Recent applications of classical theorems on holonomic distributions

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Holonomic D-modules and distributions

A D-module over a smooth affine algebraic variety X is a module over the ring $D(X)$ of differential operators on X.
A D-module M given by generators and relations can be thought of as a system of PDE. A solution of M is a D-module homomorphism of M to an appropriate space of functions.

Definition

Let M be a D-module over X with generators $m_1 \dots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree $\leq i$ and $F_i(M) := F_i(D(X))(m_1 \dots m_k)$. Define

$$SS(M) := supp(gr_F(M)) \subset T^*X.$$

For a distribution ξ on $X(\mathbb{R})$ define

$$SS(\xi) := SS(D(X)\xi) = \bigcap_{d\xi=0} Zeros(symbol(d)).$$

A D-module (or a distribution) ξ is called holonomic if

$$\dim(SS(\xi)) = \dim X.$$

Applications to analytic continuation

Theorem (Bernstein, cf. Sato)

(i) *Holonomic D-modules have finite length.*

(ii) *Let* $p \in \mathbb{R}[x_1, \dots, x_n]$ *be a non-negative polynomial, and let* $\xi \in \mathcal{S}^*(\mathbb{R}^n)$ *be a holonomic tempered distribution. Then the family of distributions* $p^\lambda \xi$ *defined for* $\operatorname{Re}\lambda > -1$ *has a meromorphic continuation to* $\lambda \in \mathbb{C}$.

This implies analytic continuation of Knap-Stein intertwining operators. Applied to symmetric pairs by van-den-Bahn, Brylinski, Delorme, Moelers-Osrted-Oshima,....

Corollary (Gourevitch-Sahi-Sayag)

Let a solvable real algebraic group B act on an affine real algebraic manifold X. *Let* χ *be a tempered character of B. Then* $\dim \mathcal{S}^*(X)^{B,\chi}$ *is at least the number of open B-orbits in X that possess* (B,χ)*-equivariant measures.*

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Let a solvable real algebraic group B act on an affine real algebraic manifold X, and let $\mathcal{O}$ be an open orbit. Then $\exists$ a B-equivariant polynomial $p \neq 0$ on X with $p|_{X \setminus \mathcal{O}} = 0$.

Corollary (Gourevitch-Sahi-Sayag)

$\forall$ *tempered* $\chi : B \to \mathbb{C}^\times$, *if* $\mathcal{S}^*(\mathcal{O})^{B,\chi} \neq 0$ *then* $\mathcal{S}^*(X)^{B,\chi} \neq 0$.

Proof.

For $\xi \in \mathcal{S}^*(\mathcal{O})^{B,\chi}$ and $n >> 0$, $p^n \xi$ extends to $\eta \in \mathcal{S}^*(X)^{B,\psi^n \chi}$. In the family $p^\lambda \eta$, take $\lambda = -n$. If this is a pole - take the principal part. □

More generally, X can be quasiprojective and B can be *MAN*.

Let solvable B act on affine X, and let $\mathcal{O}$ be an open orbit.

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Corollary (G.-Sahi-Sayag, in progress)

Let G be a real reductive group and $H \subset G$ be a real spherical subgroup. Let $C \subset P_0 \subset G$ be closed subgroup and let V be tempered fin. dim. rep. of $C \times H$. Let $U \subset G$ be open G-invariant subset. Then $\dim \mathcal{S}^*(G, V)^{C\times H} \geq \dim \mathcal{S}^*(U, V)^{C\times H}$.

This gives Knap-Stein operators on spherical spaces and (degenerate) Whittaker models for (degenerate) principal series.

Bernstein-Kashiwara theorem

Theorem (Bernstein, cf. Kashiwara)

Let X be a real algebraic manifold. Let M be a holonomic right D_X-module. Then $\dim Hom(M, \mathcal{S}^*(X)) < \infty$.

Theorem (Bernstein, Kashiwara, Aizenbud- G.- Minchenko)

Let X, Y be smooth algebraic varieties and $\mathcal{M}$ be a family of holonomic D_X-modules parameterized by Y. Then $\dim Hom(\mathcal{M}_y, \mathcal{S}^*(X))$ *is bounded when y ranges over Y.*

Corollary (Aizenbud-G.-Minchenko 2015)

Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let $\mathcal{E}$ be an algebraic G-equivariant bundle on X and χ be a character of $\mathfrak{g}$. Then,

$$\dim \mathcal{S}^*(X, \mathcal{E})^{\mathfrak{g},\chi} < \infty.$$

Moreover, it remains bounded when we change χ or tensor $\mathcal{E}$ with a representation of $\mathfrak{g}$ of a fixed dimension.

Applications for co-invariants

Theorem (Aizenbud-G.-Krötz-Liu 2016)

Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let $\mathcal{E}$ be an algebraic G-equivariant bundle on X and χ be a tempered character of G. Then,

$$\mathfrak{g}(\mathcal{S}(X,\mathcal{E}) \otimes \chi) \subset \mathcal{S}(X,\mathcal{E}) \otimes \chi$$

is closed and has finite codimension.

