be the groups of F-points of $\mathbf{H}, \mathbf{S}, \mathbf{N}$, and $\mathbf{G}$. Consider the action of $\mathbf{S} \times \mathbf{H}$ on G given by left multiplication by $\mathbf{S}$ and right multiplication by $\mathbf{H}$. Let (σ, V) be a finite-dimensional (continuous) representation of $S \times H$.

Theorem B. *Suppose that some double coset* $SgH \subset G$ *admits a non-zero tempered* $S \times H$*-equivariant* V*-valued measure. If* F *is non-archimedean suppose in addition that the restriction* $\sigma|_N$ *is the trivial representation* $\mathbb{C}$*.*

Then there exists a non-zero $S \times H$ *-equivariant* V*-valued tempered distribution on* G*.*

Taking $\mathbf{S} = \mathbf{H} = \mathbf{P}_0$ we obtain the existence of Knapp-Stein intertwining operators [KnSt80]. More generally, by taking $\mathbf{S} = \mathbf{P}_0$ and $\mathbf{H}$ arbitrary, Theorem B can be used to construct H-invariant functionals on principal series representations of G. Namely, assume that G is reductive, let σ be a finite-dimensional representation of P_0 and let $\operatorname{Ind}_{P_0}^G(\sigma)$ denote the smooth induction. For any $g \in G$ let I_g denote the group $g^{-1}Hg \cap P_0$, and let Δ_{I_g} and Δ_H denote the modular functions of I_g and H. Define a character χ_g of I_g by $\chi_g(x) := \Delta_{I_g}(x)\Delta_H^{-1}(gxg^{-1})$.

Corollary C. *Suppose that for some* $g \in G$*,* σ *has a vector that changes under the action of* I_g *by the character* χ_g*. Then there exists an* H*-invariant continuous functional on* $\operatorname{Ind}_{P_0}^G(\sigma)$*.*

Once we show that the space W of $S \times H$-equivariant tempered distributions on G is non-zero, it is natural to ask what its dimension is. In §4 below we prove a generalization of Theorem B that bounds $\dim W$ from below by the number of $(S \times H, V)$-measurable double cosets that lie in the same $\mathbf{S} \times \mathbf{H}$-double coset (see Theorem 4.5). Our method provides no upper bounds on $\dim W$.

Another natural question that arises is whether one can extend distributions supported on an orbit, and not just measures defined on an orbit. While we do not have a general result in this direction, in Example 4.8 below we show how Theorem A can be used for this purpose.

1.1. **Applications to generalized Whittaker models.** Let G be a real reductive group. As mentioned above, Theorem A implies the well known result regarding the existence of Whittaker models for principal series representations of G [Jac67, Kos78]. More generally, we deduce from Theorem B the non-vanishing of generalized Whittaker spaces for degenerate principal series representations. In §5 we recall the notion of generalized Whittaker spaces, and prove the following theorem.

Theorem D. *Let $P \subset G$ be a parabolic subgroup and let e be an element of the nilradical of the Lie algebra of P. Let V be a finite-dimensional (continuous) representation of P, and let $\mathrm{Ind}_P^G(V)$ denote the smooth induction of V to G. Then the generalized Whittaker space $\mathcal{W}_e(\mathrm{Ind}_P^G(V))$ does not vanish.*

Note that if P is a minimal parabolic then Theorem D implies the non-vanishing of $\mathcal{W}_e(\mathrm{Ind}_P^G(V))$ for all nilpotent e in the Lie algebra of G.

1.2. **Related results.** Let G be a real reductive group, and H be a (not necessarily compact) symmetric subgroup. In [vdB88, BD92] similar constructions were performed to give H-invariant functionals on principal series and generalized principal series representations of G. A recent preprint [MOO] deals with a related problem of constructing symmetry-breaking operators. Namely, [MOO] construct H-intertwining operators from certain principal and degenerate principal series representations of G to certain (degenerate or not) principal series representations of H. Note that the space of such intertwining operators (a.k.a. *symmetry braking operators*) is isomorphic to the space of ΔH-invariant functionals on (degenerate or not) principal series representations of $G \times H$, where ΔH denotes the image of the diagonal embedding of H into $G \times H$. Thus, Corollary C extends some results of [vdB88, BD92, MOO] by considering functionals invariant under arbitrary algebraic subgroups. On the other hand, [vdB88, BD92, MOO] allow inductions from non-minimal parabolic subgroups.

The point of departure in the above-mentioned works is the theory of Knapp-Stein intertwining operators, and is thus directly based on analytic considerations. Our approach is more algebraic, and covers also spaces that do not arise from symmetric pairs, in particular non-affine homogeneous spaces.

One can view both [MOO] and the $\Delta H \subset G \times H$-case of Corollary C above as part of the general program of constructing symmetry braking operators, see [Kob] and references therein. Some of the operators are constructed in that project through their kernel distribution (as in Corollary C), while some others are given by explicit differential operators.

We remark that a special case of our key Lemma 4.1 was formulated in [GSS15, Remark 3(ii)].

Our main motivation for the study of generalized Whittaker spaces comes from [MW87], which characterizes the existence of generalized Whittaker spaces for representations of p-adic reductive groups in terms of the wave-front sets of the representations. In [Mat90] a certain partial analog of [MW87] is provided for complex reductive groups. However, for $F = \mathbb{R}$ only very partial analogs of [MW87, Mat90] are proven. We view Theorem D as one more partial result of this kind, since it establishes the existence of all the generalized Whittaker models for degenerate principal series that were expected to exist.

1.3. **Structure of the paper.** In Section §2 we collect some basic results on invariants of quasi-elementary groups and some basic facts about Schwartz distributions.

In Section §3 we collect some facts about holonomic D modules and, following Bernstein, show their utility in meromorphic continuation of families of distributions.

In Section §4 we prove a key result concerning extension of equivariant distributions (Lemma 4.1) and deduce from it generalizations of Theorems A and B. We also discuss the possibilities of further generalizations. Finally, we deduce Corollary C from Theorem B.

In §5 we recall the notion of generalized Whittaker spaces and prove a generalization of Theorem D.

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2. Preliminaries

Let F be a local field of characteristic zero. We will denote algebraic varieties and algebraic groups defined over F by bold letters and their F-points by the corresponding letters in regular font.

For a representation V of a group G we denote by V^G the space of invariants.

Definition 2.1. *Let (π, V) be a representation of a group G. A vector $v \in V$ is called a generalized invariant vector if there is a natural number k such that*

$$(\pi(g_0) - \mathrm{Id})(\pi(g_1) - \mathrm{Id}) \cdots (\pi(g_k) - \mathrm{Id})v = 0 \quad \forall\, g_0, g_1, \dots, g_k \in G.$$

Note that if G is a connected Lie group and π is a smooth representation then v is generalized invariant if it only if the Lie algebra of G acts nilpotently on the subrepresentation generated by v.

