

HOLONOMICITY OF RELATIVE CHARACTERS AND APPLICATIONS TO MULTIPLICITY BOUNDS FOR SPHERICAL PAIRS

AVRAHAM AIZENBUD, DMITRY GOUREVITCH, AND ANDREY MINCHENKO

to Joseph Bernstein - the teacher, the mathematician, and the person.

ABSTRACT. In this paper, we prove that any relative character (a.k.a. spherical character) of any admissible representation of a real reductive group G with respect to any pair of spherical subgroups is a holonomic distribution on G . This implies that the restriction of the relative character to an open dense subset is given by an analytic function. The proof is based on an argument from algebraic geometry and thus implies also analogous results in the p -adic case.

As an application, we give a short proof of some results from [KO13; KS] on boundedness and finiteness of multiplicities of irreducible representations in the space of functions on a spherical space.

In order to deduce this application we prove relative and quantitative analogs of the Bernstein-Kashiwara theorem, which states that the space of solutions of a holonomic system of differential equations in the space of distributions is finite-dimensional. We also deduce that, for every algebraic group G , the space of G -equivariant distributions on any algebraic G -manifold X is finite-dimensional if G has finitely many orbits on X .

CONTENTS

1. Introduction	2
1.1. The relative character	2
1.2. Bounds on the dimension of the space of solutions	3
1.3. Applications to representation theory	3
1.4. The non-Archimedean case	4
1.5. Related results	4
1.6. Future applications	5
1.7. Structure of the paper	5
1.8. Acknowledgements	5
2. Proof of Theorem B	5
2.1. Proof of Lemma 2.0.1	7
3. Dimension of the space of solutions of a holonomic system	7
3.1. Preliminaries	7
3.2. Dimension of the space of solutions of a holonomic system	10
3.3. Families of D -modules	11
4. Proof of Theorems A and E	12
4.1. Preliminaries	12
4.2. Proof of Theorem A and Proposition 1.1.4	12
4.3. Proof of Theorem E	13

Date: August 24, 2015.

Key words and phrases. Real reductive group, Holonomic D -module, equivariant distribution, Bessel function, spherical pair, moment map. 2010 MS Classification: 17B08, 22E45, 22E46, 22E50, 14M27, 46F05, 32C38, 53D20.

The three authors were partially supported by the Minerva foundation with funding from the Federal German Ministry for Education and Research; A.A. was also partially supported by ISF grant 687/13, and D.G. and A.M. by ERC StG grant 637912 and ISF grant 756/12. At the beginning of this project D.G. was a team member in the ERC grant 291612.

Appendix A. Proof of Lemma 3.1.1

13

References

14

1. INTRODUCTION

1.1. The relative character. In this paper, we prove that a relative character (a.k.a. spherical character) of a smooth admissible Fréchet representation of moderate growth of a real reductive group is holonomic. The relative character is a basic notion of relative representation theory that generalizes the notion of a character of a representation. By a real reductive group we mean a connected algebraic reductive group defined over \mathbb{R} . Unless confusion is possible, we will not distinguish between such a group and the group of its real points. Let us now recall the notions of spherical pair, relative character and holonomic distribution. For the notion of smooth admissible Fréchet representation of moderate growth and its relation to Harish-Chandra modules we refer the reader to [Cas89] or [Wal88, Chapter 11].

Definition 1.1.1. Let G be a real reductive group and let $H \subset G$ be its (algebraic) subgroup. Let P denote a minimal parabolic subgroup of G and B denote a Borel subgroup of the complexification $G_{\mathbb{C}}$. The subgroup H is called *real spherical* if it has finitely many orbits on G/P and *spherical* if its complexification has finitely many orbits on $G_{\mathbb{C}}/B$.

It is known that a pair (G, H) is spherical if and only if $H_{\mathbb{C}}$ has an open orbit on $G_{\mathbb{C}}/B$ and real spherical if and only if H has an open orbit on G/P , see [KS2].

Definition 1.1.2. Let G be a real reductive group and let $H_1, H_2 \subset G$ be its (algebraic) spherical subgroups and let \mathfrak{h}_i be the Lie algebras of H_i . Let χ_i be characters of \mathfrak{h}_i . Let π be a smooth admissible Fréchet representation of moderate growth of G , π^* be the continuous dual of π , and $\hat{\pi} \subset \pi^*$ be the smooth contragredient representation to π (i.e. the only smooth admissible Fréchet representation of moderate growth with the same space underlying Harish-Chandra module as π^*). Let $\phi_1 \in (\pi^*)^{\mathfrak{h}_1, \chi_1}$ and $\phi_2 \in (\hat{\pi}^*)^{\mathfrak{h}_2, \chi_2}$ be equivariant functionals. Fix a Haar measure on G . It gives rise to an action of the space of Schwartz functions $\mathcal{S}(G)$ on π^* and $\hat{\pi}^*$, and this action maps elements of π^* and $\hat{\pi}^*$ to elements of $\hat{\pi}$ and $\hat{\hat{\pi}} = \pi$ respectively. For the definition of the space of Schwartz functions $\mathcal{S}(G)$ see, e.g., [Cas89; Wal88; AG08].

The relative character ξ_{ϕ_1, ϕ_2} of π , with respect to ϕ_1 and ϕ_2 , is the tempered distribution on G (i.e. a continuous functional on $\mathcal{S}(G)$) defined by $\langle \xi_{\phi_1, \phi_2}, f \rangle = \langle \phi_1, \pi(f) \cdot \phi_2 \rangle$.

Definition 1.1.3. The singular support¹ $\text{SS}(\xi)$ of a distribution ξ on a real algebraic manifold X is the joint zero set in T^*X of all the symbols of (algebraic) differential operators that annihilate ξ . The distribution ξ is called holonomic if $\dim \text{SS}(\xi) = \dim X$.

In this paper we prove the following theorem.

Theorem A (See §4.2). *In the situation of Definition 1.1.2, the relative character ξ_{ϕ_1, ϕ_2} is holonomic.*

We prove Theorem A using the following well-known statement.

Proposition 1.1.4 (See §4.2). *Let $\mathfrak{g}, \mathfrak{h}_i$ be the Lie algebras of G and H_i , $i = 1, 2$. Identify T^*G with $G \times \mathfrak{g}^*$ and let*

$$S := \{(g, \alpha) \in G \times \mathfrak{g}^* \mid \alpha \text{ is nilpotent, } \langle \alpha, \mathfrak{h}_1 \rangle = 0, \langle \alpha, \text{Ad}^*(g)(\mathfrak{h}_2) \rangle = 0\}.$$

Then $\text{SS}(\xi_{\phi_1, \phi_2}) \subset S$.

¹a.k.a. characteristic variety

Note that the Bernstein inequality states that the dimension of the singular support of any non-zero distribution is at least the dimension of the underlying manifold. Thus Theorem A follows from the following more precise version, which is the core of this paper.

Theorem B (See §2). *We have $\dim S = \dim G$.*

Let $U := \{g \in G \mid S \cap T_g^*G = \{(g, 0)\}\}$. Note that U is Zariski open since S is conic and closed. It is easy to see that Theorem B implies the following corollary.

Corollary C. *The set U is a Zariski open dense subset of G .*

This corollary is useful in view of the next proposition, which follows from Proposition 1.1.4 and Corollary 3.1.3 below.

Proposition 1.1.5. *The restriction $\xi_{\phi_1, \phi_2}|_U$ is an analytic function.*

Remark 1.1.6. In general, S has irreducible components that can not lie in $\text{SS}(\xi_{\phi_1, \phi_2})$ for any ϕ_1, ϕ_2 . Indeed, $\text{SS}(\xi_{\phi_1, \phi_2})$ is coisotropic by [Gab81; KKS73; Mal79], and thus, by Theorem A, Lagrangian. On the other hand, one can show that when $G = \text{GL}(4, \mathbb{R})$, and $H_1 = H_2 = \text{GL}(2, \mathbb{R}) \times \text{GL}(2, \mathbb{R})$ embedded as block matrices inside G , the variety S has non-isotropic (and thus non-Lagrangian) components.

1.2. Bounds on the dimension of the space of solutions. Next we apply our results to representation theory. For this, we use the following theorem.

Theorem 1.2.1 (Bernstein-Kashiwara). *Let X be a real algebraic manifold. Let*

$$\{D_i \xi = 0\}_{i=1 \dots n}$$

be a system of linear PDE on X with algebraic coefficients. Suppose that the joint zero set of the symbols of D_i is $\dim X$ -dimensional. Then the space of solutions of this system in $\mathcal{S}^(X)$ is finite-dimensional.*

It seems that this theorem is not found in the literature in this formulation, however it has two proofs, one due to Kashiwara (see [Kas74; KK76] for similar statements) and another due to Bernstein (unpublished).

