GENERALIZED HARISH-CHANDRA DESCENT AND APPLICATIONS TO GELFAND PAIRS

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ABSTRACT. In the first part of the paper we generalize a descent technique due to Harish-Chandra to the case of a reductive group acting on a smooth affine variety both defined over arbitrary local field F of characteristic zero. Our main tool is Luna slice theorem.

In the second part of the paper we apply this technique to symmetric pairs. In particular we prove that the pair $(GL_n(\mathbb{C}), GL_n(\mathbb{R}))$ is a Gelfand pair. We also prove that any conjugation invariant distribution on $GL_n(F)$ is invariant with respect to transposition. For non-archimedean F the later is a classical theorem of Gelfand and Kazhdan.

We use the techniques developed here in our subsequent work [AG3] where we prove an archimedean analog of the theorem on uniqueness of linear periods by H. Jacquet and S. Rallis.

CONTENTS

1. Introduction

Harish-Chandra developed a technique based on Jordan decomposition that allows to reduce certain statements on conjugation invariant distributions on a reductive group to the set of unipotent elements, provided that the statement is known for certain subgroups (see e.g. [HCh]).

In this paper we generalize part of this technique to the setting of a reductive group acting on a smooth affine algebraic variety, using Luna slice theorem. Our technique is oriented towards proving Gelfand property for pairs of reductive groups.

Our approach is uniform for all local fields of characteristic zero - both archimedean and non-archimedean.

1.1. **Main results.**

The core of this paper is Theorem 3.1.1:

Theorem. *Let a reductive group G act on a smooth affine variety X, both defined over a local field F of characteristic zero. Let χ be a character of $G(F)$.*

Suppose that for any $x \in X(F)$ with closed orbit there are no non-zero distributions on the normal space to the orbit $G(F)x$ at x which are equivariant with respect to the stabilizer of x with the character χ.

Then there are no non-zero $(G(F), \chi)$-equivariant distributions on $X(F)$.

Using this theorem we obtain its stronger version (Corollary 3.2.2). This stronger version is based on an inductive argument which shows that it is enough to prove that there are no non-zero equivariant distributions on the normal space to the orbit $G(F)x$ at x under the assumption that all such distributions are supported in a certain closed subset which is an analog of the cone of nilpotent elements.

Then we apply this stronger version to problems of the following type. Let a reductive group G acts on a smooth affine variety X, and τ be an involution of X which normalizes the action of G. We want to check whether any $G(F)$-invariant distribution on $X(F)$ is also τ-invariant. Evidently, there is the following necessary condition on τ:

(*) Any closed orbit in $X(F)$ is τ-invariant.

In some cases this condition is also sufficient. In these cases we call the action of G on X *tame*.

The property of being tame is weaker than the property called "density" in [RR]. However, it is sufficient for the purpose of proving Gelfand property for pairs of reductive groups.

In section 6 we give criteria for tameness of actions. In particular, we have introduced the notion of "special" action. This notion can be used in order to show that certain actions are tame (see Theorem 6.0.5 and Proposition 7.3.5). Also, in many cases one can verify that an action is special using purely algebraic - geometric means.

Then we restrict our attention to the case of symmetric pairs. There we introduce a notion of regular symmetric pair (see Definition 7.4.2), which also helps to prove Gelfand property. Namely, we prove Theorem 7.4.5.

Theorem. *Let G be a reductive group defined over a local field F and θ be an involution of G. Let $H := G^\theta$ and let σ be the anti-involution defined by $\sigma(g) := \theta(g^{-1})$. Consider the symmetric pair (G, H).*

Suppose that all its "descendants" (including itself, see Definition 7.2.2) are regular. Suppose also that any closed $H(F)$-double coset in $G(F)$ is σ- invariant.

Then every $H(F)$ double invariant distribution on $G(F)$ is σ-invariant. In particular, the pair (G, H) is a Gelfand pair (see section 8).

Also, we formulate an algebraic-geometric criterion for regularity of a pair (Proposition 7.3.7).

Using our technique we prove (in section 7.6) that the pair $(G(E), G(F))$ is *tame* for any reductive group G over F and a quadratic field extension E/F. This means that the two-sided action of $G(F) \times G(F)$ on $G(E)$ is tame. This implies that the pair $(GL_n(E), GL_n(F))$ is a Gelfand pair. In the non-archimedean case this was proven in [Fli].

Also we prove that the adjoint action of a reductive group on itself is tame. This is a generalization of a classical theorem by Gelfand and Kazhdan, see [GK].

In our subsequent work [AG3] we use the results of this paper to prove that the pair $(GL_{n+k}, GL_n \times GL_k)$ is a Gelfand pair by proving that it is regular. In the non-archimedean case this was proven in [JR] and our proof follows their lines.

In general, we conjecture that any symmetric pair is regular. This would imply van Dijk conjecture:

Conjecture (van Dijk). *Any symmetric pair (G, H) over $\mathbb{C}$ such that G/H is connected is a Gelfand pair.*

1.2. **Related works on this topic.**
This paper was inspired by the paper [JR] by Jacquet and Rallis where they prove that the pair $(GL_{n+k}(F), GL_n(F) \times GL_k(F))$ is a Gelfand pair for non-archimedean local field F of characteristic zero. Our aim was to see to what extent their techniques generalize.

Another generalization of Harish-Chandra descent using Luna slice theorem has been done in the non-archimedean case in [RR]. In that paper Rader and Rallis investigated spherical characters of H-distinguished representations of G for symmetric pairs (G, H) and checked the validity of what they call "density principle" for rank one symmetric pairs. They found out that usually it holds, but also found counterexamples.

In [vD], van-Dijk investigated rank one symmetric pairs in the archimedean case and gave the full answer to the question which of them are Gelfand pairs. In [BvD], van-Dijk and Bosman studied the non-archimedean case and gave the answer for the same question for most rank one symmetric pairs. We hope that the second part of our paper will enhance the understanding of this question for symmetric pairs of higher rank.

1.3. **Structure of the paper.**

In section 2 we introduce notation that allows us to speak uniformly about spaces of points of smooth algebraic varieties over archimedean and non-archimedean local fields, and equivariant distributions on those spaces.

In subsection 2.1 we formulate a version of Luna slice theorem for points over local fields (Theorem 2.1.16). In subsection 2.3 we formulate theorems on equivariant distributions and equivariant Schwartz distributions.

In section 3 we formulate and prove the generalized Harish-Chandra descent theorem and its stronger version.

Section 4 is relevant only to the archimedean case. In that section we prove that in cases that we consider if there are no equivariant Schwartz distributions then there are no equivariant distributions at all. Schwartz distributions are discussed in Appendix C.

In section 5 we formulate homogeneity theorem that helps us to check the conditions of the generalized Harish-Chandra descent theorem. In the non-archimedean case this theorem had been proved earlier (see e.g. [JR], [RS2] or [AGRS]). We provide the proof for the archimedean case in Appendix D.

In section 6 we introduce the notion of tame actions and provide tameness criteria.

In section 7 we apply our tools to symmetric pairs. In subsection 7.3 we provide criteria for tameness of a symmetric pair. In subsection 7.4 we introduce the notion of regular symmetric pair and prove Theorem 7.4.5 that we mentioned above. In subsection 7.5 we discuss conjectures about regularity and Gelfand property of symmetric pairs. In subsection 7.6 we prove that certain symmetric pairs are tame.

In section 8 we give preliminaries on Gelfand pairs an their connections to invariant distributions. We also prove that the pair $(GL_n(E), GL_n(F))$ is Gelfand pair for any quadratic extension E/F.

In Appendix A we formulate and prove a version of Bernstein's localization principle (Theorem 4.0.1). This is relevant only for archimedean F since for l-spaces a more general version of this principle had been proven in [Ber]. This appendix is used in section 4.

In [AGS2] we formulated localization principle in the setting of differential geometry. Currently we do not have a proof of this principle in such general setting. In Appendix A we present a proof in the case of a reductive group G acting on a smooth affine variety X. This generality is wide enough for all applications we had up to now, including the one in [AGS2].

We start Appendix B from discussing different versions of the inverse function theorem for local fields. Then we prove a version of Luna slice theorem for points over local fields (Theorem 2.1.16). For archimedean F it was done by Luna himself in [Lun2].

Appendices C and D are relevant only to the archimedean case.

In Appendix C we discuss Schwartz distributions on Nash manifolds. We prove for them Frobenius reciprocity and construct a pullback of a Schwartz distribution under Nash submersion. Also we prove that K invariant distributions which are (Nashly) compactly supported modulo K are Schwartz distributions.

In Appendix D we prove the archimedean version of the homogeneity theorem discussed in section 5.

In Appendix E we present a diagram that illustrates the interrelations of various properties of symmetric pairs.

Finally we thank **Anna Gourevitch** for the graphical design of Appendix E.

2. Preliminaries and notations

- From now and till the end of the paper we fix a local field F of characteristic zero. All the algebraic varieties and algebraic groups that we will consider will be defined over F.
- For a group G acting on a set X and an element $x \in X$ we denote by G_x the stabilizer of x.
- By a reductive group we mean an algebraic reductive group.

We treat an algebraic variety X defined over F as algebraic variety over $\overline{F}$ together with action of the Galois group $Gal(\overline{F}, F)$. On X we will consider only the Zariski topology. On $X(F)$ we consider only the analytic (Hausdorff) topology. We treat finite dimensional linear spaces defined over F as algebraic varieties.

Usually we will use letters X, Y, Z, Δ to denote algebraic varieties and letters G, H to denote algebraic groups. We will usually use letters $V, W, U, K, M, N, C, O, S, T$ to denote analytic spaces and in particular F points of algebraic varieties and the letter K to denote analytic groups. Also we will use letters L, V, W to denote vector spaces of all kinds.

Definition 2.0.1. *Let an algebraic group G act on an algebraic variety X. A pair consisting of an algebraic variety Y and a G-invariant morphism $\pi : X \to Y$ is called **the quotient of X by the action of** G if for any pair (π', Y'), there exists a unique morphism $\phi : Y \to Y'$ such that $\pi' = \phi \circ \pi$. Clearly, if such pair exists it is unique up to canonical isomorphism. We will denote it by $(\pi_X, X/G)$.*

Theorem 2.0.2. *Let a reductive group G act on an affine variety X. Then the quotient X/G exists, and every fiber of the quotient map π_X contains a unique closed orbit.*

Proof. In [Dre] it is proven that the variety $Spec\mathcal{O}(X)^G$ satisfies the universal condition of X/G. Clearly, this variety is defined over F and hence we can take $X/G := Spec\mathcal{O}(X)^G$. $\square$

2.1. **Preliminaries on algebraic geometry over local fields.**

2.1.1. *Analytic manifolds.*
In this paper we will consider distributions over l-spaces, smooth manifolds and Nash manifolds. l-spaces are locally compact totally disconnected topological spaces and Nash manifolds are semi-algebraic smooth manifolds.

For basic preliminaries on l-spaces and distributions over them we refer the reader to [BZ], section 1.

For preliminaries on Nash manifolds and Schwartz functions and distributions over them see Appendix C and [AG1]. In this paper we will consider only separated Nash manifolds.

We will now give notations which will allow a uniform exposition of archimedean and non-archimedean cases.

We will use the notion of analytic manifold over a local field (see e.g. [Ser], Part II, Chapter III). When we say "**analytic manifold**" we mean analytic manifold over some local field. Note that an analytic manifold over a non-archimedean field is in particular an l-space and analytic manifold over an archimedean field is in particular a smooth manifold.

Definition 2.1.1. *A **B-analytic manifold** is either an analytic manifold over a non-archimedean local field, or a Nash manifold.*

Remark 2.1.2. *If X is a smooth algebraic variety, then $X(F)$ is a B-analytic manifold and $(T_x X)(F) = T_x(X(F))$.*

Notation 2.1.3. *Let M be an analytic manifold and S be an analytic submanifold. We denote by $N_S^M := (T_M|_Y)/T_S$ the normal bundle to S in M. The conormal bundle is defined by $CN_S^M := (N_S^M)^*$. Denote by $Sym^k(CN_S^M)$ the k-th symmetric power of the conormal bundle. For a point $y \in S$ we denote by $N_{S,y}^M$ the normal space to S in M at the point y and by $CN_{S,y}^M$ the conormal space.*

2.1.2. *G-orbits on X and $G(F)$-orbits on $X(F)$.*

Lemma 2.1.4. *Let G be an algebraic group. Let $H \subset G$ be a closed subgroup. Then $G(F)/H(F)$ is open and closed in $(G/H)(F)$.*

For proof see Appendix B.1.

Corollary 2.1.5. *Let an algebraic group G act on an algebraic variety X. Let $x \in X(F)$. Then*
$$N_{Gx,x}^X(F) \cong N_{G(F)x,x}^{X(F)}.$$

Proposition 2.1.6. *Let an algebraic group G act on an algebraic variety X. Suppose that $S \subset X(F)$ is non-empty closed $G(F)$-invariant subset. Then S contains a closed orbit.*

Proof. The proof is by Noetherian induction on X. Choose $x \in S$. Consider $Z := \overline{Gx} - Gx$.

If $Z(F) \cap S$ is empty then $Gx(F) \cap S$ is closed and hence $G(F)x \cap S$ is closed by Lemma 2.1.4. Therefore $G(F)x$ is closed.

If $Z(F) \cap S$ is non-empty then $Z(F) \cap S$ contains a closed orbit by the induction assumption. $\square$

Corollary 2.1.7. *Let an algebraic group G act on an algebraic variety X. Let U be an open $G(F)$-invariant subset of $X(F)$. Suppose that it includes all closed $G(F)$-orbits. Then $U = X(F)$.*

Theorem 2.1.8. *Let a reductive group G act on an affine variety X. Let $x \in X(F)$. Then the following are equivalent:*
(i) $G(F)x \subset X(F)$ is closed (in the analytic topology).
(ii) $Gx \subset X$ is closed (in the Zariski topology).

For proof see [RR], section 2 fact A, pages 108-109.

Definition 2.1.9. *Let a reductive group G act on an affine variety X. We call an element $x \in X$ G-semisimple if its orbit Gx is closed. In particular, in the case of G acting on itself by the adjoint action, the notion of G-semisimple element coincides with the usual notion of semisimple element.*

Notation 2.1.10. *Let V be an algebraic finite dimensional representation over F of a reductive group G. We denote*
$$Q(V) := (V/V^G)(F).$$
Since G is reductive, there is a canonical embedding $Q(V) \hookrightarrow V(F)$. Let $\pi : V(F) \to (V/G)(F)$ be the standard projection. We denote
$$\Gamma(V) := \pi^{-1}(\pi(0)).$$
Note that $\Gamma(V) \subset Q(V)$. We denote also
$$R(V) := Q(V) - \Gamma(V).$$

Notation 2.1.11. *Let a reductive group G act on an affine variety X. Let an element $x \in X(F)$ be G-semisimple. We denote*
$$S_x := \{y \in X(F) \,|\, \overline{G(F)y} \ni x\}.$$

Lemma 2.1.12. *Let V be an algebraic finite dimensional representation over F of a reductive group G. Then $\Gamma(V) = S_0$.*

This lemma follows from fact A on page 108 in [RR] for non-archimedean F and Theorem 5.2 on page 459 in [Brk].