2.1. Meromorphic families.

Let $\mathbb{C}((\lambda))$ denote the field of Laurent power series, and let E be a complex vector space. For any real $a > 0$ define $a^\lambda := \sum_{i\geq 0} (\ln a)^i/(i!)\lambda^i \in \mathbb{C}((\lambda))$.

Lemma 2.2. *Let a group G act on E linearly, and let ψ be a character of G. Extend this action to $E \otimes_{\mathbb{C}} \mathbb{C}((\lambda))$ in the natural way. Let*

$$f = \sum_{i=-n}^{\infty} a_i \lambda^i \in E \otimes_{\mathbb{C}} \mathbb{C}((\lambda)) \text{ satisfy } gf = |\psi(g)|^\lambda f.$$

Then a_{-n} is G-invariant. Moreover, a_{-n+l} is generalized G-invariant for any $l \geq 0$.

Proof. From $gf = |\psi(g)|^\lambda f$, comparing term by term, we obtain

$$ga_i = \chi(g) \sum_{j=-n}^{i} (\ln |\psi(g)|^{i-j}/(i!))a_j \tag{1}$$

This implies that a_{-n} is G-invariant. The "moreover" part follows by induction on l. □

Definition 2.3. *For $f = \sum_{i=-n}^{\infty} a_i\lambda^i \in E \otimes_{\mathbb{C}} \mathbb{C}((\lambda))$ with $a_{-n} \neq 0$ we say that a_{-n} is the leading coefficient and a_0 is the constant term.*

Lemma 2.4. *Let $k \subset \mathbb{C}((\lambda))$ be a subfield that contains λ. Let $L \subset E \otimes_{\mathbb{C}} k$ be a finite-dimensional k-vector space, and $W \subset E$ be the $\mathbb{C}$-subspace given by the leading coefficients at 0 of the series in L. Then*

$$\dim_{\mathbb{C}} W = \dim_k L.$$

Proof. Let $l := \dim_k L$. Let us first show that $\dim W \leq l$. Indeed, any $l + 1$ vectors in W are leading coefficients of some $l+1$ vector series in L that are linearly dependent. Their dependence induces a linear dependence of their leading coefficients.

Let us now show that $\dim_{\mathbb{C}} W = l$. Since $\lambda \in k$, we can choose a basis $w_1, \dots, w_n$ for W that consists of constant terms of vectors $v_1, \dots, v_n$ in L with no poles or zeros at 0. Complete $v_1, \dots, v_n$ to a set $v_1, \dots, v_m$ that spans L (over k) and consists of vectors with no poles and no zeros at 0. Let $A \subset k$ denote the subalgebra of all series in k with no poles at 0. Then the $\{v_i\}_{i=1}^m$ generate an A-module M, and $\{v_i\}_{i=1}^n$ generate a submodule N. Since the zero terms of $\{v_i\}_{i=1}^n$ span W, we have $\lambda(M/N) = M/N$. Since λ generates the only maximal ideal of A, Nakayama's lemma implies that $M/N = 0$, hence $M = N$ and thus L is spanned by $\{v_i\}_{i=1}^n$. Thus $n = l$. □

2.2. Semi-invariants of quasi-elementary groups.

Proposition 2.5 ([Pop14, 1(A)(2) and Proposition 2.6]). *Let $\mathbf{X}$ be an irreducible F-variety that has a smooth F-point. Then $\mathbf{X}$ is F-dense, i.e. the set of F-points in $\mathbf{X}$ is (Zariski) dense in $\mathbf{X}$.*

Let $\mathbf{Q}$ be a linear algebraic F-group. Following [KK, §3] we say that $\mathbf{Q}$ is quasi-elementary if it does not contain a proper parabolic F-subgroup. Actually, [KK, §3] give a different definition, but explain why it is equivalent to the one given here. Note that all solvable groups are quasi-elementary and if $F = \mathbb{R}$ then so are all the groups with compact-by-solvable $\mathbb{R}$-points.

Assume that $\mathbf{Q}$ is quasi-elementary and let $\mathbf{Q}$ act on an affine F-variety $\mathbf{X}$.

Proposition 2.6 ([KK, Proposition 3.10]). *Suppose that $\mathbf{X}$ is F-dense. Let $I \subset \mathcal{O}(\mathbf{X})$ be a non-zero $\mathbf{Q}$-stable ideal. Then I contains a non-zero $\mathbf{Q}$-equivariant polynomial defined over F.*

Corollary 2.7. *Let $\mathbf{O}$ be an open orbit for $\mathbf{Q}$ on $\mathbf{X}$ that has an F-point. Then $\mathbf{O}$ is a basic open subset of $\mathbf{X}$ defined by a $\mathbf{Q}$-equivariant polynomial defined over F.*

Proof. Suppose first that $\mathbf{X}$ is irreducible. Then Proposition 2.5 implies that $\mathbf{X}$ is F-dense. Let I be the ideal defining $\mathbf{X} \setminus \mathbf{O}$. By Proposition 2.6, I contains a non-zero $\mathbf{Q}$-equivariant polynomial p. Then $p(x) \neq 0$ for some $x \in \mathbf{O}$ and thus for all $x \in \mathbf{O}$.

For general $\mathbf{X}$, let $\mathbf{Z}$ be the irreducible component that includes $\mathbf{O}$, and $\mathbf{Y}$ be the union of all other components. Let p be the polynomial on $\mathbf{Z}$ that defines $\mathbf{O}$. Then p vanishes on $\mathbf{Z} \cap \mathbf{Y}$ and thus there exists a polynomial q on $\mathbf{X}$ that equals some power of p on $\mathbf{Z}$ and vanishes on $\mathbf{Y}$. Clearly q is $\mathbf{Q}$-equivariant. $\square$

Example 2.8. *Let $\mathbf{G}$ be a reductive group defined and quasi-split over F, $\mathbf{B}$ be its Borel subgroup, and consider the two-sided action of $\mathbf{Q} := \mathbf{B} \times \mathbf{B}$ on $\mathbf{X} := \mathbf{G}$. Let us describe the polynomial p that defines the open Bruhat cell in $\mathbf{G}$.*

Let $\mathfrak{n}$ be the nilradical of the Lie algebra of $\mathbf{B}$ and $\bar{\mathfrak{n}}$ be the nilradical of the Lie algebra of the opposite Borel subgroup $\bar{B}$. Let $d := \dim \mathfrak{n} = \dim \bar{\mathfrak{n}}$. Let $\mathfrak{g}$ denote the Lie algebra of $\mathbf{G}$. Let $v \in \Lambda^d(\mathfrak{n}) \subset \Lambda^d(\mathfrak{g})$ and $\bar{v} \in \Lambda^d(\bar{\mathfrak{n}}) \subset \Lambda^d(\mathfrak{g})$ be non-zero vectors. Let (σ, V) and $(\bar{\sigma}, \bar{V})$ be the subrepresentations of $\Lambda^d(\mathfrak{g})$ generated by v and $\bar{v}$. Then V and $\bar{V}$ are irreducible and contragredient to each other, v is the highest weight vector of V and $\bar{v}$ is the lowest weight vector of $\bar{V}$. They define a matrix coefficient function $p(g) := \langle \bar{v}, \sigma(gw)v \rangle$, where w is a representative of the longest Weyl group element.