In order to make our applications in representation theory more precise, we need an effective version of this theorem. We prove such a version (see Theorem 3.2.2 below) following Bernstein's approach, as it is more appropriate for effective bounds. We use this effective version to derive a relative version. Namely, we show that if the system depends on a parameter in an algebraic way, then the dimension of the space of solutions is bounded (see §3.3 below).

This relative version allows us to deduce the following theorem.

Theorem D (See §3.3). *Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let \mathfrak{g} be the Lie algebra of G . Let \mathcal{E} be an algebraic G -equivariant bundle on X . Then, for any natural number $n \in \mathbb{N}$, there exists $C_n \in \mathbb{N}$ such that for every n -dimensional representation τ of \mathfrak{g} we have*

$$\dim \text{Hom}_{\mathfrak{g}}(\tau, \mathcal{S}^*(X, \mathcal{E})) \leq C_n,$$

where $\text{Hom}_{\mathfrak{g}}$ denotes the space of all continuous \mathfrak{g} -equivariant maps.

1.3. Applications to representation theory. Using §3, we give a short proof of some results from [KO13; KS]. Namely, we prove:

Theorem E (See §4). *Let G be a real reductive group, H be a Zariski closed subgroup, and \mathfrak{h} be the Lie algebra of H .*

- (i) If H is a real spherical subgroup then, for every irreducible smooth admissible Fréchet representation of moderate growth $\pi \in \text{Irr}(G)$, and natural number $n \in \mathbb{N}$ there exists $C_n \in \mathbb{N}$ such that for every n -dimensional representation τ of \mathfrak{h} we have

$$\dim \text{Hom}_{\mathfrak{h}}(\pi, \tau) \leq C_n.$$

- (ii) If H is a spherical subgroup and we consider only one-dimensional τ then the space is universally bounded, i.e. there exists $C \in \mathbb{N}$ such that $\dim(\pi^*)^{\mathfrak{h}, \chi} \leq C$ for any $\pi \in \text{Irr}(G)$ and any character χ of \mathfrak{h} .

Corollary F. Let G be a real reductive group, H be a Zariski closed reductive subgroup, and \mathfrak{h} be the Lie algebra of H .

- (i) If the diagonal ΔH is a real spherical subgroup in $G \times H$ then for every $\pi \in \text{Irr}(G)$ and $\tau \in \text{Irr}(H)$ we have finite multiplicities, i.e.

$$\dim \text{Hom}_{\mathfrak{h}}(\pi, \tau) < \infty.$$

- (ii) If the diagonal ΔH is a spherical subgroup in $G \times H$ then the multiplicities are universally bounded, i.e., there exists $C \in \mathbb{N}$ such that for every $\pi \in \text{Irr}(G)$, $\tau \in \text{Irr}(H)$ we have

$$\dim \text{Hom}_{\mathfrak{h}}(\pi, \tau) \leq C.$$

This corollary follows from Theorem E since $\text{Hom}_{\mathfrak{h}}(\pi, \tau)$ lies in the space of $\Delta \mathfrak{h}$ -invariant functionals on the completed tensor product $\pi \widehat{\otimes} \hat{\tau} \in \text{Irr}(G \times H)$ (see [AG09a, Corollary A.0.7 and Lemma A.0.8]). All symmetric pairs satisfying the conditions of the corollary were classified in [KM14].

The inverse implications for Theorem E and Corollary F are proven in [KO13].

An advantage of Theorem E(i) over [KO13; KS] is that C_n does not depend on τ . On the other hand, the results on multiplicities in [KO13; KS] are slightly stronger than Theorem E since they allow H to be any closed Lie subgroup and consider maps from the Harish-Chandra space of π to τ . In addition, [KO13, Theorem B] implies that if $H \subset G$ is an algebraic spherical subgroup there exists $C \in \mathbb{N}$ such that $\dim \text{Hom}_{\mathfrak{h}}(\pi, \tau) \leq C \dim \tau$, for every $\pi \in \text{Irr}(G)$ and every finite-dimensional continuous representation τ of H . It is easy to modify our proof of Theorem E(ii) to show the boundedness of multiplicities for any $\pi \in \text{Irr}(G)$ and any τ of a fixed dimension, but the proof that the bound depends linearly on this dimension would require more work.

Our methods are different from the methods of [KO13], which in turn differ from the ones of [KS], and the bounds given in the three works are probably very different.

1.4. The non-Archimedean case. Theorem B and Corollary C hold over arbitrary fields of characteristic zero. They are useful also for p -adic local fields F , since the analogs of Propositions 1.1.4 and 1.1.5 hold in this case, see [AGS, Theorem A and Corollary F]. Namely, we have the following theorem.

Theorem 1.4.1 ([AGS]). Let G be a reductive group defined over a non-Archimedean field F of characteristic 0 and let ξ be a relative character of a smooth admissible representation with respect to two spherical subgroups $H_1, H_2 \subset G$. Let S and U be the sets defined in Proposition 1.1.4 and Corollary C. Then

- (i) The wave front set of ξ lies in S .
(ii) The restriction of ξ to U is given by a locally constant function.

1.5. Related results. In the group case, i.e. the case when $G = H \times H$ and $H_1 = H_2 = \Delta H \subset H \times H$, Theorem A essentially becomes the well-known fact that characters of admissible representations are holonomic distributions.

As we mentioned above, Theorem E was proven earlier in [KO13; KS] using different methods. An analog of Theorem E(i) over non-Archimedean fields is proven in [Del10] and [SV, Theorem 5.1.5] for many spherical pairs, including arbitrary symmetric pairs.

The group case of Corollary C, Proposition 1.1.5, and Theorem 1.4.1(ii) is (the easy part of) the Harish-Chandra regularity theorem (see [HC63; HC65]). Another known special case of these results is the regularity of Bessel functions, see [LM; AGK; AG].

1.6. Future applications. Our proof of Theorem E(ii) does not use the Casselman embedding theorem (Theorem 4.1.3). This gives us hope that it can be extended to the non-Archimedean case. The main difficulty is the fact that our proof heavily relies on the theory of modules over the ring of differential operators, which does not act on distributions in the non-Archimedean case. However, in view of Theorem 1.4.1 we believe that this difficulty can be overcome. Namely, one can deduce an analog of Theorem E(ii) for many spherical pairs from the following conjecture .

Conjecture 1.6.1. Let G be a reductive group defined over a non-Archimedean field F of characteristic 0 and let $H_1, H_2 \subset G$ be its (algebraic) spherical subgroups. Let χ_i be characters of H_i . Fix a character λ of the Bernstein center $\mathfrak{z}(G)$.

Then the space of distributions which are:

- (1) left (H_1, χ_1) -equivariant,
- (2) right (H_2, χ_2) -equivariant,
- (3) $(\mathfrak{z}(G), \lambda)$ -eigen,

is finite-dimensional. Moreover, this dimension is uniformly bounded when λ varies.

Note that Theorem B and Theorem 1.4.1(i) imply that the dimension of (the Zariski closure of) the wave front set of a distribution that satisfies (1-3) does not exceed $\dim G$. In many ways the wave front set replaces the singular support, in absence of the theory of differential operators (see, e.g., [Aiz13; AD; AGS; AGK]). Thus, in order to prove Conjecture 1.6.1, it is left to prove analogs of Theorems 1.2.1 and 3.2.2 for the integral system of equations (1-3).

1.7. Structure of the paper. In §2, we prove Theorem B using a theorem of Steinberg [Ste76] concerning the Springer resolution.

In §3, we prove an effective version of Theorem 1.2.1, and then adapt it to algebraic families. We also derive Theorem D.

In §4, we derive Theorem E from Theorem B and §3. We do that by embedding the multiplicity space into a certain space of relative characters.

In Appendix A, we prove Lemma 3.1.1 which computes the pullback of the D-module of distributions with respect to a closed embedding. We use this lemma in §3.

1.8. Acknowledgements. We thank Eitan Sayag and Bernhard Kroetz for fruitful discussions. We thank Joseph Bernstein for telling us the sketch of his proof of Theorem 1.2.1.

A.A. and D.G. will always be grateful to Joseph Bernstein for introducing them to the amazing world of algebra, for sharing his knowledge, his approaches to problems and his philosophy for already more than half of their lives, and for being a shining example forever.