Proposition 2.1.13. *Let a reductive group G act on an affine variety X. Let $x, z \in X(F)$ be G-semisimple elements with different orbits. Then there exist disjoint $G(F)$-invariant open neighborhoods U_x of x and U_z of z.*

For proof of this proposition see [Lun2] for archimedean F and [RR], fact B on page 109 for non-archimedean F.

Corollary 2.1.14. *Let a reductive group G act on an affine variety X. Let an element $x \in X(F)$ be G-semisimple. Then the set S_x is closed.*

Proof. Let $y \in \overline{S_x}$. By proposition 2.1.6, $\overline{G(F)y}$ contains a closed orbit $G(F)z$. If $G(F)z = G(F)x$ then $y \in S_x$.

Otherwise, choose disjoint open G-invariant neighborhoods U_z of z and U_x of x. Since $z \in \overline{G(F)y}$, U_z intersects $G(F)y$ and hence includes y. Since $y \in \overline{S_x}$, this means that U_z intersects S_x. Let $t \in U_z \cap S_x$. Since U_z is $G(F)$-invariant, $G(F)t \subset U_z$. By the definition of S_x, $x \in \overline{G(F)t}$ and hence $x \in \overline{U_z}$. Hence U_z intersects U_x - contradiction! $\qquad\square$

2.1.3. *Analytic Luna slice.*

Definition 2.1.15. *Let a reductive group G act on an affine variety X. Let $\pi : X(F) \to X/G(F)$ be the standard projection. An open subset $U \subset X(F)$ is called **saturated** if there exists an open subset $V \subset X/G(F)$ such that $U = \pi^{-1}(V)$.*

We will use the following corollary from Luna slice theorem (for proof see Appendix B.2):

Theorem 2.1.16. *Let a reductive group G act on a smooth affine variety X. Let $x \in X(F)$ be G-semisimple. Then there exist*
(i) an open $G(F)$-invariant B-analytic neighborhood U of $G(F)x$ in $X(F)$ with a G-equivariant B-analytic retract $p : U \to G(F)x$ and
(ii) a G_x-equivariant B-analytic embedding $\psi : p^{-1}(x) \hookrightarrow N_{Gx,x}^X(F)$ with open saturated image such that $\psi(x) = 0$.

Definition 2.1.17. *In the notations of the previous theorem, denote $S := p^{-1}(x)$ and $N := N_{Gx,x}^X(F)$. We call the quintet (U, p, ψ, S, N) an **analytic Luna slice at** x.*

Corollary 2.1.18. *In the notations of the previous theorem, let $y \in p^{-1}(x)$. Denote $z := \psi(y)$. Then*
(i) $(G(F)_x)_z = G(F)_y$
(ii) $N_{G(F)y,y}^{X(F)} \cong N_{G(F)_x z, z}^N$ as $G(F)_y$-spaces
(iii) y is G-semisimple if and only if z is G_x-semisimple.

2.2. **Vector systems.**
In this subsection we introduce the term "vector system". This term allows to formulate statements in wider generality. However, often this generality is not necessary and therefore the reader can skip this subsection and ignore vector systems during the first reading.

Definition 2.2.1. *For an analytic manifold M we define the notions of **vector system** and **B-vector system** over it.*
For a smooth manifold M, a vector system over M is a pair (E, B) where B is a smooth locally trivial fibration over M and E is a smooth vector bundle over B.
For a Nash manifold M, a B-vector system over M is a pair (E, B) where B is a Nash fibration over M and E is a Nash vector bundle over B.

For an l-space M, a **vector system** over M (or a **B-vector system** over M) is an l-sheaf, that is locally constant sheaf, over M.

Definition 2.2.2. Let $\mathcal{V}$ be a vector system over a point pt. Let M be an analytic manifold. A **constant vector system with fiber** $\mathcal{V}$ is the pullback of $\mathcal{V}$ with respect to the map $M \to pt$. We denote it by $\mathcal{V}_M$.

2.3. **Preliminaries on distributions.**

Definition 2.3.1. Let M be an analytic manifold over F. We define $C_c^\infty(M)$ in the following way.

If F is non-archimedean, $C_c^\infty(M)$ is the space of locally constant compactly supported complex valued functions on M. We consider no topology on it.

If F is archimedean, $C_c^\infty(M)$ is the space of smooth compactly supported complex valued functions on M. We consider the standard topology on it.

For any analytic manifold M, we define the space of distributions $\mathcal{D}(M)$ by $\mathcal{D}(M) := C_c^\infty(M)^*$. We consider the weak topology on it.

Definition 2.3.2. Let M be a B-analytic manifold. We define $\mathcal{S}(M)$ in the following way.

If M is an analytic manifold over non-archimedean field, $\mathcal{S}(M) := C_c^\infty(M)$.

If M is a Nash manifold, $\mathcal{S}(M)$ is the space of Schwartz functions on M. Schwartz functions are smooth functions that decrease rapidly together with all their derivatives. For precise definition see [AG1]. We consider $\mathcal{S}(M)$ as a Fréchet space.

For any B-analytic manifold M, we define the space of Schwartz distributions $\mathcal{S}^*(M)$ by $\mathcal{S}^*(M) := \mathcal{S}(M)^*$.

Definition 2.3.3. Let M be an analytic manifold and let $N \subset M$ be a closed subset. We denote

$$\mathcal{D}_M(N) := \{\xi \in \mathcal{D}(M) | \mathrm{Supp}(\xi) \subset N\}.$$

For locally closed subset $N \subset M$ we denote $\mathcal{D}_M(N) := \mathcal{D}_{M \setminus (\overline{N} \setminus N)}(N)$.

Similarly we introduce the notation $\mathcal{S}_N^*(M)$ for a B-analytic manifold M.

Definition 2.3.4. Let M be an analytic manifold over F and $\mathcal{E}$ be a vector system over M. We define $C_c^\infty(M, \mathcal{E})$ in the following way:

If F is non-archimedean then $C_c^\infty(M, \mathcal{E})$ is the space of compactly supported sections of $\mathcal{E}$.

If F is archimedean and $\mathcal{E} = (E, B)$ where B is a fibration over M and E is a vector bundle over B, then $C_c^\infty(M, \mathcal{E})$ is the complexification of the space of smooth compactly supported sections of E over B.

If $\mathcal{V}$ is a vector system over a point, we denote $C_c^\infty(M, \mathcal{V}) := C_c^\infty(M, \mathcal{V}_M)$.

We define $\mathcal{D}(M, \mathcal{E})$, $\mathcal{D}_M(N, \mathcal{E})$, $\mathcal{S}(M, \mathcal{E})$, $\mathcal{S}^*(M, \mathcal{E})$ and $\mathcal{S}_M^*(N, \mathcal{E})$ in the natural way.

Theorem 2.3.5. Let an l-group K act on an l-space M. Let $M = \bigcup_{i=0}^l M_i$ be a K-invariant stratification of M. Let χ be a character of K. Suppose that $\mathcal{S}^*(M_i)^{K,\chi} = 0$. Then $\mathcal{S}^*(M)^{K,\chi} = 0$.

This theorem is a direct corollary from corollary 1.9 in [BZ].

For the proof of the next theorem see e.g. [AGS1, §B.2].

Theorem 2.3.6. Let a Nash group K act on a Nash manifold M. Let N be a locally closed subset. Let $N = \bigcup_{i=0}^l N_i$ be a Nash K-invariant stratification of N. Let χ be a character of K. Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq l$,

$$\mathcal{S}^*(N_i, \mathrm{Sym}^k(CN_{N_i}^M))^{K,\chi} = 0.$$

Then $\mathcal{S}_M^*(N)^{K,\chi} = 0$.

Theorem 2.3.7 (Frobenius reciprocity)**.** *Let an analytic group K act on an analytic manifold M. Let N be a K-transitive analytic manifold. Let $\phi : M \to N$ be a K-equivariant map.*

Let $z \in N$ be a point and $M_z := \phi^{-1}(z)$ be its fiber. Let K_z be the stabilizer of z in K. Let Δ_K and Δ_{K_z} be the modular characters of K and K_z.

Let $\mathcal{E}$ be a K-equivariant vector system over M. Then
(i) there exists a canonical isomorphism

$$Fr : \mathcal{D}(M_z, \mathcal{E}|_{M_z} \otimes \Delta_K|_{K_z} \cdot \Delta_{K_z}^{-1})^{K_z} \cong \mathcal{D}(M, \mathcal{E})^K.$$

In particular, Fr commutes with restrictions to open sets.
(ii) For B-analytic manifolds Fr maps $\mathcal{S}^*(M_z, \mathcal{E}|_{M_z} \otimes \Delta_K|_{K_z} \cdot \Delta_{K_z}^{-1})^{K_z}$ to $\mathcal{S}^*(M, \mathcal{E})^K$.

For proof of (i) see [Ber] 1.5 and [BZ] 2.21 - 2.36 for the case of l-spaces and theorem 4.2.3 in [AGS1] or [Bar] for smooth manifolds. For proof of (ii) see Appendix C.

We will also use the following straightforward proposition.

Proposition 2.3.8. *Let $\Omega_i \subset K_i$ be analytic groups acting on analytic manifolds M_i for $i = 1 \ldots n$. Let $\mathcal{E}_i \to M_i$ be K_i-equivariant vector systems. Suppose that $\mathcal{D}(M_i, E_i)^{\Omega_i} = \mathcal{D}(M_i, E_i)^{K_i}$ for all i. Then*

$$\mathcal{D}(\prod M_i, \boxtimes E_i)^{\prod \Omega_i} = \mathcal{D}(\prod M_i, \boxtimes E_i)^{\prod K_i},$$

where $\boxtimes$ denotes the external product.

Moreover, if Ω_i, K_i, M_i and $\mathcal{E}_i$ are B-analytic then the same statement holds for Schwartz distributions.

For proof see e.g. [AGS1], proof of Proposition 3.1.5.

3. Generalized Harish-Chandra descent

3.1. **Generalized Harish-Chandra descent.**
In this subsection we will prove the following theorem.

Theorem 3.1.1. *Let a reductive group G act on a smooth affine variety X. Let χ be a character of $G(F)$. Suppose that for any G-semisimple $x \in X(F)$ we have*

$$\mathcal{D}(N_{Gx,x}^X(F))^{G(F)_x, \chi} = 0.$$

Then

$$\mathcal{D}(X(F))^{G(F), \chi} = 0.$$

Remark 3.1.2. *In fact, the converse is also true. We will not prove it since we will not use it.*

For the proof of this theorem we will need the following lemma

Lemma 3.1.3. *Let a reductive group G act on a smooth affine variety X. Let χ be a character of $G(F)$. Let $U \subset X(F)$ be an open saturated subset. Suppose that $\mathcal{D}(X(F))^{G(F), \chi} = 0$. Then $\mathcal{D}(U)^{G(F), \chi} = 0$.*

Proof. Consider the quotient X/G. It is an affine algebraic variety. Embed it to an affine space $\mathbb{A}^n$. This defines a map $\pi : X(F) \to F^n$. Let $V \subset X/G(F)$ be an open subset such that $U = \pi^{-1}(V)$. There exists an open subset $V' \subset F^n$ such that $V' \cap X/G(F) = V$.

Let $\xi \in \mathcal{D}(U)^{G(F), \chi}$. Suppose that ξ is non-zero. Let $x \in \operatorname{Supp}\xi$ and let $y := \pi(x)$. Let $g \in C_c^\infty(V')$ be such that $g(y) = 1$. Consider $\xi' \in \mathcal{D}(X(F))$ defined by $\xi'(f) := \xi(f \cdot (g \circ \pi))$. Clearly, $x \in \operatorname{Supp}(\xi')$ and $\xi' \in \mathcal{D}(X(F))^{G(F), \chi}$. Contradiction. $\square$

Proof of the theorem. Choose a G-semisimple $x \in X(F)$. Let $(U_x, p_x, \psi_x, S_x, N_x)$ be an analytic Luna slice at x.

Let $\xi' = \xi|_{U_x}$. Then $\xi' \in \mathcal{D}(U_x)^{G(F),\chi}$. By Frobenius reciprocity it corresponds to $\xi'' \in \mathcal{D}(S_x)^{G_x(F),\chi}$.

The distribution ξ'' corresponds to a distribution $\xi''' \in \mathcal{D}(\psi_x(S_x))^{G_x(F),\chi}$.

However, by the previous lemma the assumption implies that $\mathcal{D}(\psi_x(S_x))^{G_x(F),\chi} = 0$. Hence $\xi' = 0$.

Let $S \subset X(F)$ be the set of all G-semisimple points. Let $U = \bigcup_{x \in S} U_x$. We saw that $\xi|_U = 0$. On the other hand, U includes all the closed orbits, and hence by Proposition 2.1.7 $U = X$. $\qquad\square$

The following generalization of this theorem is proven in the same way.

Theorem 3.1.4. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let χ be a character of K. Suppose that for any G-semisimple $x \in X(F)$ we have*

$$\mathcal{D}(N^X_{Gx,x}(F))^{K_x,\chi} = 0.$$

Then

$$\mathcal{D}(X(F))^{K,\chi} = 0.$$

Now we would like to formulate a slightly more general version of this theorem concerning K-equivariant vector systems. During first reading of this paper one can skip to the next subsection.

Definition 3.1.5. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup. Let $\mathcal{E}$ be a K-equivariant vector system on $X(F)$. Let $x \in X(F)$ be G-semisimple. Let $\mathcal{E}'$ be a K_x-equivariant vector system on $N^X_{Gx,x}(F)$. We say that $\mathcal{E}$ and $\mathcal{E}'$ are **compatible** if there exists an analytic Luna slice (U, p, ψ, S, N) such that $\mathcal{E}|_S = \psi^*(\mathcal{E}')$.*

Note that if $\mathcal{E}$ and $\mathcal{E}'$ are constant with the same fiber then they are compatible.

The following theorem is proven in the same way as Theorem 3.1.1.

