Representation theory and non-commutative harmonic analysis on G-spaces

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- Advanced Linear Algebra
- Study of linear symmetries

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Definition

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Example

An action of G on a set X defines a representation on the space $\mathbb{C}[X]$ of functions on X by $(\pi(g)f)(x) := f(g^{-1}x)$.

One-dimensional representations and Fourier series

- For the cyclic finite group $\mathbb{Z}/n\mathbb{Z}$, the space $\mathbb{C}[G]$ has a basis consisting of joint eigenvectors for the whole representation. The basis vectors are
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- In general, this works for any locally compact commutative group - Pontryagin duality.
- For the group $SO(3)$ of rotations in the space this does not hold, neither for $\mathbb{C}[SO(3)]$ nor for $\mathbb{C}[S^2]$ (functions on the sphere)

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- $\operatorname{Ker} T, \operatorname{Im} T$ are subrepresentations.
- T has an eigenvalue λ, thus $T - \lambda \operatorname{Id}$ is not invertible, thus $T - \lambda \operatorname{Id} = 0$.

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Spherical harmonics

H_n :=the space of homogeneous harmonic polynomials of degree n in three variables. Harmonic means that they vanish under the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

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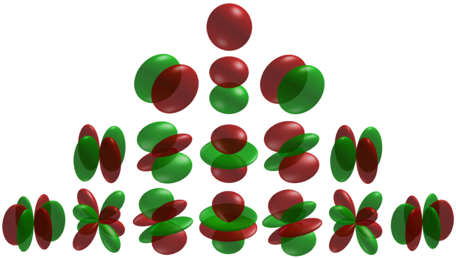

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A **model** is an explicitly defined representation that includes all irreducible representations with a certain property, each with multiplicity one.

Example: $L^2(S^2)$ is a model for all irreducible representations of $SO(3)$.

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- Applications: representation theory, integral geometry, physics, analytic number theory.
- We use algebraic geometry and functional analysis.