Representation theory and non-commutative harmonic analysis on G-spaces

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Representation theory

- Advanced Linear Algebra
- Study of linear symmetries

Definition

A representation π of a group G on a (complex) vector space V is an assignment to every element $g \in G$ of an invertible linear operator $\pi(g)$ such that $\pi(gh)$ is the composition of $\pi(g)$ and $\pi(h)$.

Example

An action of G on a set X defines a representation on the space $\mathbb{C}[X]$ of functions on X by $(\pi(g)f)(x) := f(g^{-1}x)$.

One-dimensional representations and Fourier series

- For the cyclic finite group $\mathbb{Z}/n\mathbb{Z}$, the space $\mathbb{C}[G]$ has a basis consisting of joint eigenvectors for the whole representation. The basis vectors are

$$f_k(m) = \exp(2\pi i k m/n).$$

  The decomposition of a function with respect to this basis is called discrete Fourier transform.
- The same holds for the compact group S^1. The basis vectors are

$$f_k(\theta) = \exp(ik\theta).$$

  The decomposition of a function with respect to this basis is called Fourier series.
- In general, this works for any locally compact commutative group - Pontryagin duality.
- For the group $SO(3)$ of rotations in the space this does not hold, neither for $\mathbb{C}[SO(3)]$ nor for $\mathbb{C}[S^2]$ (functions on the sphere)

Definition

A representation is called irreducible if the space does not have non-zero invariant proper subspaces.

Definition

A morphism between representations (π, V) and (τ, W) of a group G is a linear operator $T : V \to W$ s. t. $T \circ \pi(g) = \tau(g) \circ T$ for any $g \in G$.

Lemma (Schur)

- *Any non-zero morphism of irreducible representations is invertible.*
- *Any morphism of an irreducible finite-dimensional representation into itself is scalar.*

Proof.

- $\operatorname{Ker} T, \operatorname{Im} T$ are subrepresentations.
- T has an eigenvalue λ, thus $T - \lambda \operatorname{Id}$ is not invertible, thus $T - \lambda \operatorname{Id} = 0$.

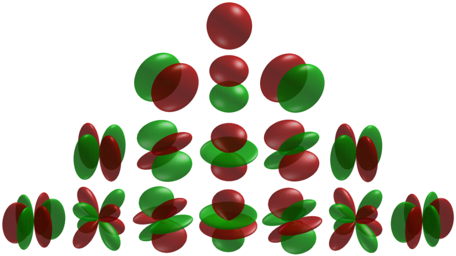
□

Spherical harmonics

H_n :=the space of homogeneous harmonic polynomials of degree n in three variables. Harmonic means that they vanish under the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

Theorem

- *H_n is an irreducible representation of $SO(3)$*
- $L^2(S^2) = \widehat{\bigoplus}_{n=0}^{\infty} H_n$,
- *Every irreducible representation of $SO(3)$ is isomorphic to H_n for some n.*



Goals of representation theory, and models

Theorem (Mashke, Schur, Peter-Weyl)

Any representation of a compact group decomposes into a direct sum of irreducible representations.

Goals:

1. Classify all the irreducible representations of G.
2. Given a representation of G find its decomposition to irreducible ones.
3. Given a group homomorphism $H \to G$, find the relation between the representation theories of G and H.

A **model** is an explicitly defined representation that includes all irreducible representations with a certain property, each with multiplicity one.

Example: $L^2(S^2)$ is a model for all irreducible representations of $SO(3)$.

Non-commutative harmonic analysis on G-spaces

Let a (finite) group G act on a set X.

- Goal: study $\mathbb{C}[X]$ as a representation of G.
- Equivalently: what irreducible representations are inside, and with what multiplicities?
- Actually, we study infinite non-compact groups like the group $\mathrm{GL}_n(\mathbb{R})$ of invertible matrices, or the Lorentz group $\mathrm{SO}(3,1)$, and their actions on algebraic varieties.
- Applications: representation theory, integral geometry, physics, analytic number theory.
- We use algebraic geometry and functional analysis.