Theorem 3.1.6. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let $\mathcal{E}$ be a K-equivariant vector system on $X(F)$. Suppose that for any G-semisimple $x \in X(F)$ there exists a K-equivariant vector system $\mathcal{E}'$ on $N^X_{Gx,x}(F)$, compatible with $\mathcal{E}$ such that*

$$\mathcal{D}(N^X_{Gx,x}(F), \mathcal{E}')^{K_x} = 0.$$

Then

$$\mathcal{D}(X(F), \mathcal{E})^K = 0.$$

If $\mathcal{E}$ and $\mathcal{E}'$ are B-vector systems and K is open B-analytic subgroup[1] then the theorem holds also for Schwartz distributions. Namely, if $\mathcal{S}^*(N^X_{Gx,x}(F), \mathcal{E}')^{K_x} = 0$ for any x then $\mathcal{S}^*(X(F), \mathcal{E})^K = 0$, and the proof is the same.

3.2. **A stronger version.**

In this section we give a way to validate the conditions of theorems 3.1.1, 3.1.4 and 3.1.6 by induction.

The goal of this section is to prove the following theorem.

Theorem 3.2.1. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let χ be a character of K. Suppose that for any G-semisimple $x \in X(F)$ such that*

$$\mathcal{D}(R(N^X_{Gx,x}))^{K_x,\chi} = 0$$

---

[1] In fact, any open subgroup of a B-analytic group is B-analytic.

we have

$$\mathcal{D}(Q(N_{Gx,x}^X))^{K_x,\chi} = 0.$$

Then for any for any G-semisimple $x \in X(F)$ we have

$$\mathcal{D}(N_{Gx,x}^X(F))^{K_x,\chi} = 0.$$

This theorem together with Theorem 3.1.4 give the following corollary.

Corollary 3.2.2. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let χ be a character of K. Suppose that for any G-semisimple $x \in X(F)$ such that*

$$\mathcal{D}(R(N_{Gx,x}^X))^{K_x,\chi} = 0$$

we have

$$\mathcal{D}(Q(N_{Gx,x}^X))^{K_x,\chi} = 0.$$

Then $\mathcal{D}(X(F))^{K,\chi} = 0$.

From now till the end of the section we fix G, X, K and χ. Let us introduce several definitions and notations.

Notation 3.2.3. *Denote*

- $T \subset X(F)$ *the set of all G-semisimple points.*
- *For $x, y \in T$ we say that $x > y$ if $G_x \supsetneqq G_y$.*
- $T_0 := \{x \in T | \mathcal{D}(Q(N_{Gx,x}^X))^{K_x,\chi} = 0\}$.

Note that if $x \in T_0$ then $\mathcal{D}(N_{Gx,x}^X(F))^{K_x,\chi} = 0$.

Proof of Theorem 3.2.1. We have to show that $T = T_0$. Assume the contrary.

Note that every chain in T with respect to our ordering has a minimum. Hence by Zorn's lemma every non-empty set in T has a minimal element. Let x be a minimal element of $T - T_0$. To get a contradiction, it is enough to show that $\mathcal{D}(R(N_{Gx,x}^X))^{K_x,\chi} = 0$.

Denote $R := R(N_{Gx,x}^X)$. By Theorem 3.1.4, it is enough to show that for any $y \in R$ we have

$$\mathcal{D}(N_{G(F)_x y,y}^R)^{(K_x)_y,\chi} = 0.$$

Let (U, p, ψ, S, N) be an analytic Luna slice at x.

We can assume that $y \in \psi(S)$ since $\psi(S)$ is open, includes 0, and we can replace y by λy for any $\lambda \in F^\times$. Let $z \in S$ be such that $\psi(z) = y$. By corollary 2.1.18, $(G(F)_x)_y = G(F)_z$ and $N_{G(F)_x y,y}^R \cong N_{Gz,z}^X(F)$. Hence $(K_x)_y = K_z$ and therefore

$$\mathcal{D}(N_{G(F)_x y,y}^R)^{(K_x)_y,\chi} \cong \mathcal{D}(N_{Gz,z}^X(F))^{K_z,\chi}.$$

However $z < x$ and hence $z \in T_0$ which means $\mathcal{D}(N_{Gz,z}^X(F))^{K_z,\chi} = 0$. $\square$

Remark 3.2.4. *As before, Theorem 3.2.1 and Corollary 3.2.5 hold also for Schwartz distributions, and the proof is the same.*

Again, we can formulate a more general version of Corollary 3.2.2 concerning vector systems. During first reading of this paper one can skip to the next subsection.

Theorem 3.2.5. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let $\mathcal{E}$ be a K-equivariant vector system on $X(F)$.*

Suppose that for any G-semisimple $x \in X(F)$ such that
(*) any $K_x \times F^\times$-equivariant vector system $\mathcal{E}'$ on $R(N_{Gx,x}^X)$ compatible with $\mathcal{E}$ satisfies $\mathcal{D}(R(N_{Gx,x}^X), \mathcal{E}')^{K_x} = 0$ (where the action of $F^\times$ is the homothety action),

we have
(**) there exists a $K_x \times F^\times$-equivariant vector system $\mathcal{E}'$ on $Q(N_{Gx,x}^X)$ compatible with $\mathcal{E}$ such that

$$\mathcal{D}(Q(N_{Gx,x}^X), \mathcal{E}')^{K_x} = 0.$$

Then $\mathcal{D}(X(F), \mathcal{E})^K = 0$.

The proof is the same as the proof of Theorem 3.2.1 using the following lemma that follows from the definitions.

Lemma 3.2.6. *Let a reductive group G act on a smooth affine variety X. Let $K \subset G(F)$ be an open subgroup and let $\mathcal{E}$ be a K-equivariant vector system on $X(F)$. Let $x \in X(F)$ be G-semisimple. Let (U, p, ψ, S, N) be an analytic Luna slice at x.*

Let $\mathcal{E}'$ be a K_x-equivariant vector system on N compatible with $\mathcal{E}$. Let $y \in S$ be G-semisimple. Let $z := \psi(y)$. Let $\mathcal{E}''$ be a $(K_x)_z$-equivariant vector system on $N_{G_x z, z}^N$ compatible with $\mathcal{E}'$. Consider the isomorphism $N_{G_x z, z}^N(F) \cong N_{Gy,y}^X(F)$ and let $\mathcal{E}'''$ be the corresponding K_y-equivariant vector system on $N_{Gy,y}^X(F)$.

Then $\mathcal{E}'''$ is compatible with $\mathcal{E}$.

Again, if $\mathcal{E}$ and $\mathcal{E}'$ are B-vector systems then the theorem holds also for Schwartz distributions.

4. Distributions versus Schwartz distributions

The tools developed in the previous section enabled us to prove the following version of localization principle.

Theorem 4.0.1 (Localization principle)**.** *Let a reductive group G act on a smooth algebraic variety X. Let Y be an algebraic variety and $\phi : X \to Y$ be an affine algebraic G-invariant map. Let χ be a character of $G(F)$. Suppose that for any $y \in Y(F)$ we have $\mathcal{D}_{X(F)}(\phi(F)^{-1}(y))^{G(F),\chi} = 0$. Then $\mathcal{D}(X(F))^{G(F),\chi} = 0$.*

For proof see Appendix A.

In this section we use this theorem to show that if there are no $G(F)$-equivariant Schwartz distributions on $X(F)$ then there are no $G(F)$-equivariant distributions on $X(F)$.

Theorem 4.0.2. *Let a reductive group G act on a smooth affine variety X. Let V be a finite dimensional continuous representation of $G(F)$ over $\mathbb{R}$. Suppose that $\mathcal{S}^*(X(F), V)^{G(F)} = 0$. Then $\mathcal{D}(X(F), V)^{G(F)} = 0$.*

For the proof we will need the following definition and theorem.

Definition 4.0.3.
*(i) Let a topological group K act on a topological space M. We call a closed K-invariant subset $C \subset M$ **compact modulo** K if there exists a compact subset $C' \subset M$ such that $C \subset KC'$.*

*(ii) Let a Nash group K act on a Nash manifold M. We call a closed K-invariant subset $C \subset M$ **Nashly compact modulo** K if there exist a compact subset $C' \subset M$ and semi-algebraic closed subset $Z \subset M$ such that $C \subset Z \subset KC'$.*

Remark 4.0.4. *Let a reductive group G act on a smooth affine variety X. Let $K := G(F)$ and $M := X(F)$. Then it is easy to see that the notions of compact modulo K and Nashly compact modulo K coincide.*

Theorem 4.0.5. *Let a Nash group K act on a Nash manifold M. Let E be a K-equivariant Nash bundle over M. Let $\xi \in \mathcal{D}(M, E)^K$ such that $\mathrm{Supp}(\xi)$ is Nashly compact modulo K. Then $\xi \in \mathcal{S}^*(M, E)^K$.*

The formulation and the idea of the proof of this theorem are due to J. Bernstein. For the proof see Appendix C.4.

Proof of Theorem 4.0.2. Fix any $y \in X/G(F)$ and denote $M := \pi_X^{-1}(y)(F)$.

By localization principle (Theorem 4.0.1 and Remark A.0.4), it is enough to prove that

$$\mathcal{S}^*_{X(F)}(M, V)^{G(F)} = \mathcal{D}_{X(F)}(M, V)^{G(F)}.$$

Choose $\xi \in \mathcal{D}_{X(F)}(M, V)^{G(F)}$. M has a unique stable closed G-orbit and hence a finite number of closed $G(F)$-orbits. By Theorem 4.0.5, it is enough to show that M is Nashly compact modulo $G(F)$. Clearly M is semi-algebraic. Choose representatives x_i of the closed $G(F)$ orbits in M. Choose compact neighborhoods C_i of x_i. Let $C' := \bigcup C_i$. By corollary 2.1.7, $G(F)C' \supset M$. □

5. Applications of Fourier transform and Weil representation

Let G be a reductive group. Let V be a finite dimensional algebraic representation of G over F. Let χ be a character of $G(F)$. In this section we provide some tools to verify $\mathcal{S}^*(Q(V))^{G(F),\chi} = 0$ if we know that $\mathcal{S}^*(R(V))^{G(F),\chi} = 0$.

5.1. **Preliminaries.**

From now till the end of the paper we fix an additive character κ of F. If F is archimedean we fix κ to be defined by $\kappa(x) := e^{2\pi i \operatorname{Re}(x)}$.

Notation 5.1.1. *Let V be a vector space over F. Let B be a non-degenerate bilinear form on V. We denote by $\mathcal{F}_B : \mathcal{S}^*(V) \to \mathcal{S}^*(V)$ the Fourier transform given by B with respect to the self-adjoint Haar measure on V. For any B-analytic manifold M over F we also denote by $\mathcal{F}_B : \mathcal{S}^*(M \times V) \to \mathcal{S}^*(M \times V)$ the partial Fourier transform.*

Notation 5.1.2. *Let V be a vector space over F. Consider the homothety action of $F^\times$ on V by $\rho(\lambda)v := \lambda^{-1}v$. It gives rise to an action ρ of $F^\times$ on $\mathcal{S}^*(V)$.*

Also, for any $\lambda \in F^\times$ denote $|\lambda| := \frac{dx}{\rho(\lambda)dx}$, where dx denotes the Haar measure on F. Note that for $F = \mathbb{R}$, $|\lambda|$ is equal to the classical absolute value but for $F = \mathbb{C}$, $|\lambda| = (Re\lambda)^2 + (Im\lambda)^2$.

Notation 5.1.3. *Let V be a vector space over F. Let B be a non-degenerate symmetric bilinear form on V. We denote by $\gamma(B)$ the Weil constant. For its definition see e.g. [Gel], section 2.3 for non-archimedean F and [RS1], section 1 for archimedean F.*

For any $t \in F^\times$ denote $\delta_B(t) = \gamma(B)/\gamma(tB)$.

Note that $\gamma_B(t)$ is an eights root of unity and if $dimV$ is odd and $F \neq \mathbb{C}$ then δ_B is **not** a multiplicative character.

Notation 5.1.4. *Let V be a vector space over F. Let B be a non-degenerate symmetric bilinear form on V. We denote*

$$Z(B) := \{x \in V | B(x,x) = 0\}.$$

Theorem 5.1.5 (non-archimedean homogeneity). *Suppose that F is **non-archimedean**. Let V be a vector space over F. Let B be a non-degenerate symmetric bilinear form on V. Let M be a B-analytic manifold over F. Let $\xi \in \mathcal{S}^*_{V \times M}(Z(B) \times M)$ be such that $\mathcal{F}_B(\xi) \in \mathcal{S}^*_{V \times M}(Z(B) \times M)$. Then for any $t \in F^\times$, we have $\rho(t)\xi = \delta_B(t)|t|^{dimV/2}\xi$ and $\xi = \gamma(B)^{-1}\mathcal{F}_B\xi$. In particular, if $dimV$ is odd then $\xi = 0$.*

For proof see [RS2], section 8.1.

For the archimedean version of this theorem we will need the following definition.

Definition 5.1.6. *Let V be a finite dimensional vector space over F. Let B be a non-degenerate symmetric bilinear form on V. Let M be a B-analytic manifold over F. We say that a distribution $\xi \in \mathcal{S}^*(V \times M)$ is **adapted to** B if either*
(i) for any $t \in F^\times$ we have $\rho(t)\xi = \delta(t)|t|^{dimV/2}\xi$ and ξ is proportional to $\mathcal{F}_B\xi$ or
(ii) F is archimedean and for any $t \in F^\times$ we have $\rho(t)\xi = \delta(t)t|t|^{dimV/2}\xi$.

Note that if $dimV$ is odd and $F \neq \mathbb{C}$ then every B-adapted distribution is zero.

Theorem 5.1.7 (archimedean homogeneity)**.** *Let V be a vector space over F. Let B be a non-degenerate symmetric bilinear form on V. Let M be a Nash manifold. Let $L \subset \mathcal{S}^*_{V \times M}(Z(B) \times M)$ be a non-zero subspace such that $\forall \xi \in L$ we have $\mathcal{F}_B(\xi) \in L$ and $B\xi \in L$ (here B is interpreted as a quadratic form).*

Then there exists a non-zero distribution $\xi \in L$ which is adapted to B.

For archimedean F we prove this theorem in Appendix D. For the non-archimedean F it follows from Theorem 5.1.5.

We will also use the following trivial observation.

Lemma 5.1.8. *Let V be a finite dimensional vector space over F. Let a B-analytic group K act linearly on V. Let B be a K-invariant non-degenerate symmetric bilinear form on V. Let M be a B-analytic K-manifold over F. Let $\xi \in \mathcal{S}^*(V \times M)$ be a K-invariant distribution. Then $\mathcal{F}_B(\xi)$ is also K-invariant.*

5.2. **Applications.**
The following two theorems easily follow form the results of the previous subsection.