2. PROOF OF THEOREM B

It is enough to prove the theorem for a reductive group G defined over an algebraically closed field of characteristic 0. Since S includes the zero section of $T^*G \cong G \times \mathfrak{g}^*$, we have $\dim S \geq \dim G$. Thus, it is enough to prove that $\dim S \leq \dim G$. Let \mathcal{B} denote the flag variety of G and $\mathcal{N} \subset \mathfrak{g}^*$ denote the nilpotent cone. Since G is reductive, we can identify

$$T^*\mathcal{B} \cong \{(B, X) \in \mathcal{B} \times \mathfrak{g}^* \mid X \in (\text{Lie} B)^\perp\}.$$

Recall the Springer resolution $\mu : T^*\mathcal{B} \rightarrow \mathcal{N}$ defined by $\mu(B, X) = X$ and consider the following diagram.

$$(1) \quad \begin{array}{ccc} T^*\mathcal{B} \times T^*\mathcal{B} & & G \times \mathcal{N} \\ & \searrow^{\mu \times \mu} & \swarrow^{\alpha} \\ & \mathcal{N} \times \mathcal{N} & \\ & \downarrow^{res} & \\ & \mathfrak{h}_1^* \times \mathfrak{h}_2^* & \end{array}$$

Here, α is defined by $\alpha(g, X) = (X, \text{Ad}^*(g^{-1})X)$, and res is the restriction. Passing to the fiber of $0 \in \mathfrak{h}_1^* \times \mathfrak{h}_2^*$, we obtain the following diagram.

$$(2) \quad \begin{array}{ccc} L_1 \times L_2 & & S \\ & \searrow^{\mu'} & \swarrow^{\alpha'} \\ & \mathcal{N}_{\mathfrak{h}_1} \times \mathcal{N}_{\mathfrak{h}_2} & \end{array}$$

Here, $\mathcal{N}_{\mathfrak{h}_i} := \mathcal{N} \cap \mathfrak{h}_i^\perp$ and $L_i := \{(B, X) \in T^*\mathcal{B} \mid X \in \mathfrak{h}_i^\perp\}$. We need to estimate $\dim S$. We do it using the following lemma.

Lemma 2.0.1 (See §2.1 below). *Let $\varphi_i : X_i \rightarrow Y$, $i = 1, 2$, be morphisms of algebraic varieties. Suppose that φ_2 is surjective. Then there exists $y \in Y$ such that*

$$\dim X_1 \leq \dim X_2 + \dim \varphi_1^{-1}(y) - \dim \varphi_2^{-1}(y).$$

By this lemma, applied to $\phi_1 = \alpha'$ and $\phi_2 = \mu'$, it is enough to estimate the dimensions of L_i and of the fibers of μ' and α' .

Lemma 2.0.2. *We have $\dim L_1 = \dim L_2 = \dim \mathcal{B}$.*

Proof. Since H_i has finitely many orbits in \mathcal{B} , it is enough to show that L_i is the union of the conormal bundles to the orbits of H_i in \mathcal{B} . Let $B \in \mathcal{B}$, and $\mathfrak{b} = \text{Lie} B$, and identify $T_B\mathcal{B} \cong \mathfrak{g}/\mathfrak{b}$. Then $T_B(H_i \cdot B) \cong \mathfrak{h}_i/(\mathfrak{b} \cap \mathfrak{h}_i)$ and the conormal space at B to the H_i -orbit of B is identified with $\mathfrak{b}^\perp \cap \mathfrak{h}_i^\perp$. \square

Let $(\eta, \text{Ad}^*(g)\eta) \in \text{Im}(\alpha')$. The fiber $(\alpha')^{-1}(\eta, \text{Ad}^*(g)\eta)$ is isomorphic to the stabilizer G_η , and the dimension of the fiber $(\mu')^{-1}(\eta, \text{Ad}^*(g)\eta)$ is twice the dimension of the Springer fiber $\mu^{-1}(\eta)$. Recall the following theorem of Steinberg (conjectured by Grothendieck):

Theorem 2.0.3 ([Ste76, Theorem 4.6]).

$$\dim G_\eta - 2 \dim \mu^{-1}(\eta) = \text{rk } G.$$

Using Lemma 2.0.1, we obtain for some $(\eta, \text{ad}^*(g)\eta)$:

$$\begin{aligned} \dim S &\leq \dim(L_1 \times L_2) + \dim(\alpha')^{-1}(\eta, \text{ad}^*(g)\eta) - \dim(\mu')^{-1}(\eta, \text{ad}^*(g)\eta) = \\ &= 2 \dim \mathcal{B} + \dim G_\eta - 2 \dim \mu^{-1}(\eta) = 2 \dim \mathcal{B} + \text{rk } G = \dim G. \end{aligned}$$

\square

2.1. Proof of Lemma 2.0.1. Recall that, for a dominant morphism $\varphi : X \rightarrow Y$ of irreducible varieties, there exists an open dense $U \subset Y$ such that $\dim X = \dim Y + \dim \varphi^{-1}(y)$ for all $y \in U$ (see, e. g., [Mum99, Theorem 1.8.3]). Let Z be an irreducible component of X_1 of maximal dimension and $W \subset Y$ be the Zariski closure of $\varphi_1(Z)$. Since W is irreducible, there exists an open dense $U \subset W$ such that

$$(3) \quad \dim X_1 = \dim Z \leq \dim W + \dim \varphi_1^{-1}(y)$$

for all $y \in U$. Let $V \subset U$ be an open dense subset such that $\varphi_2^{-1}(V)$ intersects those and only those irreducible components C_1, \dots, C_j of $\varphi_2^{-1}(W)$ that map dominantly to W . Note that $j > 0$ since φ_2 is surjective. Moreover, without loss of generality, we may assume that, for every $1 \leq i \leq j$, all fibers over V of the restriction of φ_2 to C_i are of the same dimension. Since one of these dimensions has to be equal to $\dim \varphi_2^{-1}(V) - \dim V$, we have, that there is an $1 \leq i \leq j$ such that, for all $y \in V$,

$$(4) \quad \dim V = \dim C_i - \dim(\varphi_2|_{C_i})^{-1}(y) \leq \dim \varphi_2^{-1}(V) - \dim \varphi_2^{-1}(y) \leq \dim X_2 - \dim \varphi_2^{-1}(y).$$

Thanks to $\dim V = \dim W$, taking any $y \in V$, formulas (3) and (4) imply the statement. \square

3. DIMENSION OF THE SPACE OF SOLUTIONS OF A HOLONOMIC SYSTEM

In this section, we prove an effective version of Theorem 1.2.1, and then adapt it to algebraic families. We also derive Theorem D.

3.1. Preliminaries.

3.1.1. D-modules. In this section, we will use the theory of D-modules on algebraic varieties over an arbitrary field k of characteristic zero. We will now recall some facts and notions that we will use. For a good introduction to the algebraic theory of D-modules, we refer the reader to [Ber] and [Bor87]. For a short overview, see [AG09a, Appendix B].

By a *D-module* on a smooth algebraic variety X we mean a quasi-coherent sheaf of right modules over the sheaf D_X of algebras of algebraic differential operators. By a *finitely generated D-module* on a smooth algebraic variety X we mean a coherent sheaf of right modules over the sheaf D_X . Denote the category of D_X -modules by $\mathcal{M}(D_X)$.

For a smooth affine variety V , we denote $D(V) := D_V(V)$. Note that the category $\mathcal{M}(D_V)$ of D-modules on V is equivalent to the category of $D(V)$ -modules. We will thus identify these categories.

The algebra $D(V)$ is equipped with a filtration which is called the geometric filtration and defined by the degree of differential operators. The associated graded algebra with respect to this filtration is the algebra $\mathcal{O}(T^*V)$ of regular functions on the total space of the cotangent bundle of V . This allows us to define the *singular support* of a finitely generated D-module M on V in the following way. Choose a good filtration on M , i.e. a filtration such that the associated graded module is a finitely-generated module over $\mathcal{O}(T^*V)$, and define the singular support $SS(M)$ to be the support of this module. One can show that the singular support does not depend on the choice of a good filtration on M .

This definition easily extends to the non-affine case. A finitely generated D-module M on X is called *smooth* if $SS(M)$ is the zero section of T^*X . This is equivalent to being coherent over \mathcal{O}_X and to be coherent and locally free over \mathcal{O}_X . The Bernstein inequality states that, for any non-zero finitely generated M , we have $\dim SS(M) \geq \dim X$. If the equality holds then M is called *holonomic*.

