Representation theory and Gelfand pairs

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December 2013

Representation theory

- Advanced Linear Algebra
- Linear Algebra in presence of symmetries

Definition

A representation π of a group G on a (complex) vector space V is an assignment to every element $g \in G$ of an invertible linear operator $\pi(g)$ such that $\pi(gh)$ is the composition of $\pi(g)$ and $\pi(h)$.

Example

An action of G on a set X defines a representation on the space $\mathbb{C}[X]$ of functions on X by $(\pi(g)f)(x) := f(g^{-1}x)$.

One-dimensional representations and Fourier series

- For the cyclic finite group $\mathbb{Z}/n\mathbb{Z}$, the space $\mathbb{C}[G]$ has a basis consisting of joint eigenvectors for the whole representation. The basis vectors are

$$f_k(m) = \exp(2\pi ikm/n).$$

  The decomposition of a function with respect to this basis is called discrete Fourier transform.

- The same holds for the compact group S^1. The basis vectors are

$$f_k(\theta) = \exp(ik\theta).$$

  The decomposition of a function with respect to this basis is called Fourier series.

- For the group $SO(3)$ of rotations in the space this does not hold, neither for $\mathbb{C}[SO(3)]$ nor for $\mathbb{C}[S^2]$ (functions on the sphere)

Definition

A representation is called irreducible if the space does not have invariant subspaces.

Definition

A morphism between representations (π, V) and (τ, W) of a group G is a linear operator $T : V \to W$ s. t. $T \circ \pi(g) = \tau(g) \circ T$ for any $g \in G$.

Lemma (Schur)

- *Any non-zero morphism of irreducible representations is invertible.*
- *Any morphism of an irreducible finite-dimensional representation into itself is scalar.*

Proof.

- $\operatorname{Ker} T, \operatorname{Im} T$ are subrepresentations.
- T has an eigenvalue λ, thus $T - \lambda \operatorname{Id}$ is not invertible, thus $T - \lambda \operatorname{Id} = 0$.

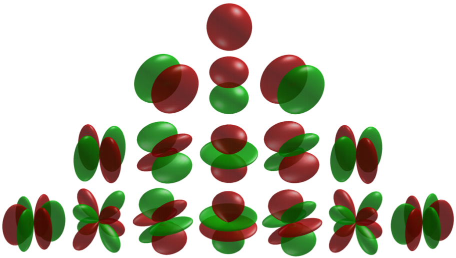
□

Spherical harmonics

H_n :=the space of homogeneous harmonic polynomials of degree n in three variables. Harmonic means that they vanish under the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

Theorem

- *H_n is an irreducible representation of $SO(3)$*
- $L^2(S^2) = \widehat{\bigoplus}_{n=0}^{\infty} H_n$,
- *Every irreducible representation of $SO(3)$ is isomorphic to H_n for some n.*



Gelfand pairs

Let G be a (finite) group and $H \subset G$ be a subgroup.

Lemma

The following conditions are equivalent

- *The representation* $\mathbb{C}[G/H]$ *is multiplicity free, i.e. includes each irreducible representation of* G *with multiplicity at most one.*
- *For any irreducible representation* (π, V) *of* G*, the space* V^H *of* H*-invariant vectors is at most one-dimensional.*
- *The algebra* $\mathbb{C}[G]^{H \times H}$ *of functions on* G *that are invariant under the action of* H *on both sides is commutative with respect to convolution.*

The convolution is defined on the basis of δ-functions by $\delta_g * \delta_{g'} = \delta_{gg'}$, or explicitly by

$$f * h(x) = \sum_{y \in G} f(y) g(y^{-1} x)$$

If the above conditions are satisfied, the pair (G, H) is called a Gelfand pair.

Example: $G = SO(3)$, $H = SO(2)$, $G/H = S^2$.

Lemma (Gelfand-Selberg trick)

Suppose that there exists $\sigma : G \to G$ such that

1. $\sigma(gg') = \sigma(g')\sigma(g)$
2. $\sigma(g) \in HgH$

Then the pair (G, H) is a Gelfand pair.

Proof.

Define σ on $\mathbb{C}[G]$ by $\delta_g^\sigma = \delta_{\sigma(g)}$.From (1) we see that $(a * b)^\sigma = b^\sigma * a^\sigma$. On the other hand, for any $a \in \mathbb{C}[G]^{H\times H}$ we have

$$a^\sigma(x) = a(\sigma(x)) = a(hxh') = a(x),$$

and thus $a * b = (a * b)^\sigma = b^\sigma * a^\sigma = b * a$ □

Using the anti-involution $\sigma(g) = g^t = g^{-1}$ one can show that $(SO(n+1), SO(n))$ is a Gelfand pair, and thus $L^2(S^n)$ is a multiplicity-free representation of $SO(n+1)$.