Theorem 5.2.1. *Suppose that F is non-archimedean. Let G be a reductive group. Let V be a finite dimensional algebraic representation of G over F. Let χ be character of $G(F)$. Suppose that $\mathcal{S}^*(R(V))^{G(F),\chi} = 0$. Let $V = V_1 \oplus V_2$ be a G-invariant decomposition of V. Let B be a G-invariant symmetric non-degenerate bilinear form on V_1. Consider the action ρ of $F^\times$ on V by homothety on V_1.*

Then any $\xi \in \mathcal{S}^*(Q(V))^{G(F),\chi}$ satisfies $\rho(t)\xi = \delta_B(t)|t|^{dimV_1/2}\xi$ and $\xi = \gamma(B)\mathcal{F}_B\xi$. In particular, if $dimV_1$ is odd then $\xi = 0$.

Theorem 5.2.2. *Let G be a reductive group. Let V be a finite dimensional algebraic representation of G over F. Let χ be character of $G(F)$. Suppose that $\mathcal{S}^*(R(V))^{G(F),\chi} = 0$. Let $Q(V) = W \oplus (\bigoplus_{i=1}^k V_i)$ be a G-invariant decomposition of $Q(V)$. Let B_i be G-invariant symmetric non-degenerate bilinear forms on V_i. Suppose that any $\xi \in \mathcal{S}^*_{Q(V)}(\Gamma(V))^{G(F),\chi}$ which is adapted to each B_i is zero.*

Then $\mathcal{S}^*(Q(V))^{G(F),\chi} = 0$.

Remark 5.2.3. *One can easily generalize theorems 5.2.2 and 5.2.1 to the case of constant vector systems.*

6. Tame actions

In this section we consider problems of the following type. A reductive group G acts on a smooth affine variety X, and τ is an automorphism of X which normalizes the action of G. We want to check whether any $G(F)$-invariant Schwartz distribution on $X(F)$ is also τ-invariant.

Definition 6.0.1. *Let π be an action of a reductive group G on a smooth affine variety X. We say that an algebraic automorphism τ of X is G-**admissible** if*
(i) $\pi(G(F))$ is of index ≤ 2 in the group of automorphisms of X generated by $\pi(G(F))$ and τ.
(ii) For any closed $G(F)$ orbit $O \subset X(F)$, we have $\tau(O) = O$.

Proposition 6.0.2. *Let π be an action of a reductive group G on a smooth affine variety X. Let τ be a G-admissible automorphism of X. Let $K := \pi(G(F))$ and let $\widetilde{K}$ be the group generated by $\pi(G(F))$ and τ. Let $x \in X(F)$ be a point with closed $G(F)$ orbit. Let $\tau' \in \widetilde{K}_x - K_x$. Then $d\tau'|_{N_{Gx,x}^X}$ is G_x-admissible.*

Proof. Let $\widetilde{G}$ denote the group generated by G and τ.
(i) is obvious.
(ii) Let $y \in N_{Gx,x}^X(F)$ be an element with closed G_x orbit. Let $y' = d\tau'(y)$. We have to show that there exists $g \in G_x(F)$ such that $gy = gy'$. Let (U, p, ψ, S, N) be analytic Luna slice at x with respect to the action of $\widetilde{G}$. We can assume that there exists $z \in S$ such that $y = \psi(z)$. Let $z' = \tau'(z)$. By corollary 2.1.18, z is G-semisimple. Since τ is admissible, this implies that there exists $g \in G(F)$ such that $gz = z'$. Clearly, $g \in G_x(F)$ and $gy = y'$. $\qquad\square$

Definition 6.0.3. *We call an action of a reductive group G on a smooth affine variety X **tame** if for any G-admissible $\tau : X \to X$, we have $\mathcal{S}^*(X(F))^{G(F)} \subset \mathcal{S}^*(X(F))^\tau$.*

Definition 6.0.4. *We call an algebraic representation of a reductive group G on a finite dimensional linear space V over F **linearly tame** if for any G-admissible linear map $\tau : V \to V$, we have $\mathcal{S}^*(V(F))^{G(F)} \subset \mathcal{S}^*(V(F))^\tau$.*

*We call a representation **weakly linearly tame** if for any G-admissible linear map $\tau : V \to V$, such that $\mathcal{S}^*(R(V))^{G(F)} \subset \mathcal{S}^*(R(V))^\tau$ we have $\mathcal{S}^*(Q(V))^{G(F)} \subset \mathcal{S}^*(Q(V))^\tau$.*

Theorem 6.0.5. *Let a reductive group G act on a smooth affine variety X. Suppose that for any G-semisimple $x \in X(F)$, the action of G_x on $N_{Gx,x}^X$ is weakly linearly tame. Then the action of G on X is tame.*

The proof is rather straightforward except of one minor complication: the group of automorphisms of $X(F)$ generated by the action of $G(F)$ is not necessarily a group of F points of any algebraic group.

Proof. Let $\tau : X \to X$ be an admissible automorphism.

Let $\widetilde{G} \subset Aut(X)$ be the algebraic group generated by the actions of G and τ. Let $K \subset Aut(X(F))$ be the B-analytic group generated by the action of $G(F)$. Let $\widetilde{K} \subset Aut(X(F))$ be the B-analytic group generated by the actions of G and τ. Note that $\widetilde{K} \subset \widetilde{G}(F)$ is an open subgroup of finite index. Note that for any $x \in X(F)$, x is $\widetilde{G}$-semisimple if and only if it is G-semisimple. If $K = \widetilde{K}$ we are done, so we will assume $K \neq \widetilde{K}$. Let χ be the character of $\widetilde{K}$ defined by $\chi(K) = \{1\}$, $\chi(\widetilde{K} - K) = \{-1\}$.

It is enough to prove that $\mathcal{S}^*(X)^{\widetilde{K},\chi} = 0$. By generalized Harish-Chandra descent (corollary 3.2.2) it is enough to prove that for any G-semisimple $x \in X$ such that $\mathcal{S}^*(R(N_{Gx,x}^X))^{\widetilde{K}_x,\chi} = 0$ we have $\mathcal{S}^*(Q(N_{Gx,x}^X))^{\widetilde{K}_x,\chi} = 0$. Choose any automorphism $\tau' \in \widetilde{K}_x - K_x$. Note that τ' and K_x generate $\widetilde{K}_x$. Denote

$$\eta := d\tau'|_{N_{Gx,x}^X(F)}.$$

By Proposition 6.0.2, η is G- admissible. Note that

$$\mathcal{S}^*(R(N_{Gx,x}^X))^{K_x} = \mathcal{S}^*(R(N_{Gx,x}^X))^{G(F)_x} \text{ and } \mathcal{S}^*(Q(N_{Gx,x}^X))^{K_x} = \mathcal{S}^*(Q(N_{Gx,x}^X))^{G(F)_x}.$$

Hence we have

$$\mathcal{S}^*(R(N_{Gx,x}^X))^{G(F)_x} \subset \mathcal{S}^*(R(N_{Gx,x}^X))^\eta.$$

Since the action of G_x is weakly linearly tame, this implies that

$$\mathcal{S}^*(Q(N_{Gx,x}^X))^{G(F)_x} \subset \mathcal{S}^*(Q(N_{Gx,x}^X))^\eta$$

and therefore $\mathcal{S}^*(Q(N_{Gx,x}^X))^{\widetilde{K}_x,\chi} = 0$. $\qquad\square$

Definition 6.0.6. *We call an algebraic representation of a reductive group G on a finite dimensional linear space V over F **special** if for any $\xi \in \mathcal{S}^*_{Q(V)}(\Gamma(V))^{G(F)}$ such that for any G-invariant decomposition $Q(V) = W_1 \oplus W_2$ and any two G-invariant symmetric non-degenerate bilinear forms B_i on W_i the Fourier transforms $\mathcal{F}_{B_i}(\xi)$ are also supported in $\Gamma(V)$, we have $\xi = 0$.*

Proposition 6.0.7. *Every special algebraic representation V of a reductive group G is weakly linearly tame.*

This proposition follows immediately from the following lemma.

Lemma 6.0.8. *Let V be an algebraic representation of a reductive group G. Let τ be an admissible linear automorphism of V. Let $V = W_1 \oplus W_2$ be a G-invariant decomposition of V and B_i be G-invariant symmetric non-degenerate bilinear forms on W_i. Then W_i and B_i are also τ-invariant.*

This lemma follows in turn from the following one.

Lemma 6.0.9. *Let V be an algebraic representation of a reductive group G. Let τ be an admissible automorphism of V. Then $\mathcal{O}(V)^G \subset \mathcal{O}(V)^\tau$.*

Proof. Consider the projection $\pi : V \to V/G$. We have to show that τ acts trivially on V/G. Let $x \in \pi(V(F))$. Let $X := \pi^{-1}(x)$. By Proposition 2.1.6 $G(F)$ has a closed orbit in $X(F)$. The automorphism τ preserves this orbit and hence preserves x. So τ acts trivially on $\pi(V(F))$, which is Zariski dense in V/G. Hence τ acts trivially on V/G. □

Now we introduce a criterion that allows to prove that a representation is special. It follows immediately from Theorem 5.1.7.

Lemma 6.0.10. *Let V be an algebraic representation of a reductive group G. Let $Q(V) = \bigoplus W_i$ be a G-invariant decomposition. Let B_i be symmetric non-degenerate G-invariant bilinear forms on W_i. Suppose that any $\xi \in \mathcal{S}^*_{Q(V)}(\Gamma(V))^{G(F)}$ which is adapted to all B_i is zero. Then V is special.*

7. SYMMETRIC PAIRS

In this section we apply our tools to symmetric pairs. We introduce several properties of symmetric pairs and discuss their interrelations. In Appendix E we present a diagram that illustrates the most important ones.

7.1. **Preliminaries and notations.**

Definition 7.1.1. *A **symmetric pair** is a triple (G, H, θ) where $H \subset G$ are reductive groups, and θ is an involution of G such that $H = G^\theta$. We call a symmetric pair **connected** if G/H is connected.*

*For a symmetric pair (G, H, θ) we define an antiinvolution $\sigma : G \to G$ by $\sigma(g) := \theta(g^{-1})$, denote $\mathfrak{g} := Lie G$, $\mathfrak{h} := Lie H$. Let θ and σ act on $\mathfrak{g}$ by their differentials and denote $\mathfrak{g}^\sigma := \{a \in \mathfrak{g} | \sigma(a) = a\} = \{a \in \mathfrak{g} | \theta(a) = -a\}$. Note that H acts on $\mathfrak{g}^\sigma$ by the adjoint action. Denote also $G^\sigma := \{g \in G | \sigma(g) = g\}$ and define a **symmetrization map** $s : G \to G^\sigma$ by $s(g) := g\sigma(g)$.*

Definition 7.1.2. *Let (G_1, H_1, θ_1) and (G_2, H_2, θ_2) be symmetric pairs. We define their **product** to be the symmetric pair $(G_1 \times G_2, H_1 \times H_2, \theta_1 \times \theta_2)$.*

Theorem 7.1.3. *For any connected symmetric pair (G, H, θ) we have $\mathcal{O}(G)^{H \times H} \subset \mathcal{O}(G)^\sigma$.*

Proof. Consider the multiplication map $H \times G^\sigma \to G$. It is étale at 1×1 and hence its image HG^σ contains an open neighborhood of 1 in G. Hence the image of HG^σ in G/H is dense. Thus $HG^\sigma H$ is dense in G. Clearly $\mathcal{O}(HG^\sigma H)^{H \times H} \subset \mathcal{O}(HG^\sigma H)^\sigma$ and hence $\mathcal{O}(G)^{H \times H} \subset \mathcal{O}(G)^\sigma$. □

Corollary 7.1.4. *For any connected symmetric pair* (G, H, θ) *and any closed* $H \times H$ *orbit* $\Delta \subset G$, *we have* $\sigma(\Delta) = \Delta$.

Proof. Denote $\Upsilon := H \times H$. Consider the action of the 2-element group $(1, \tau)$ on Υ given by $\tau(h_1, h_2) := (\theta(h_2), \theta(h_1))$. This defines the semi-direct product $\widetilde{\Upsilon} := (1, \tau) \ltimes \Upsilon$. Extend the two-sided action of Υ to $\widetilde{\Upsilon}$ by the antiinvolution σ. Note that the previous theorem implies that $G/\Upsilon = G/\widetilde{\Upsilon}$. Let Δ be a closed Υ-orbit. Let $\widetilde{\Delta} := \Delta \cup \sigma(\Delta)$. Let $a := \pi_G(\widetilde{\Delta}) \subset G/\widetilde{\Upsilon}$. Clearly, a consists of one point. On the other hand, $G/\widetilde{\Upsilon} = G/\Upsilon$ and hence $\pi_G^{-1}(a)$ contains a unique closed G-orbit. Therefore $\Delta = \widetilde{\Delta} = \sigma(\Delta)$. $\square$

Corollary 7.1.5. *Let* (G, H, θ) *be a connected symmetric pair. Let* $g \in G(F)$ *be* $H \times H$-*semisimple. Suppose that* $H^1(F, (H \times H)_g)$ *is trivial. Then* $\sigma(g) \in H(F)gH(F)$.

For example, if $(H \times H)_g$ is a product of general linear groups over some field extensions then $H^1(F, (H \times H)_g)$ is trivial.

Definition 7.1.6. *A symmetric pair* (G, H, θ) *is called* ***good*** *if for any closed* $H(F) \times H(F)$ *orbit* $O \subset G(F)$, *we have* $\sigma(O) = O$.

Corollary 7.1.7. *Any connected symmetric pair over* $\mathbb{C}$ *is good.*

Definition 7.1.8. *A symmetric pair* (G, H, θ) *is called a* ***GK pair*** *if*

$$\mathcal{S}^*(G(F))^{H(F) \times H(F)} \subset \mathcal{S}^*(G(F))^{\sigma}.$$

We will see later in section 8 that GK pairs satisfy a Gelfand pair property that we call GP2 (see Definition 8.1.2 and Theorem 8.1.4). Clearly every GK pair is good and we conjecture that the converse is also true. We will discuss it in more details in subsection 7.5.

Lemma 7.1.9. *Let* (G, H, θ) *be a symmetric pair. Then there exists a* G-*invariant* θ-*invariant non-degenerate symmetric bilinear form* B *on* $\mathfrak{g}$. *In particular,* $B|_{\mathfrak{h}}$ *and* $B|_{\mathfrak{g}^{\sigma}}$ *are also non-degenerate and* $\mathfrak{h}$ *is orthogonal to* $\mathfrak{g}^{\sigma}$.