For a closed embedding $i : X \rightarrow Y$ of smooth affine algebraic varieties, define the functor $i^! : \mathcal{M}(D_Y) \rightarrow \mathcal{M}(D_X)$ by $i^!(M) := \{m \in M \mid I_X m = 0\}$, where $I_X \subset \mathcal{O}(Y)$ is the ideal of all functions that vanish on X . It has a left adjoint functor $i_* : \mathcal{M}(D_X) \rightarrow \mathcal{M}(D_Y)$, given by tensor product with $i^!(D_Y)$. The functor i_* is an equivalence of categories between $\mathcal{M}(D_X)$

and the category of D_Y -modules supported in X . Both i_* and $i^!$ map holonomic modules to holonomic ones.

If V is an affine space then the algebra $D(V)$ has an additional filtration, called the Bernstein filtration. It is defined by $\deg(\partial/\partial x_i) = \deg(x_i) = 1$, where x_i are the coordinates in V . This gives rise to the notion of Bernstein's singular support, that we will denote $SS_b(M) \subset T^*V \cong V \oplus V^*$. It is known that $\dim SS(M) = \dim SS_b(M)$.

We will also use the theory of analytic D-modules. By an *analytic D-module* on a smooth complex analytic manifold X we mean a coherent sheaf of right modules over the sheaf D_X^{An} of algebras of differential operators with analytic coefficients. All of the above notions and statements, except those concerning the Bernstein filtration, have analytic counterparts. In addition, all smooth analytic D-modules of the same rank are isomorphic.

3.1.2. Distributions. We will use the theory of distributions on differentiable manifolds and the theory of tempered distributions on real algebraic manifolds, see e.g. [Hör90; AG08]. For a real algebraic manifold X , we denote the space of distributions on X by $\mathcal{D}'(X) := (C_c^\infty(X))^*$ and the space of tempered distributions (a.k.a. Schwartz distributions) by $\mathcal{S}^*(X) := (\mathcal{S}(X))^*$. Similarly, for an algebraic bundle \mathcal{E} over X we denote $\mathcal{D}'(X, \mathcal{E}) := (C_c^\infty(X, \mathcal{E}))^*$ and $\mathcal{S}^*(X, \mathcal{E}) := (\mathcal{S}(X, \mathcal{E}))^*$. The spaces $\mathcal{D}'(X)$ and $\mathcal{S}^*(X)$ form (right) D-modules over X . The space $\mathcal{D}'(X)$ is also an analytic D-module. We define the singular support of a distribution to be the singular support of the D-module it generates. It is well-known that this definition is equivalent to Definition 1.1.3. We say that a distribution is holonomic if it generates a holonomic D-module.

Lemma 3.1.1 (See Appendix A). *Let $i : X \rightarrow Y$ be a closed embedding of smooth affine real algebraic varieties. Then*

$$\mathcal{D}'(X) \cong i^!(\mathcal{D}'(Y)) \text{ and } \mathcal{S}^*(X) \cong i^!(\mathcal{S}^*(Y)).$$

Lemma 3.1.2. *Let M be a smooth $D(\mathbb{C}^n)$ -module of rank r . Embed the space $An(\mathbb{C}^n)$ of analytic functions on \mathbb{C}^n into $\mathcal{D}'(\mathbb{R}^n)$ using the Lebesgue measure. Then $\text{Hom}(M, \mathcal{D}'(\mathbb{R}^n)) = \text{Hom}(M, An(\mathbb{C}^n))$ and $\dim \text{Hom}(M, \mathcal{D}'(\mathbb{R}^n)) = r$.*

Proof. Let $M_{An} := M \otimes_{\mathcal{O}(\mathbb{C}^n)} An(\mathbb{C}^n)$ and $D_{An}(\mathbb{C}^n) := D(\mathbb{C}^n) \otimes_{\mathcal{O}(\mathbb{C}^n)} An(\mathbb{C}^n)$ be the analytizations of M and $D(\mathbb{C}^n)$. Then

$$\text{Hom}_{D(\mathbb{C}^n)}(M, \mathcal{D}'(\mathbb{R}^n)) \cong \text{Hom}_{D_{An}(\mathbb{C}^n)}(M_{An}, \mathcal{D}'(\mathbb{R}^n)).$$

Since M_{An} is also smooth, $M_{An} \cong An(\mathbb{C}^n)^r$. Thus it is left to prove that

$$\text{Hom}_{D_{An}(\mathbb{C}^n)}(An(\mathbb{C}^n), \mathcal{D}'(\mathbb{R}^n)) = \text{Hom}_{D_{An}(\mathbb{C}^n)}(An(\mathbb{C}^n), An(\mathbb{C}^n))$$

and the latter space is one-dimensional. This follows from the fact that a distribution with vanishing partial derivatives is a multiple of the Lebesgue measure. \square

Corollary 3.1.3. *If a distribution generates a smooth D-module, then it is analytic.*

3.1.3. Lie algebra actions.

Definition 3.1.4. Let X be an algebraic manifold over a field k and \mathfrak{g} be a Lie algebra over k .

- (i) An action of \mathfrak{g} on X is a Lie algebra map from \mathfrak{g} to the algebra of algebraic vector fields on X .
- (ii) Assume that X is affine, fix an action of \mathfrak{g} on X and let \mathcal{E} be an algebraic vector bundle on X . Let M be the space of global regular (algebraic) sections of \mathcal{E} . An action of \mathfrak{g} on \mathcal{E} is a linear map $T : \mathfrak{g} \rightarrow \text{End}_k(M)$ such that, for any $\alpha \in \mathfrak{g}$, $f \in \mathcal{O}(X)$, $v \in M$, we have

$$T(\alpha)(fv) = (\alpha f)v + fT(\alpha)v.$$

- (iii) The definition above extends to non-affine X in a straightforward way.

3.1.4. Weil representation.

Definition 3.1.5. Let V be a finite-dimensional real vector space. Let ω be the standard symplectic form on $V \oplus V^*$. Denote by $p_V : V \oplus V^* \rightarrow V$ and $p_{V^*} : V \oplus V^* \rightarrow V^*$ the natural projections. Define an action of the symplectic group $\mathrm{Sp}(V \oplus V^*, \omega)$ on the algebra $D(V)$ by

$$(\partial_v)^g := \pi(g)(\partial_v) := p_{V^*}(g(v, 0)) + \partial_{p_V(g(v, 0))}, \quad w^g := \pi(g)w := p_{V^*}(g(0, w)) + \partial_{p_V(g(0, w))}$$

where $v \in V$, $w \in V^*$, ∂_v denotes the derivative in the direction of v , and elements of V^* are viewed as linear polynomials and thus differential operators of order zero. For a $D(V)$ -module M and an element $g \in \mathrm{Sp}(V \oplus V^*)$, we will denote by M^g the $D(V)$ -module obtained by twisting the action of $D(V)$ by $\pi(g)$.

Since the above action of $\mathrm{Sp}(V \oplus V^*)$ preserves the Bernstein filtration on $D(V)$, the following lemma holds.

Lemma 3.1.6. *For a finitely generated $D(V)$ -module M and $g \in \mathrm{Sp}(V \oplus V^*)$ we have $SS_b(M^g) = gSS_b(M)$.*

Theorem 3.1.7 ([Wei64]). *There exists a two-folded cover $p : \widetilde{\mathrm{Sp}}(V \oplus V^*) \rightarrow \mathrm{Sp}(V \oplus V^*)$ and a representation Π of $\widetilde{\mathrm{Sp}}(V \oplus V^*)$ on the space $\mathcal{S}^*(V)$ of tempered distributions on V such that, for any $\alpha \in D(V)$, $g \in \widetilde{\mathrm{Sp}}(V \oplus V^*)$, $\xi \in \mathcal{S}^*(V)$, we have*

$$\Pi(g)(\xi\alpha) = (\Pi(g)\xi)\alpha^{p(g)}.$$

Corollary 3.1.8. *We have an isomorphism of $D(V)$ -modules $\mathcal{S}^*(V)^g \cong \mathcal{S}^*(V)$ for any $g \in \mathrm{Sp}(V \oplus V^*)$.*

In fact, this corollary can be derived directly from the Stone-von-Neumann theorem.

