Representation theory and Gelfand pairs

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- Advanced Linear Algebra
- Linear Algebra in presence of symmetries

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Example

An action of G on a set X defines a representation on the space $\mathbb{C}[X]$ of functions on X by $(\pi(g)f)(x) := f(g^{-1}x)$.

One-dimensional representations and Fourier series

- For the cyclic finite group $\mathbb{Z}/n\mathbb{Z}$, the space $\mathbb{C}[G]$ has a basis consisting of joint eigenvectors for the whole representation. The basis vectors are
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- For the group $SO(3)$ of rotations in the space this does not hold, neither for $\mathbb{C}[SO(3)]$ nor for $\mathbb{C}[S^2]$ (functions on the sphere)

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- $\operatorname{Ker} T, \operatorname{Im} T$ are subrepresentations.
- T has an eigenvalue λ, thus $T - \lambda \operatorname{Id}$ is not invertible, thus $T - \lambda \operatorname{Id} = 0$.

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Spherical harmonics

H_n :=the space of homogeneous harmonic polynomials of degree n in three variables. Harmonic means that they vanish under the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

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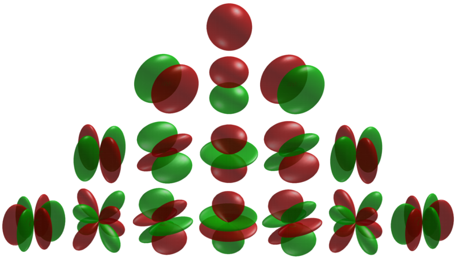

Gelfand pairs

Let G be a (finite) group and $H \subset G$ be a subgroup.

Lemma

The following conditions are equivalent

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The convolution is defined on the basis of δ-functions by $\delta_g * \delta_{g'} = \delta_{gg'}$, or explicitly by

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If the above conditions are satisfied, the pair (G, H) is called a Gelfand pair.

Example: $G = SO(3)$, $H = SO(2)$, $G/H = S^2$.

Lemma (Gelfand-Selberg trick)

Suppose that there exists $\sigma : G \to G$ *such that*

1. $\sigma(gg') = \sigma(g')\sigma(g)$
2. $\sigma(g) \in HgH$

Then the pair (G, H) *is a Gelfand pair.*

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Proof.

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Using the anti-involution $\sigma(g) = g^t = g^{-1}$ one can show that $(SO(n+1), SO(n))$ is a Gelfand pair, and thus $L^2(S^n)$ is a multiplicity-free representation of $SO(n+1)$.