Note that p is $\mathbf{Q}$-equivariant, $p(1) \neq 0$ and that p vanishes on all the Weyl group elements except the longest one. Thus the zero set of p is precisely $\mathbf{G} - \mathbf{BwB}$.

Corollary 2.9. *Let $\mathbf{G}$ be a linear algebraic F-group and let $\mathbf{Q}, \mathbf{H}$ be F-subgroups, where $\mathbf{Q}$ is quasi-elementary. Let $g \in G$. Let $\mathbf{Q} \times \mathbf{H}$ act on $\mathbf{G}$ by left multiplication by $\mathbf{Q}$ and right multiplication by $\mathbf{H}$. Then there exists a non-zero $\mathbf{Q} \times \mathbf{H}$-equivariant F-polynomial q on the (Zariski) closure $\mathbf{Z}$ of $\mathbf{Q}g\mathbf{H}$ that vanishes outside $\mathbf{Q}g\mathbf{H}$.*

Proof. By Chevalley's theorem (see e.g. [Bor87, Chapter II, Theorem 5.1]) there exist an algebraic representation $\mathbf{W}$ of $\mathbf{G}$ and a line $\mathbf{D} \subset \mathbf{W}$ both defined over F such that $\mathbf{H} = \{x \in \mathbf{G} \mid x\mathbf{D} = \mathbf{D}\}$.

Let $\mathbf{X}$ denote the closure of $\mathbf{Q}g\mathbf{D}$ in $\mathbf{W}$. By Proposition 2.5, $\mathbf{X}$ is F-dense. Note that $\mathbf{Q}g\mathbf{D}$ is open in $\mathbf{X}$ and let I be the ideal of all polynomials on $\mathbf{X}$ that vanish outside $\mathbf{Q}g\mathbf{D}$. By Proposition 2.6 I contains a non-zero $\mathbf{Q}$-equivariant polynomial p'. Let p be the leading homogeneous term of p'. Note that p is $\mathbf{Q}$-equivariant as well and $p \in I$. Let $v \in \mathbf{D}$ be a non-zero F-vector and define a map $a : \mathbf{G} \to \mathbf{W}$ by $a(x) := xv$. Then $a^{-1}(\mathbf{QgD}) = \mathbf{QgH}$ and thus $a(\mathbf{Z}) = \mathbf{X}$. Define q on $\mathbf{Z}$ by $q := p \circ a$. Note that q is non-zero, $\mathbf{Q} \times \mathbf{H}$-equivariant and vanishes outside $\mathbf{QgH}$. $\square$

2.3. **Equivariant distributions on l-spaces.** For non-archimedean F we will consider distributions on l-spaces, *i.e.* locally compact totally disconnected Hausdorff topological spaces. This generality includes F-points of algebraic varieties defined over F (see [BZ76]). For an l-space X, the space $\mathcal{S}(X)$ of test functions consists of locally constant compactly supported functions and the space of distributions is $\mathcal{S}^*(X)$, the full linear dual. All distributions on l-spaces are tempered. In this subsection we assume that F is non-archimedean and let $\mathbf{G}$ be a linear algebraic group defined over F. Let χ be a continuous character of G. A generalized χ-equivariant distribution on X is defined to be a generalized invariant vector in the representation $\mathrm{Hom}_{\mathbb{C}}(\mathcal{S}(X), \chi)$ of G. Our main tool in the non-archimedean case is the following special case of [HS, Theorem 1.5].

Theorem 2.10. *Let $\mathbf{X}$ be an algebraic variety defined over F such that $\mathbf{G}$ acts algebraically on $\mathbf{X}$ with an open orbit $\mathbf{U} \subset \mathbf{X}$. Assume that there is a $(\mathbf{G}, \psi)$-equivariant regular function f on $\mathbf{X}$ (for some character ψ of $\mathbf{G}$) such that*

$$\mathbf{U} = \mathbf{X}_f = \{x \in \mathbf{X} \textit{ with } f(x) \neq 0\}.$$

Assume also that χ vanishes on the F-points of the unipotent radical of $\mathbf{G}$. Then every generalized χ-equivariant distribution ξ on $\mathbf{U}(F)$ extends to a generalized χ-equivariant distribution on $\mathbf{X}(F)$. Moreover, there exists a meromorphic family η_λ of $(G, \chi|\psi|^\lambda)$-equivariant distributions such that the constant term η_0 of this family at 0 extends ξ.

The "moreover" part is not formulated in [HS, Theorem 1.5] but rather follows from the proof. More precisely, it follows from [HS, Propositions 5.20 and 6.22].

2.4. **Schwartz functions and tempered distributions on real algebraic manifolds.** Let $\mathbf{X}$ be an algebraic manifold (*i.e.* smooth algebraic variety) defined over $\mathbb{R}$ and $X := \mathbf{X}(\mathbb{R})$. If $\mathbf{X}$ is affine then the Fréchet space $\mathcal{S}(X)$ of Schwartz functions on X consists of smooth complex valued functions that decay, together with all their derivatives, faster than any polynomial. This is a Fréchet space, with the topology given by the system of seminorms $|\phi|_d := \max_{x\in X} |df|$, where d runs through all differential operators on X with polynomial coefficients.

For a Zariski open affine subset $\mathbf{U} \subset \mathbf{X}$, the extension by zero of a Schwartz function on U is a Schwartz function on X. This enables to define the Schwartz space on any algebraic manifold $\mathbf{X}$, as the sum of the Schwartz spaces of the open affine pieces, extended by zero to functions on X. For the precise definition of this notion see e.g. [AG08]. Elements of the dual space $\mathcal{S}^*(X)$ are called tempered distributions. The spaces $\mathcal{S}^*(U)$ for all Zariski open $\mathbf{U} \subset \mathbf{X}$ form a sheaf. We say that a measure is tempered if it defines a tempered distribution.

For a finite-dimensional complex vector space V we define the space $\mathcal{S}^*(X,V)$ of V-valued tempered distributions as the space of all continuous linear maps from $\mathcal{S}(X)$ to V. Note that $\mathcal{S}^*(X,V) \simeq \mathcal{S}^*(X)\otimes V$. If a group G acts on X and on V then we consider the diagonal action on $\mathcal{S}^*(X,V)$ and denote the space of invariants by $\mathcal{S}^*(X,V)^G$. We call the elements of this space *equivariant distributions.*

Let $\mathbf{U} \subset \mathbf{X}$ be a Zariski open subset and let Z denote the complement to U in X.