3.1.5. Flat morphisms.

Lemma 3.1.9. *Let $\phi : X \rightarrow Y$ be a proper morphism of algebraic varieties over a field k and \mathcal{M} be a coherent sheaf on X . Then there exists an open dense $U \subset Y$ such that $\mathcal{M}|_{\phi^{-1}(U)}$ is flat over U .*

Proof. By [EGA IV, Théorème II.3.I], the set V of scheme-theoretic points $x \in X$ for which \mathcal{M} is ϕ -flat at x is open in X . Since ϕ is proper, the set $Z := \phi(X \setminus V)$ is closed in Y . Note that \mathcal{M} is flat over $U := X \setminus Z$, since $\phi^{-1}(U) \subset V$. Moreover, U contains the generic points of the irreducible components of Y . Hence, $U \subset Y$ is dense. \square

Lemma 3.1.10 (See, e. g., [Mum74, Corollary on p. 50]). *Let $\phi : X \rightarrow Y$ be a proper morphism of algebraic varieties and \mathcal{M} be a coherent sheaf on X that is flat over Y . For a point $y \in Y$, let \mathcal{M}_y denote the pullback of \mathcal{M} to $\phi^{-1}(y)$. Then the function*

$$y \mapsto \chi(\mathcal{M}_y) = \sum_{i=0}^{\infty} (-1)^i \dim_{k(y)} H^i(\mathcal{M}_y)$$

is locally constant.

Corollary 3.1.11. *Let Y be an algebraic variety and \mathcal{M} be a coherent sheaf on $Y \times \mathbb{P}^n$. Then there exists an open dense $U \subset Y$ such that the Hilbert polynomial² of \mathcal{M}_y does not depend on y as long as $y \in U$.*

²For the definition of Hilbert polynomial see [Har77, Chapter III, Exercise 5.2].

3.2. Dimension of the space of solutions of a holonomic system.

Definition 3.2.1.

- (i) Let M be a finitely generated D -module over an affine space \mathbb{A}^n . Let F^i be a good filtration on M with respect to the Bernstein filtration on the ring $D_{\mathbb{A}^n}$. Let p be the corresponding Hilbert polynomial of M , i.e. $p(i) = \dim F^i$ for large enough i . Let k be the degree of p and a_k be the leading coefficient of p . Define the Bernstein degree of M to be $\deg_b(M) := k!a_k$. It is well-known that k and a_k do not depend on the choice of good filtration F^i .
- (ii) Let M be a finitely generated D -module over a smooth algebraic variety X . Let $X = \bigcup_{i=1}^l U_i$ be an open affine cover of X and let $\phi_i : U_i \hookrightarrow \mathbb{A}^{n_i}$ be closed embeddings. Denote

$$\deg_{\{(U_i, \phi_i)\}}(M) := \sum_{i=1}^l \deg_b((\phi_i)_*(M|_{U_i})).$$

Define the global degree of M by $\deg(M) := \min \deg_{\{(U_i, \phi_i)\}}(M)$, where the minimum is taken over the set of all possible affine covers and embeddings.

In this subsection, we prove

Theorem 3.2.2. *Let X be a real algebraic manifold. Let M be a holonomic right D -module. Then $\dim \text{Hom}(M, \mathcal{S}^*(X)) \leq \deg(M)$.*

We will need the following geometric lemmas

Lemma 3.2.3. *Let V be a vector space, $L \subset V$ be a subspace and $C \subset V$ be a closed conic algebraic subvariety such that $L \cap C = \{0\}$. Then the projection $p : C \rightarrow V/L$ is a finite map.*

Proof. By induction, it is enough to prove the case $\dim L = 1$. Choose coordinates x_1, \dots, x_n on V such that the coordinates x_1, \dots, x_{n-1} vanish on L . Let p be a homogeneous polynomial that vanishes on C but not on L . Write $p = \sum_{i=1}^k g_i x_n^i$, where each g_i is a homogeneous polynomial of degree $k - i$ in x_1, \dots, x_{n-1} . Then $x_n|_C$ satisfies a monic polynomial equation with coefficients in $\mathcal{O}(V/L)$. \square

Lemma 3.2.4. *Let W be a $2n$ -dimensional symplectic vector space, and $C \subset W$ be a closed conic subvariety of dimension n . Then there exists a Lagrangian subspace $L \subset W$ such that $L \cap C = \{0\}$.*

Proof. Let \mathcal{L} denote the variety of all Lagrangian subspaces of W . Note that $\dim \mathcal{L} = n(n + 1)/2$. Let $P(C) \subset \mathbb{P}(W)$ be the projectivizations of C and W . Consider the configuration space

$$X := \{(x, L) \in P(C) \times \mathcal{L} \mid x \subset L\}.$$

We have to show that $p(X) \neq \mathcal{L}$ where $p : X \rightarrow \mathcal{L}$ is the projection. Let $q : X \rightarrow P(C)$ be the other projection. Note that $\dim q^{-1}(x) = n(n - 1)/2$ for any $x \in P(C)$. Thus

$$\dim X = n(n - 1)/2 + n - 1 < n(n + 1)/2 = \dim \mathcal{L},$$

and thus $p : X \rightarrow \mathcal{L}$ cannot be onto. \square

Corollary 3.2.5. *Let V be a vector space of dimension n . Consider the standard symplectic form on $V \oplus V^*$. Let $C \subset V \oplus V^*$ be a closed conic subvariety of dimension n . Let $p : V \oplus V^* \rightarrow V$ denote the projection. Then there exists a linear symplectic automorphism $g \in \text{Sp}(V \oplus V^*)$ such that $p|_{gC}$ is a finite map.*

Proof. By Lemma 3.2.4 there exists a Lagrangian subspace $L \subset V \oplus V^*$ such that $L \cap C = \{0\}$. Since the action of $\mathrm{Sp}(V \oplus V^*)$ on Lagrangian subspaces is transitive, there exists $g \in \mathrm{Sp}(V \oplus V^*)$ such that $V^* = gL$ and thus $gC \cap V^* = \{0\}$. From Lemma 3.2.3 we get that $p|_{gC}$ is a finite map. \square

Proof of Theorem 3.2.2. Let $X = \bigcup_{i=1}^l U_i$ be an open affine cover of X and let $\phi_i : U_i \hookrightarrow \mathbb{A}^{n_i}$ be closed embeddings. Clearly

$$\dim \mathrm{Hom}(M, \mathcal{S}^*(X)) \leq \sum_{i=1}^l \dim \mathrm{Hom}(M|_{U_i}, \mathcal{S}^*(U_i)).$$

By Lemma 3.1.1

$$\mathrm{Hom}(M|_{U_i}, \mathcal{S}^*(U_i)) \cong \mathrm{Hom}(M|_{U_i}, \phi_i^!(\mathcal{S}^*(\mathbb{R}^{n_i}))) \cong \mathrm{Hom}((\phi_i)_*(M|_{U_i}), \mathcal{S}^*(\mathbb{R}^{n_i})).$$

Thus it is enough to show that for any holonomic D -module N on an affine space \mathbb{A}^n we have $\dim \mathrm{Hom}(N, \mathcal{S}^*(\mathbb{R}^n)) \leq \deg_b(N)$.

Let $C \subset \mathbb{A}^{2n}$ be the singular support of N with respect to the Bernstein filtration. By Corollary 3.2.5, there exists $g \in \mathrm{Sp}_{2n}$ such that $p|_{gC}$ is a finite map, where $p : \mathbb{A}^{2n} \rightarrow \mathbb{A}^n$ is the projection on the first n coordinates. By Corollary 3.1.8 we have

$$\dim \mathrm{Hom}(N, \mathcal{S}^*(\mathbb{R}^n)) = \dim \mathrm{Hom}(N^g, \mathcal{S}^*(\mathbb{R}^n)^g) = \dim \mathrm{Hom}(N^g, \mathcal{S}^*(\mathbb{R}^n)).$$

By Lemma 3.1.6 we have $SS_b(N^g) = gC$. Let F be a good filtration on N^g (with respect to the Bernstein filtration on $D(\mathbb{A}^n)$). We see that $\mathrm{Gr} N^g$ is finitely generated over $\mathcal{O}(\mathbb{A}^n)$, and thus so is N^g . Thus N^g is a smooth D -module. Note that $\mathrm{rk}_{\mathcal{O}(\mathbb{A}^n)} N^g \leq \deg_b N^g = \deg_b N$. By Lemma 3.1.2 $\dim \mathrm{Hom}(N^g, \mathcal{S}^*(\mathbb{R}^n)) \leq \mathrm{rk}_{\mathcal{O}(\mathbb{A}^n)} N^g$. \square

3.3. Families of D -modules. In this section we discuss families of D -modules on algebraic varieties over an arbitrary field k of zero characteristic.

Notation 3.3.1. Let $\phi : X \rightarrow Y$ be a map of algebraic varieties and \mathcal{M} be a quasi-coherent sheaf of \mathcal{O}_X -modules. For any $y \in Y$, denote by \mathcal{M}_y the pullback of \mathcal{M} to $\phi^{-1}(y)$.

Definition 3.3.2. Let X, Y be smooth algebraic varieties.