Proof.
Step 1. Proof for semisimple $\mathfrak{g}$.
Let B be the Killing form on $\mathfrak{g}$. Since it is non-degenerate, it is enough to show that $\mathfrak{h}$ is orthogonal to $\mathfrak{g}^{\sigma}$. Let $A \in \mathfrak{h}$ and $B \in \mathfrak{g}^{\sigma}$. We have to show $tr(Ad(A)Ad(B)) = 0$. This follows from the fact that $Ad(A)Ad(B)(\mathfrak{h}) \subset \mathfrak{g}^{\sigma}$ and $Ad(A)Ad(B)(\mathfrak{g}^{\sigma}) \subset \mathfrak{h}$.
Step 2. Proof in the general case.
Let $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$ such that $\mathfrak{g}'$ is semisimple and $\mathfrak{z}$ is the center. It is easy to see that this decomposition is θ invariant. Now the proposition easily follows from the previous case. $\square$

Lemma 7.1.10. *Let* (G, H, θ) *be a symmetric pair. Then there exists an* $Ad(G(F))$-*equivariant and* σ-*equivariant map* $\mathcal{U}(G) \to \mathcal{N}(\mathfrak{g})$ *where* $\mathcal{U}(G)$ *is the set of unipotent elements in* $G(F)$ *and* $\mathcal{N}(\mathfrak{g})$ *is the set of nilpotent elements in* $\mathfrak{g}(F)$.

Proof. It follows from the existence of analytic Luna slice at point $1 \in G(F)$ with respect to the action of $\widetilde{G}$ where $\widetilde{G}$ is the group generated by σ and the adjoint action of G on itself. $\square$

Lemma 7.1.11. *Let* (G, H, θ) *be a symmetric pair. Let* $x \in \mathfrak{g}^{\sigma}$ *be a nilpotent element. Then there exists a group homomorphism* $\phi : SL_2 \to G$ *such that*

$$d\phi(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})) = x, \quad d\phi(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})) \in \mathfrak{g}^{\sigma} \quad and \quad \phi(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})) \in H.$$

In particular $0 \in \overline{Ad(H)(x)}$.

This lemma was essentially proven for $F = \mathbb{C}$ in [KR]. The same proof works for any F and we repeat it here for the convenience of the reader.

Proof. By Jacobson-Morozov theorem (see [Jac], Chapter III, Theorems 17 and 10) we can complete x to an sl_2-triple (x_-, s, x). Let $s' := \frac{s + \theta(s)}{2}$. It satisfies $[s', x] = 2x$ and lies in the ideal $[x, \mathfrak{g}]$ and hence by Morozov lemma (see [Jac], Chapter III, Lemma 7), x and s' can be completed to an sl_2 triple (x_-, s', x). Let $x'_- := \frac{x_- - \theta(x_-)}{2}$. Note that (x'_-, s', x) is also an sl_2-triple. Exponentiating this sl_2-triple to a map $SL_2 \to G$ we get the required homomorphism. $\qquad\square$

Notation 7.1.12. *In the notations of the previous lemma we denote*

$$D_t(x) := \phi(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) \text{ and } d(x) := d\phi(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}).$$

Those elements depend on the choice of ϕ. However, whenever we will use this notation nothing will depend on their choice.

7.2. Descendants of symmetric pairs.

Proposition 7.2.1. *Let (G, H, θ) be a symmetric pair. Let $g \in G(F)$ be $H \times H$-semisimple. Let $x = s(g)$. Then*
(i) x is semisimple.
(ii) Consider the adjoint action of G on itself and the two-sided action of $H \times H$ on G. Then $H_x \cong (H \times H)_g$ and $(\mathfrak{g}_x)^\sigma \cong N^G_{HgH,g}$ as H_x spaces.

Proof.

(i) Let $x = x_s x_u$ be the Jordan decomposition of x. The uniqueness of Jordan decomposition implies that both x_u and x_s belong to G^σ. To show that $x_u = 1$ it is enough to show that $\overline{Ad(H)(x)} \ni x_s$. We will do that in several steps.

Step 1. Proof for the case when $x_s = 1$.
It follows immediately from the two previous lemmas (7.1.10 and 7.1.11).

Step 2. Proof for the case when $x_s \in Z(G)$.
This case follows from Step 1 since conjugation acts trivially on $Z(G)$.

Step 3. Proof in the general case.
The statement follows from Step 2 for the group G_{x_s}.

(ii) The symmetrization gives rise to an isomorphism $(H \times H)_g \cong H_x$. Let us now prove $(\mathfrak{g}_x)^\sigma \cong N^G_{HgH,g}$. First of all, $N^G_{HgH,g} \cong \mathfrak{g}/(\mathfrak{h} + Ad(g)\mathfrak{h})$. Let θ' be the involution of G defined by $\theta'(y) = x\theta(y)x^{-1}$. Note that $Ad(g)\mathfrak{h} = \mathfrak{g}^{\theta'}$. Fix a non-degenerate G-invariant symmetric bilinear form B on $\mathfrak{g}$ as in Lemma 7.1.9. Note that B is also θ' invariant and hence

$$(Ad(g)\mathfrak{h})^\perp = \{a \in \mathfrak{g} | \theta'(a) = -a\}.$$

Now

$$N^G_{HgH,g} \cong (\mathfrak{h} + Ad(g)\mathfrak{h})^\perp = \mathfrak{h}^\perp \cap Ad(g)\mathfrak{h}^\perp = \{a \in \mathfrak{g} | \theta(a) = \theta'(a) = -a\} = (\mathfrak{g}_x)^\sigma.$$

$\qquad\square$

It is easy to see that the isomorphism $N^G_{HgH,g} \cong (\mathfrak{g}_x)^\sigma$ does not depend on the choice of B.

Definition 7.2.2. *In the notations of the previous proposition we will say that the pair $(G_x, H_x.\theta|_{G_x})$ is a **descendant** of (G, H, θ).*

7.3. Tame symmetric pairs.

Definition 7.3.1. *We call a symmetric pair* (G, H, θ)
*(i) **tame** if the action of* $H \times H$ *on* G *is tame.*
*(ii) **linearly tame** if the action of* H *on* $\mathfrak{g}^\sigma$ *is linearly tame.*
*(iii) **weakly linearly tame** if the action of* H *on* $\mathfrak{g}^\sigma$ *is weakly linearly tame.*

Remark 7.3.2. *Evidently, any good tame symmetric pair is a GK pair.*

The following theorem is a direct corollary of Theorem 6.0.5.

Theorem 7.3.3. *Let* (G, H, θ) *be a symmetric pair. Suppose that all its descendants (including itself) are weakly linearly tame. Then* (G, H, θ) *is tame and linearly tame.*

Definition 7.3.4. *We call a symmetric pair* (G, H, θ) ***special*** *if* $\mathfrak{g}^\sigma$ *is a special representation of* H.

Proposition 7.3.5. *Any special symmetric pair is weakly linearly tame.*

This proposition follows immediately from Proposition 6.0.7

Proposition 7.3.6. *A product of special symmetric pairs is special.*

The proof of this proposition is straightforward using Lemma 7.1.9.

Now we would like to give a criterion of speciality for symmetric pairs.

Proposition 7.3.7 (Speciality criterion)**.** *Let* (G, H, θ) *be a symmetric pair. Suppose that for any nilpotent* $x \in \mathfrak{g}^\sigma$ *either*
(i) $\mathrm{Tr}(ad(d(x))|_{\mathfrak{h}_x}) < dim\mathfrak{g}^\sigma$ *or*
(ii) F *is non-archimedean and* $\mathrm{Tr}(ad(d(x))|_{\mathfrak{h}_x}) \neq dim\mathfrak{g}^\sigma$.
 Then the pair (G, H, θ) *is special.*

For the proof we will need the following lemmas.

Lemma 7.3.8. *Let* (G, H, θ) *be a symmetric pair. Then* $\Gamma(\mathfrak{g}^\sigma)$ *is the set of all nilpotent elements in* $Q(\mathfrak{g}^\sigma)$.

This lemma is a direct corollary from Lemma 7.1.11.

Lemma 7.3.9. *Let* (G, H, θ) *be a symmetric pair. Let* $x \in \mathfrak{g}^\sigma$ *be a nilpotent element. Then all the eigenvalues of* $ad(d(x))|_{\mathfrak{g}^\sigma/[x,\mathfrak{h}]}$ *are non-positive integers.*

This lemma follows from the existence of a natural onto map $\mathfrak{g}/[x, \mathfrak{g}] \twoheadrightarrow \mathfrak{g}^\sigma/[x, \mathfrak{h}]$ using the following straightforward lemma.

Lemma 7.3.10. *Let* V *be a representation of an* sl_2 *triple* (e, h, f). *Then all the eigenvalues of* $h|_{V/e(V)}$ *are non-positive integers.*

Now we are ready to prove the speciality criterion.

Proof of Proposition 7.3.7. We will give a proof in the case that F is archimedean. The case of non-archimedean F is done in the same way but with less complications.

Let χ be a character of $F^\times$ given by either $\chi(\lambda) = u(\lambda)|\lambda|^{dim\mathfrak{g}^\sigma/2}$ or $\chi(\lambda) = u(\lambda)|\lambda|^{dim\mathfrak{g}^\sigma/2+1}$, where u is some unitary character. By Lemma 6.0.10 it is enough to prove

$$\mathcal{S}^*_{Q(\mathfrak{g}^\sigma)}(\Gamma(\mathfrak{g}^\sigma))^{H(F) \times F^\times, (1,\chi)} = 0.$$

$\Gamma(\mathfrak{g}^\sigma)$ has a finite number of H orbits (it follows from Lemma 7.3.8 and the introduction of [KR]). Hence it is enough to show that for any $x \in \Gamma(\mathfrak{g}^\sigma)$ we have

$$\mathcal{S}^*(Ad(H(F))x, Sym^k(CN^{\mathfrak{g}^\sigma}_{Ad(H(F))x}))^{H(F) \times F^\times, (1,\chi)} = 0 \text{ for any } k.$$

Let $K := \{(D_t(x), t^2) | t \in F^\times\} \subset (H(F) \times F^\times)_x$.

Note that

$$\Delta_{(H(F) \times F^\times)_x}((D_t(x), t^2)) = |det(Ad(D_t(x))|_{\mathfrak{g}^\sigma_x})| = |t|^{\text{Tr}(ad(d(x))|_{\mathfrak{h}_x})}.$$

By Lemma 7.3.9 the eigenvalues of the action of $(D_t(x), t^2)$ on $(Sym^k(\mathfrak{g}^\sigma/[x, \mathfrak{h}]))$ are of the form t^l where l is a non-positive integer.

Now by Frobenius reciprocity (Theorem 2.3.7) we have

$$\mathcal{S}^*((H(F))x, Sym^k(CN^{\mathfrak{g}^\sigma}_{Ad(H(F))x}))^{H(F) \times F^\times, (1,\chi)} =$$
$$= \mathcal{S}^*(\{x\}, Sym^k(CN^{\mathfrak{g}^\sigma}_{Ad(H(F))x,x}) \otimes \Delta_{H(F) \times F^\times}|_{(H(F) \times F^\times)_x} \cdot \Delta^{-1}_{(H(F) \times F^\times)_x} \otimes (1, \chi))^{(H(F) \times F^\times)_x} =$$
$$= (Sym^k(\mathfrak{g}^\sigma/[x, \mathfrak{h}]) \otimes \Delta_{(H(F) \times F^\times)_x} \otimes (1, \chi)^{-1} \otimes_\mathbb{R} \mathbb{C})^{(H(F) \times F^\times)_x} \subset$$
$$\subset (Sym^k(\mathfrak{g}^\sigma/[x, \mathfrak{h}]) \otimes \Delta_{(H(F) \times F^\times)_x} \otimes (1, \chi)^{-1} \otimes_\mathbb{R} \mathbb{C})^K$$

which is zero since all the absolute values of the eigenvalues of the action of any $(D_t(x), t^2) \in K$ on

$$Sym^k(\mathfrak{g}^\sigma/[x, \mathfrak{h}]) \otimes \Delta_{(H(F) \times F^\times)_x} \otimes (1, \chi)^{-1}$$

are of the form $|t|^l$ where $l < 0$. □

7.4. Regular symmetric pairs.
In this subsection we will formulate a property which is weaker than weakly linearly tame but still enables us to prove GK property for good pairs.

Definition 7.4.1. *Let (G, H, θ) be a symmetric pair. We call an element $g \in G(F)$ **admissible** if*
(i) $Ad(g)$ commutes with θ (or, equivalently, $s(g) \in Z(G)$) and
(ii) $Ad(g)|_{\mathfrak{g}^\sigma}$ is H-admissible.

Definition 7.4.2. *We call a symmetric pair (G, H, θ) **regular** if for any admissible $g \in G(F)$ such that $\mathcal{S}^*(R(\mathfrak{g}^\sigma))^{H(F)} \subset \mathcal{S}^*(R(\mathfrak{g}^\sigma))^{Ad(g)}$ we have*

$$\mathcal{S}^*(Q(\mathfrak{g}^\sigma))^{H(F)} \subset \mathcal{S}^*(Q(\mathfrak{g}^\sigma))^{Ad(g)}.$$

Remark 7.4.3. *Clearly, every weakly linearly tame pair is regular.*

Proposition 7.4.4. *A product of regular symmetric pairs is regular.*

This is a direct corollary from Proposition 2.3.8.

The goal of this subsection is to prove the following theorem.

Theorem 7.4.5. *Let (G, H, θ) be a good symmetric pair such that all its descendants are regular. Then it is a GK pair.*

We will need several definitions and lemmas.

Definition 7.4.6. *Let (G, H, θ) be a symmetric pair. $g \in G$ is called **normal** if $\sigma(g)g = g\sigma(g)$.*

The following lemma is straightforward.

Lemma 7.4.7. *Let (G, H, θ) be a symmetric pair. Let $O \subset G(F)$ be an $H(F) \times H(F)$ orbit.*
(i) If $\sigma(O) = O$ then there exists a normal element $g \in O$.
(ii) Let $g \in G(F)$ be a normal element. Then there exists $h \in H(F)$ such that $gh = hg = \sigma(g)$.