Theorem 2.11 ([AG08, Theorem 4.6.1 and §5.3])**.** *The restriction to Z defines an epimorphism $\mathcal{S}(X) \twoheadrightarrow \mathcal{S}(Z)$.*

Dualizing the map $\mathcal{S}(X) \twoheadrightarrow \mathcal{S}(Z)$ we obtain an embedding $\mathcal{S}^*(Z) \hookrightarrow \mathcal{S}^*(X)$. We call this map extension of distributions by zero.

Theorem 2.12 ([AG08, Theorem 5.4.3])**.** *We have*

$$\mathcal{S}(U) \cong \{\phi \in \mathcal{S}(X)| \quad \phi \text{ is 0 on } Z \text{ with all derivatives}\}.$$

In particular, extension by zero defines a closed imbedding $\mathcal{S}(U) \hookrightarrow \mathcal{S}(X)$.

Corollary 2.13. *The restriction map $\mathcal{S}^*(X) \to \mathcal{S}^*(U)$ is onto.*

Remark 2.14. *Note that this corollary does not hold for arbitrary distributions. For example, the distribution $e^x dx$ does not extend from $\mathbb{R}$ to $\mathbb{R}P^1$. Indeed, since $\mathbb{R}P^1$ is compact, any distribution on it is tempered and therefore restricts to a tempered distribution on $\mathbb{R}$.*

Let us record one more corollary of this theorem. Let $\mathbf{G}$ be an algebraic group defined over $\mathbb{R}$, and V be a finite-dimensional representation of $G := \mathbf{G}(\mathbb{R})$.

Corollary 2.15. *Let $\mathbf{G}$ act on $\mathbf{X}$ algebraically let $\mathbf{U} \subset \mathbf{X}$ be a $\mathbf{G}$-invariant Zariski open subset and $U := \mathbf{U}(\mathbb{R})$. Let $\xi \in \mathcal{S}^*(U,V)^G$. Then for some natural number n, ξ extends to a G-intertwining operator $\xi' : F^n(X,U) \to V$, where $F^n(X,U)$ is the space of Schwartz functions on X that vanish on Z with first n derivatives.*

Moreover, there exist (n,ξ') as above such that for any differential operator d on $\mathbf{X}$ with polynomial coefficients satisfying $\xi \circ d = 0$, ξ' vanishes on $d(F^{n+\deg d}(X,U))$, where $\deg d$ denotes the degree of d.

Proof. We can assume that $\mathbf{X}$ is affine. Then the Fréchet space $\mathcal{S}(X)$ is the inverse limit of the Fréchet spaces $\mathcal{S}^n(X)$ consisting of n times differentiable functions that decay rapidly at infinity together with their first n derivatives. Note that $\mathcal{S}(X)$ is dense in each of these spaces. Denote by $\mathcal{S}^n(X,U)$ the closed subspace of $\mathcal{S}^n(X)$ consisting of functions that vanish on Z with first n derivatives. Then both $F^n(X,U)$ and $\mathcal{S}(U)$ are dense in $\mathcal{S}^n(X,U)$, and by Theorem 2.12 we have $\mathcal{S}(U) = \varprojlim \mathcal{S}^n(X,U)$. Thus ξ extends to a continuous linear map $\tilde{\xi} : \mathcal{S}^n(X,U) \to V$ for some n. Note that the action of G preserves all the spaces mentioned above. Thus the equivariance of $\tilde{\xi}$ follows from the equivariance of ξ and the density of $\mathcal{S}(U)$ in $\mathcal{S}^n(X,U)$. To obtain ξ' we restrict $\tilde{\xi}$ to $F^n(X,U)$. To prove the moreover part, note that $\xi(d(\mathcal{S}(U))) = 0$ and the density of $\mathcal{S}(U)$ in $\mathcal{S}^{n+\deg d}(X,U)$ implies that $\tilde{\xi}(d(\mathcal{S}^{n+\deg d}(X,U))) = 0$. Restricting to $F^{n+\deg d}(X,U)$ we deduce the vanishing of $\xi' \circ d$ on this space. $\square$

Remark 2.16. *More generally, one can define Schwartz sections of Nash (=smooth semi-algebraic) bundles on Nash manifolds. Theorem 2.12 is proven in* [AG08] *in this generality, and Corollary 2.15 stays true in this wider generality with identical proof.*

2.4.1. *Finite-dimensional representations of moderate growth.* A smooth function f on X is called tempered if for any open affine $\mathbf{U} \subset \mathbf{X}$ and any algebraic differential operator d on $\mathbf{U}$, there exists a polynomial p on $\mathbf{U}$ such that $|d(f|_U)(x)| < p(x)$ for any $x \in U$.

Let $\mathbf{G}$ be an algebraic group defined over $\mathbb{R}$ and $G := \mathbf{G}(\mathbb{R})$. We say that a character χ of G has moderate growth if it is tempered as a function on G. All unitary characters have moderate growth. If $\mathbf{G}$ is a unipotent group, then a character χ has moderate growth if and only if χ is unitary. If $\mathbf{G}$ is reductive, then all continuous characters have moderate growth. These statements reduce to the case of one-dimensional groups, and in this case they are straightforward. We say that a finite-dimensional representation has moderate growth if all its matrix coefficients are tempered.

The following lemma is standard.

Lemma 2.17. *Let V be a (continuous) finite-dimensional representation of G. Let $\mathbf{H} \subset \mathbf{G}$ be a Zariski closed subgroup. Let Δ_G and Δ_H denote the modular functions of G and H. Then the space of G-invariant V-valued measures on G/H is isomorphic to $(V \otimes \Delta_G \otimes \Delta_H^{-1})^H$. Moreover, if V has moderate growth then all G-invariant V-valued measures on G/H are tempered.*

For the next two lemmas assume that $\mathbf{G}$ is reductive and let $P \subset G$ be a parabolic subgroup. Let N be the unipotent radical of P and $\mathfrak{n}$ be the Lie algebra of N.

Lemma 2.18. *Let V be a (continuous) finite-dimensional representation of P. Then*
(i) The action of $\mathfrak{n}$ on V is nilpotent.
(ii) V has moderate growth.

Proof. (i) Note that there exists a hyperbolic semi-simple $S \in \mathfrak{p}$ such that $\mathfrak{p} = \bigoplus_{\lambda \geq 0} \mathfrak{g}(\lambda)$, where $\mathfrak{g}(\lambda)$ denotes the λ-eigenspace of the adjoint action of S. Then $\mathfrak{n} = \bigoplus_{\lambda > 0} \mathfrak{g}(\lambda)$. Decomposing V to generalized eigenspaces of S and using the finiteness of the dimension we obtain that $\mathfrak{n}$ acts nilpotently.