- If X and Y are affine we define the algebra $D(X, Y)$ to be $D(X) \otimes_k \mathcal{O}(Y)$.
- Extending this definition we obtain a sheaf of algebras $D_{X,Y}$ on $X \times Y$.
- By a family of D_X -modules parameterized by Y , we mean a sheaf of right modules over the sheaf of algebras $D_{X,Y}$ on $X \times Y$ which is quasicohherent as a sheaf of $\mathcal{O}_{X \times Y}$ -modules.
- We call a family of D_X -modules parameterized by Y coherent if it is locally finitely generated as a $D_{X,Y}$ -module.
- For a family \mathcal{M} of D_X -modules parameterized by Y and a point $y \in Y$, we call \mathcal{M}_y the specialization of \mathcal{M} at y and consider it with the natural structure of a D_X -module.
- We say that a coherent family \mathcal{M} is holonomic if every specialization is holonomic.

Theorem 3.3.3. *Let X, Y be smooth algebraic varieties and \mathcal{M} be a family of D_X -modules parametrized by Y . Then $\deg \mathcal{M}_y$ is bounded when y ranges over the k -points of Y .*

Proof. Without loss of generality, we can assume that $X = \mathbb{A}^n$ and Y is an affine variety, and prove that $\deg_b(\mathcal{M}_y)$ is bounded. We will prove this by induction on $\dim Y$.

The Bernstein filtration on $D(\mathbb{A}^n)$ gives rise to a filtration on $D(\mathbb{A}^n, Y)$. Choose a filtration F on \mathcal{M} which is good with respect to this filtration and let $N := \mathrm{Gr} \mathcal{M}$, considered as a graded $\mathcal{O}(\mathbb{A}^{2n} \times Y)$ -module. Associate to N a coherent sheaf \mathcal{N} on $\mathbb{P}^{2n-1} \times Y$. Let \mathcal{N}_y be the pullback of \mathcal{N} under the embedding of \mathbb{P}^{2n-1} into $\mathbb{P}^{2n-1} \times Y$ given by $x \mapsto (x, y)$. By definition, the Hilbert polynomial of \mathcal{M}_y with respect to the filtration induced by F is the

Hilbert polynomial of \mathcal{N}_y . By Corollary 3.1.11, there exists an open dense subset $U \subset Y$ such that the Hilbert polynomial of \mathcal{N}_y does not depend on y as long as $y \in U$. By the induction hypothesis, $\deg_b(\mathcal{M}_y)$ is bounded on $Y \setminus U$, and thus bounded on Y . \square

For an application of this theorem we will need the following lemma.

Lemma 3.3.4. *Let a real Lie algebra \mathfrak{g} act on a real algebraic manifold X and on an algebraic vector bundle \mathcal{E} on X . Fix a natural number n and let Y be the variety of all representations of \mathfrak{g} on \mathbb{C}^n . Then there exists a coherent family \mathcal{M} of D_X -modules parameterized by Y such that, for any $\tau \in Y$, we have*

- (1) $\mathrm{Hom}_{\mathfrak{g}}(\tau, \mathcal{S}^*(X, \mathcal{E})) = \mathrm{Hom}_{D_X}(\mathcal{M}_\tau, \mathcal{S}^*(X))$.
- (2) *The singular support of \mathcal{M}_τ (with respect to the geometric filtration) is included in*

$$\{(x, \phi) \in T^*X \mid \forall \alpha \in \mathfrak{g} \text{ we have } \langle \phi, \alpha(x) \rangle = 0\}.$$

Proof. It is enough to prove the lemma for affine X . Let N be the coherent sheaf of the regular (algebraic) sections of \mathcal{E} (considered as a sheaf of \mathcal{O}_X -modules). Let \mathcal{N} be the pullback of N to $X \times Y$. Let $\mathcal{N}' := \mathcal{N} \otimes_{\mathcal{O}_{X \times Y}} D_{X,Y} \otimes_{\mathbb{C}} \mathbb{C}^n$, and $\mathcal{N}'' \subset \mathcal{N}'$ be the $D_{X,Y}$ -submodule generated by elements of the form

$$\alpha n \otimes 1 \otimes v + n \otimes \xi_\alpha \otimes v + n \otimes f_\alpha(v),$$

where $\alpha \in \mathfrak{g}$, ξ_α is the vector field on X corresponding to α , and $f_\alpha(v) \in D_{X,Y} \otimes_{\mathbb{C}} \mathbb{C}^n$ is the \mathbb{C}^n -valued regular function on $X \times Y$ given by $f_\alpha(v)(x, \tau) = \tau(\alpha)v$. Then $\mathcal{M} := \mathcal{N}'/\mathcal{N}''$ satisfies the requirements. \square

Theorems 3.2.2 and 3.3.3 and Lemma 3.3.4 imply Theorem D.

4. PROOF OF THEOREMS A AND E

In this section, we derive Theorems A and E from Theorem B and §3. We do that by embedding the multiplicity space into a certain space of relative characters.

4.1. Preliminaries. For a real reductive group G , we denote by $\mathrm{Irr}(G)$ the collection of irreducible admissible smooth Fréchet representation of G of moderate growth. We refer to [Cas89; Wal88] for the background on these representations.

Theorem 4.1.1 (See [Wal88, Theorem 4.2.1]). *The center $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ of the universal enveloping algebra of the Lie algebra of G acts finitely on every admissible smooth Fréchet representation π of G of moderate growth. This means that there exists an ideal in $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ of finite codimension that annihilates π .*

Lemma 4.1.2 ([Ada, Theorem 1.2 and Corollary 1.4]). *For any real reductive group G , there exists an involution θ of G such that, for any $\pi \in \mathrm{Irr}(G)$, we have $\hat{\pi} \cong \pi^\theta$.*

Theorem 4.1.3 (Casselman embedding theorem, see [CM82, Proposition 8.23]). *Let G be a real reductive group and P be a minimal parabolic subgroup of G . Let $\pi \in \mathrm{Irr}(G)$. Then there exists a finite-dimensional representation σ of P and an epimorphism $\mathrm{Ind}_P^G(\sigma) \twoheadrightarrow \pi$.*

4.2. Proof of Theorem A and Proposition 1.1.4. Theorem A follows from Theorem B and Proposition 1.1.4.

Proof of Proposition 1.1.4. Let ξ be a relative character of a smooth admissible Fréchet representation π of G of moderate growth with respect to a pair of subgroups (H_1, H_2) and their characters χ_1, χ_2 . By Theorem 4.1.1, there exists an ideal $I \subset \mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ of finite codimension that annihilates π and thus annihilates ξ . For any element $z \in \mathfrak{z}(\mathcal{U}(\mathfrak{g}))$, there exists a polynomial p such that $p(z) \in I$ and thus $p(z)\xi = 0$. This implies that the symbol of any $z \in \mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ of positive degree vanishes on the singular support of ξ . It is well-known that the

joint zero-set of these symbols over each point $g \in G$ is the nilpotent cone $\mathcal{N}(\mathfrak{g}^*)$. Since ξ is $(\mathfrak{h}_1 \times \mathfrak{h}_2, \chi_1 \times \chi_2)$ -equivariant, this implies that the singular support of ξ lies in S . \square

4.3. Proof of Theorem E. Part (i) follows immediately from Theorem D and the Casselman embedding theorem. If G is quasi-split then so does part (ii). For the proof of part (ii) in the general case, we will need the following lemma.

Lemma 4.3.1. *Let G be a real reductive group and H_1, H_2 be spherical subgroups. Let $Y = \text{Spec}(\mathfrak{z}(\mathcal{U}(\mathfrak{g}))) \times Y_1 \times Y_2$, where Y_i is the variety of characters of $\mathfrak{h}_i = \text{Lie } H_i$. For any $\lambda \in Y(\mathbb{C})$, define $U_{\lambda, \chi_1, \chi_2} := \mathcal{S}^*(G)^{\mathfrak{h}_1 \times \mathfrak{h}_2, (\chi_1, \chi_2), (\mathfrak{z}(\mathcal{U}(\mathfrak{g})), \lambda)}$ to be the space of tempered distributions on G that are left χ_1 -equivariant with respect to \mathfrak{h}_1 , right χ_2 -equivariant with respect to \mathfrak{h}_2 and are eigendistributions with respect to the action of $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ with eigencharacter λ . Then $\dim U_{\lambda, \chi_1, \chi_2}$ is bounded over $Y(\mathbb{C})$.*

Proof. Let us construct a family of $D(G)$ -modules \mathcal{M} parameterized by Y . For any $\alpha \in \mathfrak{g}$, let r_α and l_α be the corresponding right and left invariant vector fields on G considered as elements in $D(G, Y)$. For any $\beta \in \mathfrak{z}(\mathcal{U}(\mathfrak{g}))$, $\alpha_i \in \mathfrak{h}_i$, let $f_\beta, g_{\alpha_i}^i$ be the functions on Y that send $(\mu, \gamma_1, \gamma_2) \in Y$ to $\mu(\beta), \gamma_i(\alpha_i)$ respectively. Let also d_β be the differential operator on G corresponding to β , such that $d_\beta \xi = \beta \xi$ for any distribution ξ on G . We consider $d_\beta, r_{\alpha_1}, l_{\alpha_2}, f_\beta, g_{\alpha_i}^i$ as elements of $D(G, Y)$. Let $I \subset D(G, Y)$ be the ideal generated by $r_{\alpha_1} - g_{\alpha_1}^1, l_{\alpha_2} - g_{\alpha_2}^2$ and $f_\beta - d_\beta$ where $\alpha_i \in \mathfrak{h}_i$ and $\beta \in \mathfrak{z}(\mathcal{U}(\mathfrak{g}))$. Define $\mathcal{M} := D(G, Y)/I$.