Proof.
(i). Let $g' \in O$. We know that $\sigma(g') = h_1 g' h_2$ where $h_1, h_2 \in H(F)$. Let $g := g' h_1$. Then

$$\sigma(g)g = h_1^{-1}\sigma(g')g'h_1 = h_1^{-1}\sigma(g')\sigma(\sigma(g'))h_1 =$$
$$= h_1^{-1}h_1 g'h_2\sigma(h_1 g' h_2))h_1 = g'\sigma(g') = g'h_1 h_1^{-1}\sigma(g') = g\sigma(g).$$

(ii) Follows from the fact that $g^{-1}\sigma(g) = \sigma(g)g^{-1} \in H(F)$. $\qquad\square$

Notation 7.4.8. *Let (G, H, θ) be a symmetric pair. We denote $\widetilde{H \times H} := H \times H \rtimes \{1, \sigma\}$ where $\sigma \cdot (h_1, h_2) = (\theta(h_2), \theta(h_1)) \cdot \sigma$. The two-sided action of $H \times H$ on G is extended to action of $\widetilde{H \times H}$ in the natural way. We denote by χ the character of $\widetilde{H \times H}$ defined by $\chi(\widetilde{H \times H} - H \times H) = \{-1\}$, $\chi(H \times H) = \{1\}$.*

Proposition 7.4.9. *Let (G, H, θ) be a good symmetric pair. Let $O \subset G(F)$ be a closed $H(F) \times H(F)$ orbit.*
Then for any $g \in O$ there exist $\tau \in (\widetilde{H \times H})_g(F) - (H \times H)_g(F)$ and $g' \in G_{s(g)}(F)$ such that $Ad(g')$ commutes with θ on $G_{s(g)}$ and the action of τ on $N_{O,g}^G$ corresponds via the isomorphism given by Proposition 7.2.1 to the adjoint action of g' on $\mathfrak{g}_{s(g)}^\sigma$.

Proof. Clearly, if the statement holds for some $g \in O$ then it holds for any $g \in O$.

Let $g \in O$ be a normal element. Let $h \in H(F)$ be such that $gh = hg = \sigma(g)$. Let $\tau := (h^{-1}, 1) \cdot \sigma$. Evidently, $\tau \in (\widetilde{H \times H})_g(F) - (H \times H)_g(F)$. Consider $d\tau_g : T_g G \to T_g G$. It corresponds via the identification $dg : \mathfrak{g} \cong T_g G$ to some $A : \mathfrak{g} \to \mathfrak{g}$. Clearly, $A = da$ where $a : G \to G$ is defined by $a(\alpha) = g^{-1}h^{-1}\sigma(g\alpha)$. However, $g^{-1}h^{-1}\sigma(g\alpha) = \theta(g)\sigma(\alpha)\theta(g)^{-1}$. Hence $A = Ad(\theta(g)) \circ \sigma$. Let B be a non-degenerate G-invariant σ-invariant symmetric form on $\mathfrak{g}$. By Theorem 7.1.3, A preserves B. Therefore τ corresponds to $A|_{\mathfrak{g}_{s(g)}^\sigma}$ via the isomorphism given by Proposition 7.2.1. However, σ is trivial on $\mathfrak{g}_{s(g)}^\sigma$ and hence $A|_{\mathfrak{g}_{s(g)}^\sigma} = Ad(\theta(g))|_{\mathfrak{g}_{s(g)}^\sigma}$. Since g is normal, $\theta(g) \in G_{s(g)}$. It is easy to see that $Ad(\theta(g))$ commutes with θ on $G_{s(g)}$. Hence we take $g' := \theta(g)$. $\qquad\square$

The last proposition implies Theorem 7.4.5. This implication is proven in the same way as Theorem 6.0.5.

7.5. Conjectures.

Conjecture 1 (van Dijk)**.** *If $F = \mathbb{C}$, any connected symmetric pair is a Gelfand pair (GP3, see Definition 8.1.2 below).*

By theorem 8.1.4 it follows from the following conjecture.

Conjecture 2. *If $F = \mathbb{C}$, any connected symmetric pair is a GK pair.*

By Corollary 7.1.7 it follows from the following more general conjecture.

Conjecture 3. *Every good symmetric pair is a GK pair.*

which in turn follows (by Theorem 7.4.5) from the following one.

Conjecture 4. *Any symmetric pair is regular.*

An indirect evidence for this conjecture is that one can show that every GK pair is regular.

Remark 7.5.1. *It is well known that if F is archimedean, G is connected and H is compact then the pair (G, H, θ) is good, Gelfand (GP1, see Definition 8.1.2 below) and in fact also GK.*

Remark 7.5.2. *In general, not every symmetric pair is good. For example, $(SL_2(\mathbb{R}), T)$ where T is the split torus. Also, it is not a Gelfand pair (even not GP3, see Definition 8.1.2 below).*

Remark 7.5.3. *We do not believe that any symmetric pair is special. However, in the next subsection we will prove that certain symmetric pairs are special.*

7.6. The pairs $(G \times G, \Delta G)$ and $(G_{E/F}, G)$ are tame.

Notation 7.6.1. *Let E be a quadratic extension of F. Let G be an algebraic group defined over F. We denote by $G_{E/F}$ the canonical algebraic group defined over F such that $G_{E/F}(F) = G(E)$.*

In this section we will prove the following theorem.

Theorem 7.6.2. *Let G be a reductive group.*
(i)Consider the involution θ of $G \times G$ given by $\theta((g, h)) := (h, g)$. Its fixed points form the diagonal subgroup ΔG. Then the symmetric pair $(G \times G, \Delta G, \theta)$ is tame.
(ii) Let E be a quadratic extension of F. Consider the involution γ of $G_{E/F}$ given by the nontrivial element of $Gal(E/F)$. Its fixed points form G. Then the symmetric pair $(G_{E/F}, G, \gamma)$ is tame.

Corollary 7.6.3. *Let G be a reductive group. Then the adjoint action of G on itself is tame. In particular, every conjugation invariant distribution on $GL_n(F)$ is transposition invariant [2].*

For the proof of the theorem we will need the following straightforward lemma.

Lemma 7.6.4.
(i) Every descendant of $(G \times G, \Delta G, \theta)$ is of the form $(H \times H, \Delta H, \theta)$ for some reductive group H.
(ii) Every descendant of $(G_{E/F}, G, \gamma)$ is of the form $(H_{E/F}, H, \gamma)$ for some reductive group H.

Now Theorem 7.6.2 follows from the following theorem.

Theorem 7.6.5. *The pairs $(G \times G, \Delta G, \theta)$ and $(G_{E/F}, G, \gamma)$ are special for any reductive group G.*

By the speciality criterion (Proposition 7.3.7) this theorem follows from the following lemma.

Lemma 7.6.6. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Let $\{e, h, f\} \subset \mathfrak{g}$ be an sl_2 triple. Then $tr(Ad(h)|_{\mathfrak{g}_e})$ is an integer smaller than $dim\mathfrak{g}$.*

Proof. Consider $\mathfrak{g}$ as a representation of sl_2 via the triple (e, h, f). Decompose it into irreducible representations $\mathfrak{g} = \bigoplus V_i$. Let λ_i be the highest weights of V_i. Clearly

$$tr(Ad(h)|_{\mathfrak{g}_e}) = \sum \lambda_i \text{ and } dim\mathfrak{g} = \sum (\lambda_i + 1).$$

$\square$

8. APPLICATIONS TO GELFAND PAIRS

8.1. Preliminaries on Gelfand pairs and distributional criteria.
In this section we recall a technique due to Gelfand and Kazhdan which allows to deduce statements in representation theory from statements on invariant distributions. For more detailed description see [AGS1], section 2.

Definition 8.1.1. *Let G be a reductive group. By an **admissible representation of** G we mean an admissible representation of $G(F)$ if F is non-archimedean (see [BZ]) and admissible smooth Fréchet representation of $G(F)$ if F is archimedean.*

We now introduce three notions of Gelfand pair.

---

[2]In the non-archimedean case, the later is a classical result of Gelfand and Kazhdan, see [GK].

Definition 8.1.2. *Let $H \subset G$ be a pair of reductive groups.*
- *We say that (G, H) satisfy **GP1** if for any irreducible admissible representation (π, E) of G we have*
$$dimHom_{H(F)}(E, \mathbb{C}) \leq 1$$
- *We say that (G, H) satisfy **GP2** if for any irreducible admissible representation (π, E) of G we have*
$$dimHom_{H(F)}(E, \mathbb{C}) \cdot dimHom_H(\widetilde{E}, \mathbb{C}) \leq 1$$
- *We say that (G, H) satisfy **GP3** if for any irreducible **unitary** representation $(\pi, \mathcal{H})$ of $G(F)$ on a Hilbert space $\mathcal{H}$ we have*
$$dimHom_{H(F)}(\mathcal{H}^{\infty}, \mathbb{C}) \leq 1.$$

Property GP1 was established by Gelfand and Kazhdan in certain p-adic cases (see [GK]). Property GP2 was introduced in [Gro] in the p-adic setting. Property GP3 was studied extensively by various authors under the name **generalized Gelfand pair** both in the real and p-adic settings (see e.g. [vDP], [vD], [BvD]).

We have the following straightforward proposition.

Proposition 8.1.3. $GP1 \Rightarrow GP2 \Rightarrow GP3$.

We will use the following theorem from [AGS1] which is a version of a classical theorem of Gelfand and Kazhdan (see [GK]).

Theorem 8.1.4. *Let $H \subset G$ be reductive groups and let τ be an involutive anti-automorphism of G and assume that $\tau(H) = H$. Suppose $\tau(\xi) = \xi$ for all bi $H(F)$-invariant Schwartz distributions ξ on $G(F)$. Then (G, H) satisfies GP2.*

In some cases, GP2 is equivalent to GP1. For example, see corollary 8.2.3 below.

8.2. **Applications to Gelfand pairs.**

Theorem 8.2.1. *Let G be reductive group and let σ be an $Ad(G)$-admissible anti-automorphism of G. Let θ be the automorphism of G defined by $\theta(g) := \sigma(g^{-1})$. Let (π, E) be an irreducible admissible representation of G.*

Then $\widetilde{E} \cong E^{\theta}$, where $\widetilde{E}$ denotes the smooth contragredient representation and E^{θ} is E twisted by θ.

Proof. By Theorem 8.1.5 in [Wal1], it is enough to prove that the characters of $\widetilde{E}$ and E^{θ} are identical. This follows from corollary 7.6.3. $\qquad\square$

Remark 8.2.2. *This theorem has an alternative proof using Harish-Chandra regularity theorem, which says that character of an admissible representation is a locally integrable function.*

Corollary 8.2.3. *Let $H \subset G$ be reductive groups and let τ be an $Ad(G)$-admissible anti-automorphism of G such that $\tau(H) = H$. Then GP1 is equivalent to GP2 for the pair (G, H).*

Theorem 8.2.4. *Let E be a quadratic extension of F. Then the pair $(GL_n(E), GL_n(F))$ satisfies GP1.*

For non-archimedean F this theorem is proven in [Fli].

Proof. By theorem 7.6.2 this pair is tame. Hence it is enough to show that this symmetric pair is good. This follows from the fact that for any semisimple $x \in GL_n(E)^{\sigma}$ we have $H^1(F, (GL_n)_x) = 0$. Here we consider the adjoint action of GL_n on itself. $\qquad\square$

Appendix A. Localization principle

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In this appendix we formulate and prove localization principle in the case of a reductive group G acting on a smooth affine variety X. This is relevant only over archimedean F since for l-spaces, a more general version of this principle has been proven in [Ber].

In [AGS2], we formulated localization principle in the setting of differential geometry. Currently we do not have a proof of this principle in such setting. Now we present a proof in the case of a reductive group G acting on a smooth affine variety X. This generality is wide enough for all applications we had up to now, including the one in [AGS2].

Theorem A.0.1 (Localization principle). *Let a reductive group G act on a smooth algebraic variety X. Let Y be an algebraic variety and $\phi : X \to Y$ be an affine algebraic G-invariant map. Let χ be a character of $G(F)$. Suppose that for any $y \in Y(F)$ we have $\mathcal{D}_{X(F)}(\phi(F)^{-1}(y))^{G(F),\chi} = 0$. Then $\mathcal{D}(X(F))^{G(F),\chi} = 0$.*

Proof. Clearly, it is enough to prove for the case when X is affine, $Y = X/G$ and $\phi = \pi_X(F)$. By the generalized Harish-Chandra descent (Corollary 3.2.2), it is enough to prove that for any G-semisimple $x \in X(F)$, we have

$$\mathcal{D}_{N^X_{Gx,x}(F)}(\Gamma(N^X_{Gx,x}))^{G_x(F),\chi} = 0.$$

Let (U, p, ψ, S, N) be an analytic Luna slice at x. Clearly,

$$\mathcal{D}_{N^X_{Gx,x}(F)}(\Gamma(N^X_{Gx,x}))^{G_x(F),\chi} \cong \mathcal{D}_{\psi(S)}(\Gamma(N^X_{Gx,x}))^{G_x(F),\chi} \cong \mathcal{D}_S(\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G_x(F),\chi}.$$

By Frobenius reciprocity,

$$\mathcal{D}_S(\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G_x(F),\chi} = \mathcal{D}_U(G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G(F),\chi}.$$

By lemma 2.1.12,

$$G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})) = \{y \in X(F) | x \in \overline{G(F)y}\}.$$

Hence by Corollary 2.1.14, $G(F)\psi^{-1}(\Gamma(N^X_{Gx,x}))$ is closed in $X(F)$. Hence

$$\mathcal{D}_U(G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G(F),\chi} = \mathcal{D}_{X(F)}(G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})))^{G(F),\chi}.$$

Now,

$$G(F)\psi^{-1}(\Gamma(N^X_{Gx,x})) \subset \pi_X(F)^{-1}(\pi_X(F)(x))$$

and we are given

$$\mathcal{D}_{X(F)}(\pi_X(F)^{-1}(\pi_X(F)(x)))^{G(F),\chi} = 0$$

for any G-semisimple x. $\qquad\qquad\square$

Remark A.0.2. *An analogous statement holds for Schwartz distributions and the proof is the same.*

Corollary A.0.3. *Let a reductive group G act on a smooth algebraic variety X. Let Y be an algebraic variety and $\phi : X \to Y$ be an affine algebraic G-invariant submersion. Suppose that for any $y \in Y(F)$ we have $\mathcal{S}^*(\phi^{-1}(y))^{G(F),\chi} = 0$. Then $\mathcal{D}(X(F))^{G(F),\chi} = 0$.*

Proof. For any $y \in Y(F)$, denote $X(F)_y := (\phi^{-1}(y))(F)$. Since ϕ is a submersion, for any $y \in Y(F)$ the set $X(F)_y$ is a smooth manifold. Moreover, $d\phi$ defines an isomorphism between $N^{X(F)}_{X(F)_y,z}$ and $T_{Y(F),y}$ for any $z \in X(F)_y$. Hence the bundle $CN^{X(F)}_{X(F)_y}$ is a trivial $G(F)$-equivariant bundle.