(ii) We can assume that V is irreducible. By (i) this implies that N acts trivially on V, and thus the reductive quotient P/N acts on V. Thus $V = W \otimes \chi$, where W is an algebraic representation of P/N and χ is a character of P/N. Since both W and χ are tempered, so is V. $\square$

Let $\operatorname{Ind}_P^G(V)$ denote the smooth induction, and let $(\operatorname{Ind}_P^G(V))^*$ denote the continuous linear dual.

Lemma 2.19 (See e.g. [GSS15, Lemma 6])**.** *We have a natural isomorphism of G - representations*

$$(\operatorname{Ind}_P^G(V))^* \cong (C^\infty(G, V)^P)^* \cong \mathcal{S}^*(G, V \otimes \Delta_P^{-1})^P,$$

where G acts on $C^\infty(G, V)$ and $\mathcal{S}^*(G, V \otimes \Delta_P^{-1})$ from the left and P from the right.

3. Preliminaries on holonomic D-modules and distributions

We will now recall some facts and notions from the theory of D-modules on smooth affine algebraic varieties. For the proofs and for further details we refer the reader to [Ber72, Bor87, HTT08].

By a D-module on a smooth affine algebraic variety $\mathbf{X}$ we mean a module over the algebra $D(\mathbf{X})$ of differential operators. The algebra $D(\mathbf{X})$ is equipped with a filtration, defined by the order of differential operators. This filtration is called the geometric filtration. The associated graded algebra with respect to this filtration is the algebra $\mathcal{O}(T^*\mathbf{X})$ of regular functions on the total space of the cotangent bundle of $\mathbf{X}$. This implies that the algebra $D(\mathbf{X})$ is Noetherian.

This allows us to define the singular support of a finitely generated D-module M on $\mathbf{X}$ in the following way. Choose a good filtration on M, *i.e.* a filtration compatible with the filtration on $D(\mathbf{X})$ such that the associated graded module is a finitely-generated module over $\mathcal{O}(T^*\mathbf{X})$. Define the singular support $SS(M)$ to be the support of the associated graded module. One can show that the singular support does not depend on the choice of a good filtration on M, and that a good filtration exists if and only if M is finitely generated.

The Bernstein inequality states that, for any non-zero finitely generated M, we have $\dim SS(M) \geq \dim X$. If the equality holds then M is called holonomic. This property is closed under submodules, quotients and extensions, and implies finite length.

For any finite-dimensional vector space V, the space $\mathcal{S}^*(X,V)$ of tempered V-valued distributions has a structure of a right $D(\mathbf{X})$-module, given by $(\xi d)(f) = \xi(df)$, where $d \in D(\mathbf{X})$, $X = \mathbf{X}(\mathbb{R})$, $\xi \in \mathcal{S}^*(X,V)$ and $f \in \mathcal{S}(X)$. A distribution $\xi \in \mathcal{S}^*(X,V)$ is called holonomic if the submodule $\xi D(\mathbf{X}) \subset \mathcal{S}^*(X,V)$ generated by ξ is holonomic. Note that if ξ is holonomic then so is ξp, for any polynomial p on $\mathbf{X}$.

Lemma 3.1 (See *e.g.* [Bor87, Theorem VI.7.11] and [AG09, Facts 2.3.8 and 2.3.9])**.** *Let $\mathbf{Z} \subset \mathbf{X}$ be a closed smooth subvariety, let $\xi \in \mathcal{S}^*(Z,V)$, and let $\eta \in \mathcal{S}^*(X,V)$ be the extension of ξ to X by zero. Then*

(i) η is holonomic if and only if ξ is holonomic.

(ii) Let an algebraic group $\mathbf{G}$ act transitively on $\mathbf{Z}$, and its $\mathbb{R}$-points G act linearly on V. Suppose that ξ is G-equivariant. Then ξ is holonomic.

For any polynomial $p \in \mathcal{O}(\mathbf{X})$, the algebra $D(\mathbf{X}_p)$ of differential operators on the basic open affine set $\mathbf{X}_p := \{x \in \mathbf{X} \text{ with } p(x) \neq 0\}$ is isomorphic to the localization $D(\mathbf{X})_p$, i.e. the algebra of fractions of the form dp^{-i}. In order to define the latter algebra one proves that the family p^i satisfies Ore conditions. This follows from the next lemma.

Lemma 3.2. *For any index n and any $d \in D(\mathbf{X})$ there exists $d' \in D(\mathbf{X})$ such that $p^n d' = dp^{n+\deg d}$.*

This lemma is proven by induction on $\deg d$.

Theorem 3.3 (See *e.g.* [Bor87, §VI.5.2 and Theorem VII.10.1])**.** *Let $p \in \mathcal{O}(\mathbf{X})$ and let M be a holonomic module over $D(\mathbf{X}_p)$. Then M is holonomic also as a module over $D(\mathbf{X})$.*

In the case when $\mathbf{X}$ is an affine space, another natural filtration on $D(\mathbf{X})$ is possible. This filtration is called the arithmetic filtration, or the Bernstein filtration. It leads to a different definition of singular support and thus could a priori lead to a different notion of a holonomic module. However, the two definitions of holonomicity are equivalent, since both are equivalent to a certain homological property, see [Bor87, §V.2].

Let p be a polynomial on $\mathbf{X}$ with real coefficients, which takes non-negative values on $X := \mathbf{X}(\mathbb{R})$. For any $N \in \mathbb{R}$ denote $\mathbb{C}_{>N} := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > N\}$. For any $\lambda \in \mathbb{C}_{>0}$ denote by p^λ the function on X given by 0 on the zeros of p and by $p(x)^\lambda$ elsewhere. Note that for any natural number N and any $\lambda \in \mathbb{C}_{>N}$, p^λ is N times differentiable. Thus, for any $\xi \in \mathcal{S}^*(X,V)$ there exists N such that for any $\lambda \in \mathbb{C}_{>N}$, the distribution ξp^λ is well defined. For any open subset $\Lambda \subset \mathbb{C}$ denote by $\mathcal{S}^*_\Lambda(X,V)$ the space of meromorphic families of distributions in $\mathcal{S}^*(X,V)$ parameterized by $\lambda \in \Lambda$. We regard the functions ξ_λ and ξ'_λ as defining the same element in $\mathcal{S}^*_\Lambda(X,V)$ if they agree in some open subset of Λ.

Our main tool is the following theorem, essentially proven in [Ber72].