Corollary

Let G be a real reductive group, H be a real spherical subgroup, and $\mathfrak{h}$ be the Lie algebra of H. Let χ be a tempered character of H. Then for any admissible representation π of G, $\mathrm{H}_0(\mathfrak{h}, \pi \otimes \chi)$ is separated and is non-degenerately paired with $(\pi^*)^{\mathfrak{h},-\chi}$. In particular, the following conj. of Casselman are equivalent

- *Automatic continuity:* $((\pi^{HC})^*)^{\mathfrak{h}} \cong (\pi^*)^{\mathfrak{h}}$
- *Comparison:* $\mathrm{H}_0(\mathfrak{h}, \pi^{HC}) \cong \mathrm{H}_0(\mathfrak{h}, \pi)$

Applications to multiplicities

We reprove the following theorem

Theorem (Kobayashi-Oshima, Krötz-Schlichtkrull 2013)

Let G be a real reductive group, H be a Zariski closed subgroup, and $\mathfrak{h}$ be the Lie algebra of H.

1. *If H is a spherical subgroup (i.e. HB is open for some Borel subgroup B) then there exists $C \in \mathbb{N}$ such that $\dim(\pi^*)^{\mathfrak{h},\chi} \leq C$ for any $\pi \in Irr(G)$ and any character χ of $\mathfrak{h}$.*
2. *If H is a real spherical subgroup (i.e. HP is open for some minimal parabolic subgroup P) then, for every irreducible admissible representation $\pi \in Irr(G)$, and natural number $n \in \mathbb{N}$ there exists $C_n \in \mathbb{N}$ such that for every n-dimensional representation τ of $\mathfrak{h}$ we have*

$$\dim Hom_{\mathfrak{h}}(\pi, \tau) \leq C_n.$$

Sketch of the proof of Bernstein-Kashiwara theorem

- Enough to prove for the case X is a vector space.
- Stone von-Neumann: The group $Sp(T^*(X))$ acts on the category of D-modules on X stabilizing $\mathcal{S}^*(X)$.
- $\dim SS_b = \dim SS_g$
- For $g \in Sp(T^*(X))$ we have, $g(SS_b(M)) = SS_b(gM)$
- $\exists g \in Sp(T^*(X))$ s.t. $g(SS_b(M)) \cap X^* = 0$
- This implies that $p : g(SS_b(M)) \to X$ is finite.
- This implies that gM is smooth.

Relation with multiplicity

Theorem (Aizenbud-G.-Minchenko 2015)

Let G be an algebraic reductive group, $H_1, H_2 \subset G$ be spherical subgroups. The following system of equations on a distribution ξ on G is holonomic:

- *ξ is left H_1 invariant*
- *ξ is right H_2 invariant*
- *ξ is eigen w.r.t. the center $\mathfrak{z}(u(\mathfrak{g}))$ of the universal enveloping algebra of the Lie algebra of G.*

The spherical character

Definition

Let (π, V) be an admissible representation of $G(\mathbb{R})$ and $v_1 \in (V^*)^{H_1}$, $v_2 \in (\tilde{V}^*)^{H_2}$. Define the spherical character of π w.r.t. v_1 and v_2 by:

$$\langle \xi, f \rangle := \langle \pi^*(f) v_1, v_2 \rangle.$$

Corollary

A spherical character of admissible representation w.r.t. pair of spherical groups is a holonomic distribution.

Corollary (Aizenbud, Gourevitch, Minchenko, Sayag)

For any local field F, any spherical character of an admissible representation of $G(F)$ is smooth in a Zariski open dense set.

Relation with Hausdorffness

Theorem: If $\#X/G < \infty$ then $\mathfrak{g}\mathcal{S}(X) \subset \mathcal{S}(X)$ is closed and has finite codimension.

Proof:

Lemma (Aizenbud-Gourevitch-Krötz-Liu)

$H_*(\mathfrak{g}, \mathcal{S}(G/H))$ *are finite dimensional (and Hausdorff).*

Assume that $X = U \cup Z$ is a union of an open orbit and a closed one. It is enough to prove that $\mathfrak{g}(\mathcal{S}(X)/\mathcal{S}(Z)) \subset \mathcal{S}(X)/\mathcal{S}(Z)$ is closed and of finite co-dimension. Let $V := (\mathcal{S}(X)/\mathcal{S}(Z))$. The Borel's lemma and the lemma above implies that V is an inverse limit (with epimorphisms) of representations with finite dimensional co-homologies.

Lemma

Such inverse limit commutes with homologies.

On the other hand the Bernstein-Kashiwara theorem implies that $\dim(V^*)^{\mathfrak{g}} \leq \mathcal{S}^*(X)^{\mathfrak{g}} < \infty$.