It is easy to see that $U_{\lambda, \chi_1, \chi_2} \cong \text{Hom}(\mathcal{M}_{(\lambda, \chi_1, \chi_2)}, \mathcal{S}^*(G))$. As in the proof of Proposition 1.1.4, the singular support of $\mathcal{M}_{(\lambda, \chi_1, \chi_2)}$ lies in S , for any λ, χ_1, χ_2 . By Theorem B, $\mathcal{M}_{(\lambda, \chi_1, \chi_2)}$ is holonomic and, therefore, \mathcal{M} is holonomic. By Theorem 3.2.2, we have $\dim U_{\lambda, \chi_1, \chi_2} \leq \deg \mathcal{M}_{(\lambda, \chi_1, \chi_2)}$. By Theorem 3.3.3, $\deg \mathcal{M}_{(\lambda, \chi_1, \chi_2)}$ are bounded. \square

Proof of Theorem E(ii). We choose an involution θ as in Lemma 4.1.2, let $H_1 := H, H_2 := \theta(H)$, and define the spaces U_λ as in Lemma 4.3.1.

Now let $\pi \in \text{Irr}(G)$ and let χ be a character of \mathfrak{h} such that $(\pi^*)^{\mathfrak{h}, \chi} \neq 0$. Let λ stand for the infinitesimal character of π . By Lemma 4.1.2, $(\hat{\pi}^*)^{d\theta(\mathfrak{h}), d\theta(\chi)} \neq 0$. Fix a non-zero $\phi \in (\hat{\pi}^*)^{d\theta(\mathfrak{h}), d\theta(\chi)}$. Then ϕ defines an embedding $(\pi^*)^{\mathfrak{h}, \chi} \hookrightarrow U_{\lambda, \chi, d\theta(\chi)}$ by $\psi \mapsto \xi_{\psi, \phi}$, where $\xi_{\psi, \phi}$ is the relative character, which is defined by $\xi_{\psi, \phi}(f) := \langle \psi, \pi(f)\phi \rangle$. Thus, $\dim(\pi^*)^{\mathfrak{h}, \chi} \leq \dim U_{\lambda, \chi, d\theta(\chi)}$, which is bounded by Lemma 4.3.1. \square

APPENDIX A. PROOF OF LEMMA 3.1.1

For the proof, we will need the following standard lemmas. Let M be a smooth manifold and $N \subset M$ be a closed smooth submanifold.

Lemma A.0.1. *Denote $I_N := \{f \in C_c^\infty(M) \mid f|_N = 0\}$. Let $J \subset I_N$ be an ideal in $C_c^\infty(M)$ such that*

- (1) *For any $x \in N$, the space $\{d_x f \mid f \in J\}$ is the conormal space to N in M at the point x .*
- (2) *For any $x \in M \setminus N$, there exists $f \in J$ such that $f(x) \neq 0$.*

Then $J = I_N$.

Proof. Using partition of unity, it is enough to show that, for any $f \in I_N$ and $x \in M$, there exists $f' \in J$ such that f coincides with f' in a neighborhood of x . For $x \notin N$ this is obvious, so we assume that $x \in N$. We prove the statement by induction on the codimension d of N in M . The base case $d = 1$ follows, using the implicit function theorem, from the case $N = \mathbb{R}^{n-1} \subset \mathbb{R}^n = M$, which is obvious.

For the induction step, take an element $g \in J$ such that $d_x g \neq 0$. Let

$$Z := \{y \in M \mid g(y) = 0\} \text{ and } U := \{y \in M \mid d_y g \neq 0\}.$$

By the implicit function theorem, $U \cap Z$ is a closed submanifold of U . Choose $\rho \in C_c^\infty(M)$ such that $\rho = 1$ in a neighborhood of x and $\text{Supp}(\rho) \subset U$. Let $\bar{f} := (\rho f)|_{U \cap Z}$. Let

$$\bar{J} := \{\alpha|_{U \cap Z} \mid \alpha \in J \text{ and } \text{Supp } \alpha \subset U\}.$$

By the induction hypothesis, $\bar{f} \in \bar{J}$. Thus, there exists $f'' \in J$ such that $f - f''$ vanishes in a neighborhood of x in Z . Now, the case $d = 1$ implies that there exists $\alpha \in C_c^\infty(M)$ such that $f - f''$ coincides with αg in a neighborhood of x . \square

Lemma A.0.2. *The restriction $C_c^\infty(M) \rightarrow C_c^\infty(N)$ is an open map.*

Proof. Let $K \subset M$ be a compact subset. It is easy to see that there exists a compact $K' \supset K$ such that the restriction map $C_{K'}^\infty(M) \rightarrow C_{K' \cap N}^\infty(N)$ is onto, using the partition of unity. By the Banach open map theorem this map is open. Thus, the restriction $C_c^\infty(M) \rightarrow C_c^\infty(N)$ is an open map. \square

Let Y be a real algebraic manifold and X be a closed algebraic submanifold. Let $i : X \rightarrow Y$ denote the embedding.

Lemma A.0.3. *Let ξ be a distribution on X such that $i_*\xi$ is a tempered distribution. Then ξ is a tempered distribution.*

Proof. The map i_* is dual to the pullback map $C_c^\infty(Y) \rightarrow C_c^\infty(X)$. This can be extended to a continuous map $i^* : \mathcal{S}(Y) \rightarrow \mathcal{S}(X)$ which is onto by [AG08, Theorem 4.6.1]. The Banach open map theorem implies that i^* is an open map. It is easy to see that $i_*\xi : \mathcal{S}(Y) \rightarrow \mathbb{C}$ vanishes on $\text{Ker}(i^*)$, and thus it gives rise to a continuous map $\mathcal{S}(X) \rightarrow \mathbb{C}$, which extends ξ . \square

Lemma A.0.4. *Let ξ be a distribution on Y such that $p\xi = 0$ for any polynomial p on Y that vanishes on X . Then ξ is a pushforward of a distribution on X .*

Proof. Let $J(X)$ be the ideal of all polynomials on Y that vanish on X . Let $J := J(X)C_c^\infty(Y)$. By Lemma A.0.1 we have $J = I_X$. Thus, ξ vanishes on I_X and thus, by Lemma A.0.2, ξ is a pushforward of a distribution on X . \square

Lemma 3.1.1 follows from Lemmas A.0.3 and A.0.4 and the definition of $i^!$ for closed embedding of smooth affine varieties.

REFERENCES

- [AD] Avraham Aizenbud and Vladimir Drinfeld. “The wave front set of the Fourier transform of algebraic measures”. In: *Israel Journal of Mathematics* (to appear). arXiv: 1212.3630 [math.AG].
- [Ada] Jeffrey Adams. “The real Chevalley involution”. In: *Compositio Mathematica* (to appear). DOI: 10.1112/S0010437X14007374. arXiv: 1203.1901 [math.RT].
- [AG] Avraham Aizenbud and Dmitry Gourevitch. “Vanishing of certain equivariant distributions on spherical spaces”. In: *Mathematische Zeitschrift* (online). DOI: 10.1007/s00209-014-1391-6. arXiv: 1311.6111 [math.RT].
- [AG08] Avraham Aizenbud and Dmitry Gourevitch. “Schwartz functions on Nash manifolds”. In: *International Mathematics Research Notices* (2008), 37 pages. DOI: 10.1093/imrn/rnm155. arXiv: 0704.2891 [math.RT].
- [AG09a] Avraham Aizenbud and Dmitry Gourevitch. “Multiplicity one theorem for $(GL_{n+1}(\mathbb{R}), GL_n(\mathbb{R}))$ ”. In: *Selecta Mathematica* 15.2 (2009), pp. 271–294. DOI: 10.1007/s00029-009-0544-7. arXiv: 0808.2729 [math.RT].
- [AGK] Avraham Aizenbud, Dmitry Gourevitch, and Alexander Kemarsky. “Vanishing of certain equivariant distributions on p -adic spherical spaces, and non-vanishing of spherical Bessel functions”. In: *International Mathematics Research Notices* (2014). DOI: 10.1093/imrn/RNU198. arXiv: 1404.3383 [math.RT].