Theorem 3.4. *Let $\xi \in \mathcal{S}^*(X,V)$ be a holonomic distribution. Then the family $\xi_\lambda \in \mathcal{S}^*_{\mathbb{C}_{>0}}(X)$ defined by $\xi_\lambda = \xi p^\lambda$ for $\lambda \in \mathbb{C}_{>0}$ has a meromorphic continuation to the entire complex plane. Moreover, all the distributions in the extended family and all the Laurent coefficients at any $\lambda \in \mathbb{C}$ are holonomic.*

Proof. Note that we can assume $V = \mathbb{C}$. Then, for the case when X is the affine space $\mathbb{A}^n$, the theorem is [Ber72, Corollary 4.6 and Proposition 4.2(3)]. For the general case, consider a closed embedding $i : \mathbf{X} \hookrightarrow \mathbb{A}^n$ and extend ξ to a distribution η on $\mathbb{R}^n$ by $\eta(f) := \xi(f \circ i)$. Also, extend p to a polynomial p' on $\mathbb{R}^n$ with real coefficients. By Lemma 3.1 η is holonomic, and thus the family $\eta_\lambda \in \mathcal{S}^*_{\mathbb{C}_{>0}}(\mathbb{R}^n)$ defined by $\eta_\lambda = \eta((p')^2)^{\lambda/2}$ has a meromorphic continuation to the entire complex plane.

Note that for any polynomial $q \in \mathcal{O}(\mathbb{R}^n)$ that vanishes on X and $\lambda \in \mathbb{C}_{>0}$ we have $\eta_\lambda q = 0$. Thus this holds for all $\lambda \in \mathbb{C}$ and thus the family η_λ can be restricted to a family $\xi_\lambda \in \mathcal{S}^*_{\mathbb{C}}(X)$. The family η_λ and all its Laurent coefficients at any $\lambda \in \mathbb{C}$ are holonomic, and by Lemma 3.1 the same holds for the family ξ_λ and all its Laurent coefficients. ☐

4. Main results

Let a linear algebraic group $\mathbf{Q}$ act (algebraically) on an affine algebraic variety $\mathbf{X}$, both defined over F. Let (σ, V) be a (continuous) finite-dimensional representation of Q. For any polynomial $p \in \mathcal{O}(\mathbf{X})$ we denote by $\mathbf{X}_p$ the basic open subset

$$\mathbf{X}_p := \{x \in \mathbf{X} \text{ with } p(x) \neq 0\}.$$

Lemma 4.1. *Assume that $F = \mathbb{R}$ and $\mathbf{X}$ is smooth. Let $p \in \mathcal{O}(\mathbf{X})$ be real-valued on X and let $U := \mathbf{X}_p(\mathbb{R})$. Assume that U is Q-invariant. Let $\xi \in \mathcal{S}^*(U, V)$ be holonomic. Let ψ be an algebraic character of Q and assume that ξp^n is $(Q, \psi^n\sigma)$-equivariant for any $n \geq 0$.*

Then ξ can be extended to a holonomic generalized G-invariant $\eta \in \mathcal{S}^*(X, V)$. Moreover, there exists a meromorphic $(Q, |\psi|^\lambda\sigma)$-equivariant holonomic family $\eta_\lambda \in \mathcal{S}^*(X, V)$ such that $\eta_\lambda|_U = \xi|p|^\lambda$ and the constant term of η_λ at zero is η.

Proof. Replacing p by p^2 if needed we assume that p is non-negative on X, and thus ψ is positive on Q.

By Corollary 2.15, ξ extends to a (Q, σ)-equivariant functional ξ' on

$$F^n(X, U) = \{\phi \in \mathcal{S}(X, V) | \, \phi \equiv 0 \text{ on } X \setminus U \text{ with first } m \text{ derivatives}\},$$

for some n, such that for any $d \in D(\mathbf{X})$ we have

$$\text{if } \xi d = 0 \text{ then } \xi'(d(F^{n+\deg d}(X, U))) = 0. \tag{2}$$

Define $\eta_n \in \mathcal{S}^*(X, V)$ by $\eta_n(\phi) := \xi'(p^n\phi)$. Note that η_n is $(Q, \psi^n\sigma)$ - equivariant.

Let us show that η_n is a holonomic distribution. Let I be the annihilator ideal of ξ in $D(\mathbf{X})$, i.e. the ideal of all $d \in D(\mathbf{X})$ with $\xi(d|_{\mathbf{U}}) = 0$. Let $d_1, \dots d_l$ be a finite set of generators of I Let J denote the annihilator ideal of η_n in $D(\mathbf{X})$. By Lemma 3.2, for any $i \leq l$ we can find $d_i' \in D(\mathbf{X})$ such that $d_i p^{n+\deg d_i} = p^n d_i'$. Then for any $\phi \in \mathcal{S}(X, V)$ we have

$$(\eta_n d_i')(\phi) = \eta_n(d_i'(\phi)) = \xi'(p^n d_i'(\phi)) = \xi'(d_i(p^{n+\deg d_i}\phi)).$$

Since $p^{n+\deg d_i} f \in F^{n+\deg d_i}(X, U)$, from (2) we have $d_i' \in J$. Thus the localization J_p of J includes $p^{-n}d_i$ for all i. Note that $\{p^{-n}d_i\}$ generate the ideal $p^{-n}ID(\mathbf{U})$, which is the annihilator of ξp^n in $D(\mathbf{U})$. Since ξp^n is holonomic, we get that $D(\mathbf{U})/J_p$ is holonomic. Now, $D(\mathbf{U})/J_p = M_p$, where $M := \eta_n D(\mathbf{X})$. Thus M_p is holonomic, and Theorem 3.3 implies that M is holonomic and thus so is η_n.

Consider the analytic family of equivariant distributions $\eta_\lambda := \eta_n p^{\lambda - n}$ defined for $\operatorname{Re}\lambda$ big enough. It is easy to see that this family is $(Q, \psi^\lambda\sigma)$-equivariant. By Theorem 3.4 the family η_λ has a meromorphic continuation to the entire complex plane. Now, define η to be the constant term of this family. Note that $\eta|_U = p^0\xi = \xi$. By Lemma 2.2, η is generalized Q-equivariant. $\square$

Remark 4.2. *The distribution η gives rise to Q-equivariant distributions on X. However, the restrictions of these distributions to U might vanish. E.g. in the situation of Example 1.1, all Q-invariant distributions on X are supported at the origin. Note also that the temperedness assumption on ξ is necessary. For example, the $\mathbb{R}$-equivariant measure $e^x dx$ on $\mathbb{R}$ does not extend to $\mathbb{R}P^1$.*

Let us now formulate and prove our main results. Let $\mathbf{N_Q}$ denote the unipotent radical of $\mathbf{Q}$.

Theorem 4.3. *Assume that $\mathbf{Q}$ is quasi-elementary. Assume that there exists a Q-orbit $O \subset X$ that admits a tempered Q-equivariant V-valued measure μ. Suppose also that one of the following two conditions holds:*

(a) F is non-archimedean, and the restriction $\sigma|_{N_Q}$ is the trivial representation $\mathbb{C}$.
(b) $F = \mathbb{R}$ and X is smooth.

Then there exists a generalized Q-equivariant $\eta \in \mathcal{S}^*(X, V)$ and a Zariski open Q-invariant neighborhood U of O such that the restriction of η to U equals the extension of μ to U by zero.

