Recent applications of classical theorems on holonomic distributions

Dmitry Gourevitch

Weizmann Institute of Science

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http://www.wisdom.weizmann.ac.il/~dimagur/

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Let M be a D-module over X with generators $m_1 \dots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree $\leq i$ and $F_i(M) := F_i(D(X))(m_1 \dots m_k)$. Define

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A D-module (or a distribution) ξ is called holonomic if

$$\dim(SS(\xi)) = \dim X.$$

Applications to analytic continuation

Theorem (Bernstein, cf. Sato)

(i) *Holonomic D-modules have finite length.*

(ii) *Let* $p \in \mathbb{R}[x_1, \ldots, x_n]$ *be a non-negative polynomial, and let* $\xi \in \mathcal{S}^*(\mathbb{R}^n)$ *be a holonomic tempered distribution. Then the family of distributions* $p^\lambda \xi$ *defined for* $\operatorname{Re} \lambda > -1$ *has a meromorphic continuation to* $\lambda \in \mathbb{C}$.

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Corollary (Gourevitch-Sahi-Sayag)

Let a solvable real algebraic group B act on an affine real algebraic manifold X. Let χ be a tempered character of B. Then $\dim \mathcal{S}^*(X)^{B,\chi}$ is at least the number of open B-orbits in X that possess (B, χ)-equivariant measures.

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Let a solvable real algebraic group B act on an affine real algebraic manifold X, and let $\mathcal{O}$ be an open orbit. Then $\exists$ a B-equivariant polynomial $p \neq 0$ on X with $p|_{X \setminus \mathcal{O}} = 0$.

Corollary (Gourevitch-Sahi-Sayag)

$\forall$ *tempered* $\chi : B \to \mathbb{C}^\times$, *if* $\mathcal{S}^*(\mathcal{O})^{B,\chi} \neq 0$ *then* $\mathcal{S}^*(X)^{B,\chi} \neq 0$.

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Proof.

For $\xi \in \mathcal{S}^*(\mathcal{O})^{B,\chi}$ and $n >> 0$, $p^n \xi$ extends to $\eta \in \mathcal{S}^*(X)^{B,\psi^n \chi}$. In the family $p^\lambda \eta$, take $\lambda = -n$. If this is a pole - take the principal part. □

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Corollary (G.-Sahi-Sayag, in progress)

Let G be a real reductive group and $H \subset G$ be a real spherical subgroup. Let $C \subset P_0 \subset G$ be closed subgroup and let V be tempered fin. dim. rep. of $C \times H$. Let $U \subset G$ be open G-invariant subset. Then $\dim \mathcal{S}^*(G, V)^{C\times H} \geq \dim \mathcal{S}^*(U, V)^{C\times H}$.

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This gives Knap-Stein operators on spherical spaces and (degenerate) Whittaker models for (degenerate) principal series.

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Let X, Y be smooth algebraic varieties and $\mathcal{M}$ be a family of holonomic D_X-modules parameterized by Y. Then $\dim Hom(\mathcal{M}_y, \mathcal{S}^*(X))$ *is bounded when y ranges over Y.*

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Corollary (Aizenbud-G.-Minchenko 2015)

Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let $\mathcal{E}$ be an algebraic G-equivariant bundle on X and χ be a character of $\mathfrak{g}$. Then,

$$\dim \mathcal{S}^*(X, \mathcal{E})^{\mathfrak{g},\chi} < \infty.$$

Moreover, it remains bounded when we change χ or tensor $\mathcal{E}$ with a representation of $\mathfrak{g}$ of a fixed dimension.

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2. *If H is a real spherical subgroup (i.e. HP is open for some minimal parabolic subgroup P) then, for every irreducible admissible representation $\pi \in Irr(G)$, and natural number $n \in \mathbb{N}$ there exists $C_n \in \mathbb{N}$ such that for every n-dimensional representation τ of $\mathfrak{h}$ we have*

$$\dim Hom_{\mathfrak{h}}(\pi, \tau) \leq C_n.$$

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- This implies that $p : g(SS_b(M)) \to X$ is finite.
- This implies that gM is smooth.

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- *ξ is right H_2 invariant*
- *ξ is eigen w.r.t. the center $\mathfrak{z}(u(\mathfrak{g}))$ of the universal enveloping algebra of the Lie algebra of G.*

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Corollary

A spherical character of admissible representation w.r.t. pair of spherical groups is a holonomic distribution.

The spherical character

Definition

Let (π, V) be an admissible representation of $G(\mathbb{R})$ and $v_1 \in (V^*)^{H_1}$, $v_2 \in (\tilde{V}^*)^{H_2}$. Define the spherical character of π w.r.t. v_1 and v_2 by:

$$\langle \xi, f \rangle := \langle \pi^*(f) v_1, v_2 \rangle.$$

Corollary

A spherical character of admissible representation w.r.t. pair of spherical groups is a holonomic distribution.

Corollary (Aizenbud, Gourevitch, Minchenko, Sayag)

For any local field F, any spherical character of an admissible representation of $G(F)$ is smooth in a Zariski open dense set.

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