- [AGS] Avraham Aizenbud, Dmitry Gourevitch, and Eitan Sayag. *3-Finite distributions on p -adic groups*. arXiv: [1405.2540](https://arxiv.org/abs/1405.2540) [math.RT].
- [Aiz13] Avraham Aizenbud. “A partial analog of the integrability theorem for distributions on p -adic spaces and applications”. In: *Israel Journal of Mathematics* 193.1 (2013), pp. 233–262. DOI: [10.1007/s11856-012-0088-y](https://doi.org/10.1007/s11856-012-0088-y).
- [Ber] Joseph Bernstein. *Algebraic theory of D -modules*. URL: <http://ncatlab.org/nlab/files/BernsteinDModule.pdf>.
- [Bor87] Armand Borel. *Algebraic D -Modules*. Vol. 2. Perspectives in Mathematics. Academic Press, 1987. ISBN: 0121177408.
- [Cas89] William Casselman. “Canonical extensions of Harish-Chandra modules to representations of G ”. In: *Canadian Journal of Mathematics* 41.3 (1989), pp. 385–438. DOI: [10.4153/CJM-1989-019-5](https://doi.org/10.4153/CJM-1989-019-5).
- [CM82] William Casselman and Dragan Miličić. “Asymptotic behavior of matrix coefficients of admissible representations”. In: *Duke Mathematical Journal* 49.4 (1982), pp. 869–930. DOI: [10.1215/S0012-7094-82-04943-2](https://doi.org/10.1215/S0012-7094-82-04943-2).
- [Del10] Patrick Delorme. “Constant term of smooth H_ψ -spherical functions on a reductive p -adic group”. In: *Transactions of the American Mathematical Society* 362 (2010), pp. 933–955. DOI: [10.1090/S0002-9947-09-04925-3](https://doi.org/10.1090/S0002-9947-09-04925-3).
- [EGA IV] Alexander Grothendieck. “Éléments de géométrie algébrique : IV. Étude Locale des schémas et des morphismes de schémas, Troisième partie”. In: *Publications Mathématique de l’Institut des Hautes Études Scientifiques* 28 (1966), pp. 5–255. URL: http://www.numdam.org/numdam-bin/feuilleter?id=PMIHES_1966__28_.
- [Gab81] Ofer Gabber. “The Integrability of the Characteristic Variety”. In: *American Journal of Mathematics* 103.3 (1981), pp. 445–468. DOI: [10.2307/2374101](https://doi.org/10.2307/2374101).
- [Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics. Springer, 1977. ISBN: 0-387-90244-9.
- [HC63] Harish-Chandra. “Invariant eigendistributions on semisimple Lie groups”. In: *Bulletin of the American Mathematical Society* 69.1 (1963), pp. 117–123. URL: <http://projecteuclid.org/euclid.bams/1183525024>.
- [HC65] Harish-Chandra. “Invariant eigendistributions on a semisimple Lie group”. In: *Transactions of the American Mathematical Society* 119 (1965), pp. 457–508. DOI: [10.1090/S0002-9947-1965-0180631-0](https://doi.org/10.1090/S0002-9947-1965-0180631-0).
- [Hör90] Lars Hörmander. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*. Classics in Mathematics. Springer, 1990. ISBN: 3540006621.
- [Kas74] Masaki Kashiwara. “On the maximally overdetermined system of linear differential equations, I”. In: *Publications of the Research Institute for Mathematical Sciences* 10.2 (1974), pp. 563–579. DOI: [10.2977/prims/1195192011](https://doi.org/10.2977/prims/1195192011).
- [KK76] Masaki Kashiwara and Takahiro Kawai. “Finiteness theorem for holonomic systems of micro-differential equations”. In: *Proceedings of the Japan Academy* 52.7 (1976), pp. 341–343. DOI: [10.3792/pja/1195518268](https://doi.org/10.3792/pja/1195518268).
- [KKS73] Masaki Kashiwara, Takahiro Kawai, and Mikio Sato. “Microfunctions and Pseudo-Differential Equations”. In: *Hyperfunctions and Pseudo-Differential Equations*. Vol. 287. Lecture Notes in Mathematics. Proceedings of a Conference at Katata, 1971. Springer Berlin Heidelberg, 1973, pp. 265–529. ISBN: 978-3-540-06218-9. DOI: [10.1007/BFb0068143](https://doi.org/10.1007/BFb0068143).
- [KM14] Toshiyuki Kobayashi and Toshihiko Matsuki. “Classification of finite-multiplicity symmetric pairs”. In: *Transformation Groups* 19 (2014), pp. 457–493. DOI: [10.1007/s00031-014-9265-x](https://doi.org/10.1007/s00031-014-9265-x).
- [KO13] Toshiyuki Kobayashi and Toshio Oshima. “Finite multiplicity theorems for induction and restriction”. In: *Advances in Mathematics* 248 (2013), pp. 921–944. DOI: [10.1016/j.aim.2013.07.015](https://doi.org/10.1016/j.aim.2013.07.015).
- [KS] Bernhard Krötz and Henrik Schlichtkrull. “Multiplicity bounds and the subrepresentation theorem for real spherical spaces”. In: *Transactions of the American Mathematical Society* (to appear). arXiv: [1309.0930](https://arxiv.org/abs/1309.0930) [math.RT].

- [KS2] Bernhard Krötz and Henrik Schlichtkrull. *Finite orbit decomposition of real flag manifolds*. arXiv: [arXiv:1307.2375](https://arxiv.org/abs/1307.2375) [math.RT].
- [LM] Erez Lapid and Zhengyu Mao. *On an analogue of the Ichino–Ikeda conjecture for Whittaker coefficients on the metaplectic group*. arXiv: [1404.2905](https://arxiv.org/abs/1404.2905) [math.NT].
- [Mal79] Bernard Malgrange. “L’Involativité des caractéristiques des systèmes différentiels et microdifférentiels”. In: *Séminaire Bourbaki vol. 1977/78 Exposés 507–524*. Vol. 710. Lecture Notes in Mathematics. Exp. No. 522. Springer Berlin Heidelberg, 1979, pp. 277–289. ISBN: 978-3-540-09243-8. DOI: [10.1007/BFb0069985](https://doi.org/10.1007/BFb0069985).
- [Mum74] David Mumford. *Abelian Varieties*. 2nd. Vol. 5. Tata Institute of Fundamental Research. Oxford University Press, 1974.
- [Mum99] David Mumford. *The Red Book of Varieties and Schemes: Includes the Michigan Lectures (1974) on Curves and their Jacobians*. 2nd. Lecture Notes in Mathematics. Springer, 1999. ISBN: 354063293X.
- [Ste76] Robert Steinberg. “On the desingularization of the unipotent variety”. In: *Inventiones Mathematicae* 36.1 (1976), pp. 209–224. DOI: [10.1007/BF01390010](https://doi.org/10.1007/BF01390010).
- [SV] Yiannis Sakellaridis and Akshay Venkatesh. *Periods and harmonic analysis on spherical varieties*. arXiv: [1203.0039](https://arxiv.org/abs/1203.0039) [math.RT].
- [Wal88] Nolan R. Wallach. *Real reductive groups I, II*. Pure and Applied Mathematics. Academic Press, 1988, 1992. ISBN: 0080874517, 0080874525.
- [Wei64] André Weil. “Sur certains groupes d’opérateurs unitaires”. In: *Acta Mathematica* 111.1 (1964), pp. 143–211. DOI: [10.1007/BF02391012](https://doi.org/10.1007/BF02391012).

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, WEIZMANN INSTITUTE OF SCIENCE, 234 HERZL STREET, REHOVOT 7610001, ISRAEL

E-mail address: aizenr@gmail.com

URL: <http://www.wisdom.weizmann.ac.il/~aizenr>

E-mail address: dmitry.gourevitch@weizmann.ac.il

URL: <http://www.wisdom.weizmann.ac.il/~dimagur>

E-mail address: an.minchenko@gmail.com

URL: <http://www.wisdom.weizmann.ac.il/~andrey/>