Proof. By Corollary 2.7, there exists an algebraic character ψ of $\mathbf{Q}$ and a non-zero $(\mathbf{Q}, \psi)$-equivariant F-polynomial q on the Zariski closure $\mathbf{Z}$ of O in $\mathbf{X}$. Let μ denote the non-zero equivariant measure on O.

If F is non-archimedean and $\sigma|_{R_Q}$ is trivial then, by Theorem 2.10, μ can be extended to a generalized Q-equivariant V-valued distribution on $Z := \mathbf{Z}(F)$. Extending this distribution by zero we obtain a generalized Q-equivariant $\eta \in \mathcal{S}^*(X)$.

Now assume that $F = \mathbb{R}$ and X is smooth. Lift q to a polynomial p on $\mathbf{X}$ and let $U := \mathbf{X}_p(\mathbb{R})$. Note that O is closed in U and extend μ by zero to $\xi \in \mathcal{S}^*(U)$. Note that $p^n\xi$ is $(Q, \psi^n\sigma)$-equivariant for any

$n \geq 0$. By Lemma 3.1 ξ is holonomic. Applying Lemma 4.1 we obtain a generalized (Q,σ)-equivariant extension of ξ to X. $\Box$

Remark 4.4. *In Theorem A one can replace the assumption that $\mathbf{X}$ is affine by the weaker assumption that $\mathbf{X}$ is a locally closed subset of the projective space $\mathbb{P}(\mathbf{W})$, for some algebraic representation $\mathbf{W}$ of $\mathbf{Q}$. This condition holds for example if $\mathbf{X}$ is a simple spherical variety of some linear algebraic F-group $\mathbf{G}$ that contains $\mathbf{Q}$, see* [Sum74, Theorem 2.3.1].

To prove the theorem in this generality one first of all generalizes Theorem 3.4 and Lemma 4.1 to the case when $\mathbf{X}$ is an arbitrary smooth variety and p is a globally defined regular function on $\mathbf{X}$. These extensions follow from the versions proven here by the uniqueness of analytic continuation and the sheaf property of tempered distributions. Next we let $\mathbf{X}' := pr^{-1}(\mathbf{X}) \subset \mathbf{W}$, where $pr : \mathbf{W} \setminus \{0\} \to \mathbb{P}(\mathbf{W})$ is the natural projection. Then we have an isomorphism of Q-representations

$$\mathcal{S}^*(X,V) \simeq \mathcal{S}^*(X',V)^{F^\times}. \tag{3}$$

Let $\mathbf{Q}' := \mathbf{Q} \times \mathbf{GL_1}$. Note that $\mathbf{Q}'$ is quasi-elementary and $\mathbf{X}'$ is $\mathbf{Q}'$-invariant. Let $O' := pr^{-1}(O)$ and let μ' be the (Q',χ)-equivariant measure on O corresponding to μ. By Corollary 2.7, there exists an algebraic character ψ of $\mathbf{Q}'$ and a non-zero $(\mathbf{Q}',\psi)$-equivariant F-polynomial q on the Zariski closure $\mathbf{Z}'$ of O' in $\mathbf{W}$. If F is p-adic we proceed as in the affine case. For $F = \mathbb{R}$ we extend q to a polynomial p on the Zariski closure of $\mathbf{X}'$ in $\mathbf{W}$ and restrict p to $\mathbf{X}'$, and then proceed as in the affine case. In this way we obtain a generalized Q'-equivariant distributions on X'. This implies $\mathcal{S}^*(X',V)^{Q\times F^\times} \neq 0$, which by (3) *implies $\mathcal{S}^*(X,V)^Q \neq 0$.*

Let $\mathbf{G}$ be a linear algebraic F-group. Let $\mathbf{Q},\mathbf{H} \subset \mathbf{G}$ be F-subgroups such that $\mathbf{Q}$ is quasi-elementary. Consider the action of $\mathbf{Q}\times\mathbf{H}$ on $\mathbf{G}$ given by left multiplication by $\mathbf{Q}$ and right multiplication by $\mathbf{H}$. Let (σ,V) be a finite-dimensional (continuous) representation of $Q\times H$. Let $\mathbf{N_H}$ denote the unipotent radical of $\mathbf{H}$.

Theorem 4.5. *Let μ be a tempered $Q\times H$-equivariant V-valued measure on a double coset $QgH \subset G$. If F is non-archimedean we assume that the restriction $\sigma|_{N_Q\times N_H}$ is the trivial representation.*

Then there exists a generalized $Q\times H$-equivariant $\eta \in \mathcal{S}^*(G,V)$ and a Zariski open $Q\times H$-invariant neighborhood U of QgH such that the restriction of η to U equals the extension of μ to U by zero.

Moreover, the dimension of $\mathcal{S}^*(G,V)^{Q\times H}$ is at least the number of $Q\times H$-double cosets in $(\mathbf{Q}g\mathbf{H})(F)$ possessing non-zero $Q\times H$-equivariant V-valued tempered measures.

Proof. By Corollary 2.9, there exists an algebraic character ψ of $\mathbf{Q}\times\mathbf{H}$ and a non-zero $(\mathbf{Q}\times\mathbf{H},\psi)$-equivariant F-polynomial q on the Zariski closure $\mathbf{Z}$ of the double coset $\mathbf{Q}g\mathbf{H}$. Extend μ by zero to a $\mathbf{Q}\times\mathbf{H}$-equivariant V-valued measure on $\mathbf{Q}g\mathbf{H}(F)$, that we will also denote by μ.

If F is non-archimedean, and $\sigma|_{N_Q\times N_H}$ is trivial then, by Theorem 2.10, μ can be extended to a generalized $Q\times H$-invariant V-valued distribution on $Z := \mathbf{Z}(F)$. Extending this distribution by zero we obtain a generalized $Q\times H$-invariant distribution η on G.

Now assume that $F = \mathbb{R}$. Lift q to a polynomial p on $\mathbf{G}$ and let $U := \mathbf{G}_p(\mathbb{R})$. Let $\xi \in \mathcal{S}^*(U,V)$ be the extension of μ by zero and note that $p^n\xi$ is $(Q\times H,\psi^n\sigma)$-equivariant for any $n \geq 0$. By Lemma 3.1 ξ is holonomic. Applying Lemma 4.1 to $p,\xi,\mathbf{G}$ and $\mathbf{Q}g\mathbf{H}$, we obtain a generalized invariant extension η of ξ to $\mathcal{S}^*(G,V)$.