We know that

$$\mathcal{S}^*(X(F)_y)^{G(F),\chi} = 0.$$

Therefore for any k, we have

$$\mathcal{S}^*(X(F)_y, \mathrm{Sym}^k(CN_{X(F)_y}^{X(F)}))^{G(F),\chi} = 0.$$

Thus by Theorem 2.3.6, $\mathcal{S}_{X(F)}^*(X(F)_y)^{G(F),\chi} = 0$. Now, by Theorem A.0.1 (and Remark A.0.2) this implies that $\mathcal{S}^*(X(F))^{G(F),\chi} = 0$. Finally, by Theorem 4.0.2 this implies $\mathcal{D}(X(F))^{G(F),\chi} = 0$. $\qquad\square$

Remark A.0.4. *Theorem 4.0.1 and Corollary A.0.3 have obvious generalizations to constant vector systems, and the same proofs hold.*

APPENDIX B. ALGEBRAIC GEOMETRY OVER LOCAL FIELDS

B.1. **Implicit function theorems.**

Definition B.1.1. *An analytic map $\phi : M \to N$ is called **ètale map** if $d_x\phi : T_xM \to T_xN$ is an isomorphism for any $x \in M$. An analytic map $\phi : M \to N$ is called **submersion** if $d_x\phi : T_xM \to T_xN$ is onto for any $x \in M$.*

We will use the following version of the inverse function theorem.

Theorem B.1.2. *Let $\phi : M \to N$ be an ètale map of analytic manifolds. Then it is locally an isomorphism.*

For proof see e.g. [Ser], Theorem 2 in section 9 of Chapter III in part II.

Corollary B.1.3. *Let $\phi : X \to Y$ be a morphism of (not necessarily smooth) algebraic varieties. Suppose that ϕ is ètale at $x \in X(F)$.*
 Then there exists an open neighborhood $U \subset X(F)$ of x such that $\phi|_U$ is a homeomorphism to its open image in $Y(F)$.

For proof see e.g. [Mum], Chapter III, section 5, proof of Corolary 2. There, the proof is given for the case $F = \mathbb{C}$ but it works in the general case.

Remark B.1.4. *If F is archimedean then one can choose U to be semi-algebraic.*

The following proposition is well known (see e.g. section 10 of Chapter III in part II of [Ser]).

Proposition B.1.5. *Any submersion $\phi : M \to N$ is open.*

Corollary B.1.6. *Lemma 2.1.4 holds. Namely, for any algebraic group G and a closed algebraic subgroup $H \subset G$ the subset $G(F)/H(F)$ is open and closed in $(G/H)(F)$.*

Proof. Consider the map $\phi : G(F) \to (G/H)(F)$ defined by $\phi(g) = gH$. Clearly, it is a submersion and its image is exactly $G(F)/H(F)$. Hence, $G(F)/H(F)$ is open. Since there is a finite number of $G(F)$ orbits in $(G/H)(F)$ and each of them is open for the same reason, $G(F)/H(F)$ is also closed. $\qquad\square$

B.2. **Luna slice theorem.**
In this subsection we formulate Luna slice theorem and show how it implies Theorem 2.1.16. For a survey on Luna slice theorem we refer the reader to [Dre] and the original paper [Lun1].

Definition B.2.1. *Let a reductive group G act on affine varieties X and Y. A G-equivariant algebraic map $\phi : X \to Y$ is called **strongly ètale** if*
(i) $\phi/G : X/G \to Y/G$ is ètale
(ii) ϕ and the quotient morphism $\pi_X : X \to X/G$ induce a G-isomorphism $X \cong Y \times_{Y/G} X/G$.

Definition B.2.2. *Let G be a reductive group and H be a closed reductive subgroup. Suppose that H acts on an affine variety X. Then $G \times_H X$ denotes $(G \times X)/H$ with respect to the action $h(g, x) = (gh^{-1}, hx)$.*

Theorem B.2.3 (Luna slice theorem)**.** *Let a reductive group G act on a smooth affine variety X. Let $x \in X$ be G-semisimple.*

Then there exists a locally closed smooth affine G_x-invariant subvariety $Z \ni x$ of X and a strongly ètale algebraic map of G_x spaces $\nu : Z \to N_{Gx,x}^X$ such that the G-morphism $\phi : G \times_{G_x} Z \to X$ induced by the action of G on X is strongly ètale .

Proof. It follows from Proposition 4.18, lemma 5.1 and theorems 5.2 and 5.3 in [Dre], noting that one can choose Z and ν (in our notations) to be defined over F. $\qquad\square$

Corollary B.2.4. *Theorem 2.1.16 holds. Namely:*
Let a reductive group G act on a smooth affine variety X. Let $x \in X(F)$ be G-semisimple.
    Then there exist
(i) an open $G(F)$-invariant B-analytic neighborhood U of $G(F)x$ in $X(F)$ with a G-equivariant B-analytic retract $p : U \to G(F)x$ and
(ii) a G_x-equivariant B-analytic embedding $\psi : p^{-1}(x) \hookrightarrow N_{Gx,x}^X(F)$ with open saturated image such that $\psi(x) = 0$.

Proof. Let Z, ϕ and ν be as in the last theorem.

Let $Z' := Z/G_x \cong (G \times_{G_x} Z)/G$ and $X' := X/G$. Consider the natural map $\phi' : Z'(F) \to X'(F)$. By Corollary B.1.3 there exists a neighborhood $S' \subset Z'(F)$ of $\pi_Z(x)$ such that $\phi'|_{S'}$ is a homeomorphism to its open image.

Consider the natural map $\nu' : Z'(F) \to N_{Gx,x}^X/G_x(F)$. Let $S'' \subset Z(F)$ be a neighborhood of $\pi_Z(x)$ such that $\nu'|_{S''}$ is an isomorphism to its open image. In case that F is archimedean we choose S' and S'' to be semi-algebraic.

Let $S := \pi_Z^{-1}(S'' \cap S') \cap Z(F)$. Clearly, S is B-analytic.

Let $\rho : (G \times_{G_x} Z)(F) \to Z'(F)$ be the natural projection. Let $O = \rho^{-1}(S'' \cap S')$. Let $q : O \to G/G_x(F)$ be the natural projection. Let $O' := q^{-1}(G(F)/G_x(F))$ and $q' := q|_{O'}$.

Now put $U := \phi(O')$ and put $p : U \to G(F)x$ be the morphism that corresponds to q'. Note that $p^{-1}(x) \cong S$ and put $\psi : p^{-1}(x) \to N_{Gx,x}^X(F)$ to be the imbedding that corresponds to $\nu|_S$. $\quad\square$

Appendix C. Schwartz distributions on Nash manifolds

C.1. Preliminaries and notations.
In this appendix we will prove some properties of K-equivariant Schwartz distributions on Nash manifolds. We work in the notations of [AG1], where one can read on Nash manifolds and Schwartz distributions over them. More detailed references on Nash manifolds are [BCR] and [Shi].

Nash manifolds are equipped with **restricted topology**. This is the topology in which open sets are open semi-algebraic sets. This is not a topology in the classical sense of the word as infinite unions of open sets are not necessary open sets in the restricted topology. However, finite unions of open sets are open sets and therefore in the restricted topology we consider only finite covers. In particular, if $E \to M$ is a Nash vector bundle it means that there exists a <u>finite</u> open cover U_i of M such that $E|_{U_i}$ is trivial.

Notation C.1.1. *Let M be a Nash manifold. We denote by D_M the Nash bundle of densities on M. It is the natural bundle whose smooth sections are smooth measures, for precise definition see e.g. [AG1].*

An important property of Nash manifolds is

Theorem C.1.2 (Local triviality of Nash manifolds.)**.** *Any Nash manifold can be covered by finite number of open submanifolds Nash diffeomorphic to $\mathbb{R}^n$.*

For proof see theorem I.5.12 in [Shi].

Definition C.1.3. *Let M be a Nash manifold. We denote by $\mathcal{G}(M) := \mathcal{S}^*(M, D_M)$ the **space of Schwartz generalized functions** on M. Similarly, for a Nash bundle $E \to M$ we denote by $\mathcal{G}(M, E) := \mathcal{S}^*(M, E^* \otimes D_M)$ the **space of Schwartz generalized sections** of E.*
*In the same way, for any smooth manifold M we denote by $C^{-\infty}(M) := \mathcal{D}(M, D_M)$ the **space of generalized functions** on M and for a smooth bundle $E \to M$ we denote by $C^{-\infty}(M, E) := \mathcal{D}(M, E^* \otimes D_M)$ the **space of generalized sections** of E.*

Usual L^1 functions can be interpreted as Schwartz generalized functions but not as Schwartz distributions. We will need several properties of Schwartz functions from [AG1].

Property C.1.4. $\mathcal{S}(\mathbb{R}^n) = $ *Classical Schwartz functions on $\mathbb{R}^n$.*

For proof see theorem 4.1.3 in [AG1].

Property C.1.5. *Let $U \subset M$ be a (semi-algebraic) open subset, then*

$$\mathcal{S}(U, E) \cong \{\phi \in \mathcal{S}(M, E)| \quad \phi \text{ is 0 on } M \setminus U \text{ with all derivatives}\}.$$

For proof see theorem 5.4.3 in [AG1].

Property C.1.6. *Let M be a Nash manifold. Let $M = \bigcup U_i$ be a finite open cover of M. Then a function f on M is a Schwartz function if and only if it can be written as $f = \sum\limits_{i=1}^{n} f_i$ where $f_i \in \mathcal{S}(U_i)$ (extended by zero to M).*
Moreover, there exists a smooth partition of unity $1 = \sum\limits_{i=1}^{n} \lambda_i$ such that for any Schwartz function $f \in \mathcal{S}(M)$ the function $\lambda_i f$ is a Schwartz function on U_i (extended by zero to M).

For proof see section 5 in [AG1].

Property C.1.7. *Let M be a Nash manifold and E be a Nash bundle over it. Let $M = \bigcup U_i$ be a finite open cover of M. Let $\xi_i \in \mathcal{G}(U_i, E)$ such that $\xi_i|_{U_j} = \xi_j|_{U_i}$. Then there exists a unique $\xi \in \mathcal{G}(M, E)$ such that $\xi|_{U_i} = \xi_i$.*

For proof see section 5 in [AG1].
We will also use the following notation.

Notation C.1.8. *Let M be a metric space and $x \in M$. We denote by $B(x, r)$ the open ball with center x and radius r.*

C.2. Submersion principle.

Theorem C.2.1. *Let M and N be Nash manifolds and $s : M \to N$ be a surjective submersive Nash map. Then locally it has a Nash section, i.e. there exists a finite open cover $N = \bigcup\limits_{i=1}^{k} U_i$ such that s has a Nash section on each U_i.*

For proof see [AG2], theorem 2.4.16.

Corollary C.2.2. *An ètale map $\phi : M \to N$ of Nash manifolds is locally an isomorphism. That means that there exist a finite cover $M = \bigcup U_i$ such that $\phi|_{U_i}$ is an isomorphism to its open image.*

Theorem C.2.3. *Let $p : M \to N$ be a Nash submersion of Nash manifolds. Then there exist a finite open (semi-algebraic) cover $M = \bigcup U_i$ and isomorphisms $\phi_i : U_i \cong W_i$ and $\psi_i : p(U_i) \cong V_i$ where $W_i \subset \mathbb{R}^{d_i}$ and $V_i \subset \mathbb{R}^{k_i}$ are open (semi-algebraic) subsets, $k_i \leq d_i$ and $p|_{U_i}$ correspond to the standard projections.*

Proof. Without loss of generality we can assume that $N = \mathbb{R}^k$, M is an equidimensional closed submanifold of $\mathbb{R}^n$ of dimension d, $d \geq k$, and p is given by the standard projection $\mathbb{R}^n \to \mathbb{R}^k$.

Let Ω be the set of all coordinate subspaces of $\mathbb{R}^n$ of dimension d which contain N. For any $V \in \Omega$ consider the projection $pr : M \to V$. Define $U_V = \{x \in M | d_x pr$ is an isomorphism $\}$. It is easy to see that $pr|_{U_V}$ is ètale and $\{U_V\}_{V \in \Omega}$ gives a finite cover of M. Now the theorem follows from the previous corollary (Corollary C.2.2). $\qquad\square$

Theorem C.2.4. *Let $\phi : M \to N$ be a Nash submersion of Nash manifolds. Let E be a Nash bundle over N. Then*
(i) there exists a unique continuous linear map $\phi_* : \mathcal{S}(M, \phi^*(E) \otimes D_M) \to \mathcal{S}(N, E \otimes D_N)$ such that for any $f \in \mathcal{S}(N, E^*)$ and $\mu \in \mathcal{S}(M, \phi^*(E) \otimes D_M)$ we have

$$\int_{x \in N} \langle f(x), \phi_*\mu(x) \rangle = \int_{x \in M} \langle \phi^* f(x), \mu(x) \rangle.$$

In particular, we mean that both integrals converge.
(ii) If ϕ is surjective then ϕ_* is surjective.

Proof.
(i)
Step 1. Proof for the case when $M = \mathbb{R}^n$, $N = \mathbb{R}^k$, $k \leq n$, ϕ is the standard projection and E is trivial.
Fix Haar measure on $\mathbb{R}$ and identify $D_{\mathbb{R}^l}$ with the trivial bundle for any l. Define

$$\phi_*(f)(x) := \int_{y \in \mathbb{R}^{n-k}} f(x, y) dy.$$

Convergence of the integral and the fact that $\phi_*(f)$ is a Schwartz function follows from standard calculus.

Step 2. Proof for the case when $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^k$ are open (semi-algebraic) subsets, ϕ is the standard projection and E is trivial.
Follows from the previous step and Property C.1.5.

Step 3. Proof for the case when E is trivial.
Follows from the previous step, Theorem C.2.3 and partition of unity (Property C.1.6).

Step 4. Proof in the general case.
Follows from the previous step and partition of unity (Property C.1.6).

(ii) The proof is the same as in (i) except of Step 2. Let us prove (ii) in the case of Step 2. Again, fix Haar measure on $\mathbb{R}$ and identify $D_{\mathbb{R}^l}$ with the trivial bundle for any l. By Theorem C.2.1 and partition of unity (Property C.1.6) we can assume that there exists a Nash section $\nu : N \to M$. We can write ν in the form $\nu(x) = (x, s(x))$.