To prove the “moreover” part, let $g_1,\dots g_n \in (\mathbf{Q}g\mathbf{H})(F)$ such that the double cosets Qg_iH are distinct and posses invariant measures $\mu_1,\dots\mu_n$. These measures extend by zero to linearly independent distributions ξ_i on U. Applying Lemma 4.1 we construct meromorphic families $\xi_{i,\lambda}$ of distributions on G. It is easy to see that the support of each family $\xi_{i,\lambda}$ lies in the closure of Qg_iH and thus the families are linearly independent. Let L denote the linear span of these families and let $W \subset \mathcal{S}^*(G,V)$ denote the space spanned by the leading coefficients of the families in L. By Lemma 2.4, $\dim W = \dim L = n$ and by Lemma 2.2 all the distributions in W are $Q\times H$-equivariant. $\Box$

Remark 4.6. *The lower bound on dimension as in Theorem 4.5 can be shown to hold under the conditions of Theorem 4.3 as well.*

Proof of Corollary C. By Lemmas 2.17 and 2.18(ii), the double coset P_0gH has a tempered $P_0 \times H$ equivariant σ-valued measure. By Theorem B, this implies that $\mathcal{S}^*(G,\sigma)^{P_0\times H} \neq 0$, and thus $\mathcal{S}^*(G,\sigma)^{H\times P_0} \neq 0$. By Lemma 2.19 $\mathcal{S}^*(G,\sigma)^{H\times P_0}$ is isomorphic to the space of H-invariant continuous functionals on $\mathrm{Ind}_{P_0}^G(\sigma)$. $\square$

The next natural question that arises is whether one can extend distributions which are supported on an orbit but not defined on this orbit.

Question 4.7. *Let a quasi-elementary group* $\mathbf{Q}$ *act on an affine variety* $\mathbf{X}$ *defined over* $\mathbb{R}$. *Let* $\mathbf{U} \subset \mathbf{X}$ *be a (Zariski) open* $\mathbf{Q}$*-invariant subset. Let* χ *be a character of* Q. *Let* $O \subset U$ *be a closed* Q*-orbit and assume that there exists* $\xi \in \mathcal{S}^*(U,\chi)^Q$ *with* $\mathrm{Supp}\xi = O$. *Does this imply* $\mathcal{S}^*(X,\chi)^Q \neq 0$*?*

Example 4.8. *The answer is yes if* $\mathbf{Q} =$ *upper triangular* 2×2 *invertible matrices,* $\mathbf{X} = \mathbb{A}^2$.

Proof. The only case that does not follow from Theorem A is $\mathbf{U} = \mathbb{A}^2 \setminus 0$, $O = \mathbb{R}^\times \times \{0\}$. Fix $\xi \in \mathcal{S}^*(U,\chi)^Q$ with $\mathrm{Supp}\xi = O$. If $y\xi = 0$ then ξ is a measure on O and we use Theorem A again. Assume $\xi y^2 = 0$, $\xi y \neq 0$. Note that $\mathfrak{q} := Lie(Q)$ acts on $X = \mathbb{R}^2$ by the vector fields $\alpha = x\partial_x$, $\beta = y\partial_y$, $\gamma = y\partial_x$. Thus the coordinate y and the vector field ∂_x are Q-equivariant. Since ξ is equivariant $\xi\partial_y y$ is proportional to ξ and thus is (Q,χ)-equivariant. Now, we build the family $\xi y x^\lambda$ as in Lemma 4.1, take the leading coefficient at $\lambda = 0$ and apply ∂_y to obtain a (Q,χ)-equivariant distribution. If ξ has order n along Z, we consider the family $\xi y^{n-1}x^\lambda$ and apply ∂_y^{n-1}. $\square$

5. Application to generalized Whittaker spaces

Let G be a real reductive group, $\mathfrak{g}$ be its Lie algebra, and κ be the Killing form on $\mathfrak{g}$. For any nilpotent element $e \in \mathfrak{g}$, one defines a nilpotent subalgebra $\mathfrak{r} := \mathfrak{r}_e \subset \mathfrak{g}$ such that $\kappa(e,[\mathfrak{r},\mathfrak{r}]) = 0$ (see e.g. [GGS, §2.5]). Then e defines a character χ of $R := \mathrm{Exp}(\mathfrak{r})$ by $\chi(\mathrm{Exp}(y)) = \kappa(e,y)$. For any smooth representation π of G, one defines the generalized Whittaker space $\mathcal{W}_e(\pi)$ to be the space of (N,χ)-equivariant continuous functionals on π. While the definition of $\mathfrak{r}_e$ involves some choices, for any two different choices the spaces of functionals are canonically isomorphic.

Now let $P \subset G$ be a parabolic subgroup, $N \subset P$ be its unipotent radical and $\mathfrak{p},\mathfrak{n}$ be the Lie algebras of P and N. Let V be finite-dimensional (complex) representation of G, and let $\mathrm{Ind}_P^G(V)$ denote the smooth induction. The following theorem generalizes Theorem D.

Theorem 5.1. *Let* $e \in \mathfrak{n}$. *Let* $\mathfrak{r} \subset \mathfrak{g}$ *be a nilpotent Lie algebra with* $\kappa(e,[\mathfrak{r},\mathfrak{r}]) = 0$. *Let* $R := \mathrm{Exp}(\mathfrak{r})$ *and let define a character* χ *of* R *by* $\chi(\mathrm{Exp}(y)) = \kappa(e,y)$. *Then there exists a non-zero* (R,χ)*-equivariant continuous functional on* $\mathrm{Ind}_P^G(V)$.

Proof. By Lemma 2.19 we have

$$((\mathrm{Ind}_P^G(V))^*)^{R,\chi} = \mathcal{S}^*(G,\chi\otimes(V\otimes\Delta_P^{-1}))^{R\times P}.$$

Let us show that the double coset RP has a tempered $R\times P$-equivariant $\chi\otimes(V\otimes\Delta_P^{-1})$-valued measure. By Lemmas 2.17 and 2.18(ii) one has to show that $(\chi\otimes\Delta_L)\otimes V\otimes\Delta_C^{-1}$ has a C-invariant vector, where $C := R\cap P$ is diagonally embedded into $R\times P$ and Δ_C denotes the modular function of C. Since R is unipotent, so is $C = R\cap P$ and thus $\Delta_R = \Delta_C = 1$. Also, C lies in the unipotent radical of a minimal parabolic subgroup $P_0 \subset P$. Applying Lemma 2.18(i) to P_0 we obtain that C acts unipotently on V and thus has an invariant vector. Now, e lies in the nilradical of $\mathfrak{p}$, thus is orthogonal to $\mathfrak{p}$ under the Killing form and thus $\chi|_C = 1$. Altogether, we get that $(\chi\otimes\Delta_R)\otimes V\otimes\Delta_C^{-1}$ has a C-invariant vector. Thus the double coset RP has a tempered $R\times P$-invariant $\chi\otimes(V\otimes\Delta_P^{-1})$-valued measure, and by Theorem B we have

$$\mathcal{S}^*(G,\chi\otimes(V\otimes\Delta_P^{-1}))^{R\times P} \neq 0.$$

$\square$

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