For any $x \in N$ define $R(x) := \sup\{r \in \mathbb{R}_{\geq 0} | B(\nu(x), r) \subset M\}$. Clearly, R is continuous and positive. By Tarski - Seidenberg principle (see e.g. [AG1], theorem 2.2.3) it is semi-algebraic. Hence (by lemma A.2.1 in [AG1]) there exists a positive Nash function $r(x)$ such that $r(x) < R(x)$. Let $\rho \in \mathcal{S}(\mathbb{R}^{n-k})$ such that ρ is supported in the unit ball and its integral is 1. Now let $f \in \mathcal{S}(N)$. Let $g \in C^\infty(M)$ defined by $g(x, y) := f(x)\rho((y - s(x))/r(x))/r(x)$ where $x \in N$ and $y \in \mathbb{R}^{n-k}$. It is easy to see that $g \in \mathcal{S}(M)$ and $\phi_* g = f$. $\qquad\square$

Notation C.2.5. *Let $\phi : M \to N$ be a Nash submersion of Nash manifolds. Let E be a bundle on N. We denote by $\phi^* : \mathcal{G}(N, E) \to \mathcal{G}(M, \phi^*(E))$ the dual map to ϕ_*.*

Remark C.2.6. *Clearly, the map* $\phi^* : \mathcal{G}(N, E) \to \mathcal{G}(M, \phi^*(E))$ *extends to the map* $\phi^* : C^{-\infty}(N, E) \to C^{-\infty}(M, \phi^*(E))$ *described in* [AGS1]*, theorem A.0.4.*

Proposition C.2.7. *Let* $\phi : M \to N$ *be a surjective Nash submersion of Nash manifolds. Let* E *be a bundle on* N*. Let* $\xi \in C^{-\infty}(N)$*. Suppose that* $\phi^*(\xi) \in \mathcal{G}(M)$*. Then* $\xi \in \mathcal{G}(N)$*.*

Proof. It follows from Theorem C.2.4 and Banach open map theorem (see theorem 2.11 in [Rud]).
□

C.3. **Frobenius reciprocity.**
In this subsection we prove Frobenius reciprocity for Schwartz functions on Nash manifolds.

Proposition C.3.1. *Let* M *be a Nash manifold. Let* K *be a Nash group. Let* $E \to M$ *be a Nash bundle. Consider the standard projection* $p : K \times M \to M$*. Then the map* $p^* : \mathcal{G}(M, E) \to \mathcal{G}(M \times K, p^* E)^K$ *is an isomorphism.*

This proposition follows from Proposition 4.0.11 in [AG2].

Corollary C.3.2. *Let a Nash group* K *act on a Nash manifold* M*. Let* E *be a* K*-equivariant Nash bundle over* M*. Let* $N \subset M$ *be a Nash submanifold such that the action map* $K \times N \to M$ *is submersive. Then there exists a canonical map*

$$HC : \mathcal{G}(M, E)^K \to \mathcal{G}(N, E|_N).$$

Theorem C.3.3. *Let a Nash group* K *act on a Nash manifold* M*. Let* N *be a* K*-transitive Nash manifold. Let* $\phi : M \to N$ *be a Nash* K*-equivariant map.*

Let $z \in N$ *be a point and* $M_z := \phi^{-1}(z)$ *be its fiber. Let* K_z *be the stabilizer of* z *in* K*. Let* E *be a* K*-equivariant Nash vector bundle over* M*.*

Then there exists a canonical isomorphism

$$Fr : \mathcal{G}(M_z, E|_{M_z})^{K_z} \cong \mathcal{G}(M, \mathcal{E})^K.$$

Proof. Consider the map $a_z : K \to N$ given by $a_z(g) = gz$. It is a submersion. Hence by Theorem C.2.1 there exists a finite open cover $N = \bigcup_{i=1}^{k} U_i$ such that a_z has a Nash section s_i on each U_i. This gives an isomorphism $\phi^{-1}(U_i) \cong U_i \times M_z$ which defines a projection $p : \phi^{-1}(U_i) \to M_z$. Let $\xi \in \mathcal{G}(M_z, E|_{M_z})^{K_z}$. Denote $\xi_i := p^* \xi$. Clearly it does not depend on the section s_i. Hence $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ and hence by Property C.1.7 there exists $\eta \in \mathcal{G}(M, \mathcal{E})$ such that $\eta|_{U_i} = \xi_i$. Clearly η does not depend on the choices. Hence we can define $Fr(\xi) = \eta$.

It is easy to see that the map $HC : \mathcal{G}(M, E)^K \to \mathcal{G}(M_z, E|_{M_z})$ described in the last corollary gives the inverse map. □

Since our construction coincides with the construction of Frobenius reciprocity for smooth manifolds (see e.g. [AGS1], theorem A.0.3) we obtain the following corollary.

Corollary C.3.4. *Part (ii) of Theorem 2.3.7 holds.*

C.4. **K-invariant distributions compactly supported modulo K.**

In this subsection we prove Theorem 4.0.5. Let us first remind its formulation.

Theorem C.4.1. *Let a Nash group* K *act on a Nash manifold* M*. Let* E *be a* K*-equivariant Nash bundle over* M*. Let* $\xi \in \mathcal{D}(M, E)^K$ *such that* $\text{Supp}(\xi)$ *is Nashly compact modulo* K*. Then* $\xi \in \mathcal{S}^*(M, E)^K$*.*

For the proof we will need the following lemmas.

Lemma C.4.2. *Let M be a Nash manifold. Let $C \subset M$ be a compact subset. Then there exists a relatively compact open (semi-algebraic) subset $U \subset M$ that includes C.*

Proof. For any point $x \in C$ choose an affine chart, and let U_x be an open ball with center at x inside this chart. Those U_x give an open cover of C. Choose a finite subcover $\{U_i\}_{i=1}^n$ and let $U := \bigcup_{i=1}^n U_i$. □

Lemma C.4.3. *Let M be a Nash manifold. Let E be a Nash bundle over M. Let $U \subset M$ be a relatively compact open (semi-algebraic) subset. Let $\xi \in \mathcal{D}(M, E)$. Then $\xi|_U \in \mathcal{S}^*(U, E|_U)$.*

Proof. It follows from the fact that extension by zero $ext : \mathcal{S}(U, E|_U) \to C_c^\infty(M, E)$ is a continuous map. □

Proof of Theorem C.4.1. Let $Z \subset M$ be a semi-algebraic closed subset and $C \subset M$ be a compact subset such that $Supp(\xi) \subset Z \subset KC$.

Let $U \supset C$ be as in Lemma C.4.2. Let $\xi' := \xi|_{KU}$. Since $\xi|_{M-Z} = 0$, it is enough to show that ξ' is Schwartz.

Consider the surjective submersion $m_U : K \times U \to KU$. Let
$$\xi'' := m_U^*(\xi') \in \mathcal{D}(K \times U, m_U^*(E))^K.$$
By Proposition C.2.7, it is enough to show that
$$\xi'' \in \mathcal{S}^*(K \times U, m_U^*(E)).$$
By Frobenius reciprocity, ξ'' corresponds to $\eta \in \mathcal{D}(U, E)$. It is enough to prove that $\eta \in \mathcal{S}^*(U, E)$. Consider the submersion $m : K \times M \to M$ and let
$$\xi''' := m^*(\xi) \in \mathcal{D}(K \times M, m^*(E)).$$
By Frobenius reciprocity, ξ''' corresponds to $\eta' \in \mathcal{D}(M, E)$. Clearly $\eta = \eta'|_U$. Hence by Lemma C.4.3, $\eta \in \mathcal{S}^*(U, E)$. □

Appendix D. Proof of archimedean homogeneity theorem

The goal of this appendix is to prove Theorem 5.1.7 for archimedean F. First we remind its formulation.

Theorem D.0.1 (archimedean homogeneity). *Let V be a vector space over F. Let B be a non-degenerate symmetric bilinear form on V. Let M be a Nash manifold. Let $L \subset \mathcal{S}^*_{V \times M}(Z(B) \times M)$ be a non-zero subspace such that $\forall \xi \in L$ we have $\mathcal{F}_B(\xi) \in L$ and $B\xi \in L$ (here B is interpreted as a quadratic form).*

Then there exists a non-zero distribution $\xi \in L$ which is adapted to B.

Till the end of the section we assume that F is archimedean and we fix V and B.

First we will need some facts about the Weil representation. For a survey on the Weil representation in the archimedean case we refer the reader to [RS1], section 1.

(1) There exists a unique (infinitesimal) action π of $sl_2(F)$ on $\mathcal{S}^*(V)$ such that

   (i) $\pi(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})\xi = -i\pi Re(B)\xi$ and $\pi(\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix})\xi = -\mathcal{F}_B^{-1}(i\pi Re(B)\mathcal{F}_B(\xi))$.

   (ii) If $F = \mathbb{C}$ then $\pi(\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}) = \pi(\begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}) = 0$

(2) It can be lifted to an action of the metaplectic group $Mp(2, F)$.
   We will denote this action by Π.

(3) In case $F = \mathbb{C}$ we have $Mp(2, F) = SL_2(F)$ and in case $F = \mathbb{R}$ the group $Mp(2, F)$ is a connected 2-folded covering of $SL_2(F)$. We will denote by $\varepsilon \in Mp(2, F)$ the element of order 2 that satisfies $SL_2(F) = Mp(2, F)/\{1, \varepsilon\}$.

(4) In case $F = \mathbb{R}$ we have $\Pi(\varepsilon) = (-1)^{dimV}$ and therefore if $dimV$ is even then Π factors through $SL_2(F)$ and if $dimV$ is odd then no nontrivial subrepresentation of Π factors through $SL_2(F)$. In particular if $dimV$ is odd then Π has no nontrivial finite dimensional representations, since every finite dimensional representation of sl_2 has a unique lifting both to $SL_2(F)$ and to $Mp(2, F)$.

(5) In case $F = \mathbb{C}$ or in case $dimV$ is even we have $\Pi(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})\xi = \delta^{-1}(t)|t|^{-dimV/2}\rho(t)\xi$ and
$$\Pi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\xi = \gamma(B)^{-1}\mathcal{F}_B\xi.$$

We also need the following straightforward lemma.

Lemma D.0.2. *Let (Λ, L) be a continuous finite dimensional representation of $SL_2(\mathbb{R})$. Then there exists a non-zero $\xi \in L$ such that either*

$$\Lambda(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})\xi = \xi \text{ and } \Lambda(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\xi \text{ is proportional to } \xi$$

or

$$\Lambda(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})\xi = t\xi.$$

Now we are ready to prove the theorem.

Proof of Theorem 5.1.7. Without loss of generality assume $M = pt$.

Let $\xi \in L$ be a non-zero distribution. Let $L' := U_{\mathbb{C}}(sl_2(\mathbb{R}))\xi \subset L$. Here, $U_{\mathbb{C}}$ means the complexified universal enveloping algebra.

It is easy to see that $L' \subset \mathcal{S}^*(V)$ is finite dimensional (see Lemma D.0.3 below). Clearly, L' is also $sl_2(F)$-invariant and hence is also a subrepresentation of Π. Therefore by Fact (4), $F = \mathbb{C}$ or $dimV$ is even. Hence Π factors through $SL_2(F)$.

Now by the lemma there exists $\xi' \in L'$ which is B adapted. $\qquad\square$

Lemma D.0.3. *Let V be a representation of sl_2. Let $v \in V$ be a vector such that $e^k v = f^n v = 0$ for some n, k. Then the representation generated by v is finite dimensional.*[3]

Proof. The proof is by induction on k.

Base k=1:
It is easy to see that

$$e^l f^l v = l!(\prod_{i=0}^{l-1}(h-i))v$$

for all l. It can be checked by a direct computation, and also follows from the fact that $e^l f^l$ is of weight 0, hence it acts on the singular vector v by its Harish Chandra projection which is $HC(e^l f^l) = l!\prod_{i=0}^{l-1}(h-i)$.

Therefore $(\prod_{i=0}^{n-1}(h-i))v = 0$.

Hence $W := U_{\mathbb{C}}(h)v$ is finite dimensional and h acts on it semi-simply. Let $\{v_i\}_{i=1}^m$ be an eigenbasis of h in W. It is enough to show that $U_{\mathbb{C}}(sl_2)v_i$ is finite dimensional for any i. Note that $e|_W = f^n|_W = 0$. Now, $U_{\mathbb{C}}(sl_2)v_i$ is finite dimensional by Poincare-Birkhoff-Witt theorem.
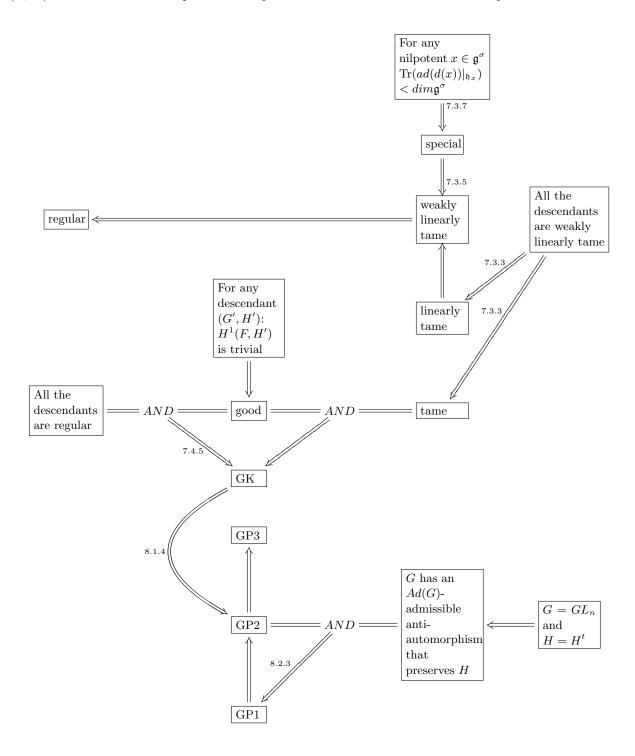
Induction step:
Let $w := e^{k-1}v$. Let us show that $f^{n+k-1}w = 0$. Consider the element $f^{n+k-1}e^{k-1} \in U_{\mathbb{C}}(sl_2)$. It

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[3]For our purposes it is enough to prove this lemma for k=1.

is of weight $-2n$, hence by Poincare-Birkhoff-Witt theorem it can be rewritten as a combination of elements of the form $e^a h^b f^c$ such that $c - a = n$ and hence $c \geq n$. Therefore $f^{n+k-1} e^{k-1} v = 0$.

Now let $V_1 := U_\mathbb{C}(sl_2)v$ and $V_2 := U_\mathbb{C}(sl_2)w$. By the base of the induction V_2 is finite dimensional, by the induction hypotheses V_1/V_2 is finite dimensional, hence V_1 is finite dimensional. $\square$

Appendix E. Diagram

The following diagram illustrates the interrelations of various properties of a symmetric pair (G, H). On the non-trivial implications we put the numbers of the statements that prove them.

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