Holonomicity of spherical characters and applications to multiplicity bounds

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Definition

Let M be a D-module over X with generators $m_1 \dots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree i and $F_i(M) := F_i(D(X))(m_1 \dots m_k)$. Define

$$SS(M) := supp(gr_F(M)) \subset T^*X.$$

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A distribution (or a D-module) ξ is called holonomic if

$$\dim(SS(\xi)) = \dim X.$$

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Corollary

Let (π, V) be an admissible representation of $G(\mathbb{R})$ and $v_1 \in (V^*)^{H_1}$, $v_2 \in (\tilde{V}^*)^{H_2}$. Let ξ be the corresponding spherical character:

$$\langle \xi, f \rangle := \langle \pi^*(f) v_1, v_2 \rangle.$$

Then ξ is a holonomic distribution.

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Corollary (Aizenbud, G., Minchenko, Sayag)

Let F be a local field of characteristic zero. Then the wave front set of any spherical character of an admissible representation of $G(F)$ is included in a conic subvariety of T^*G of middle dimension. If $F = \mathbb{R}$ then the subvariety is Lagrangian.

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Corollary

Any spherical character of an admissible representation of $G(F)$ is smooth in a Zariski open dense set.

Bernstein-Kashiwara theorem

Theorem (Bernstein, Kashiwara ∼1974)

Let X be a real algebraic manifold. Let M be a holonomic right D_X-module. Then $\dim Hom(M, \mathcal{S}^*(X)) < \infty$.

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Theorem (Bernstein, Kashiwara, Aizenbud, G., Minchenko)

Let X, Y be smooth algebraic varieties and $\mathcal{M}$ be a family of D_X-modules parameterized by Y. Suppose that $\mathcal{M}_y$ is holonomic. Then $\dim Hom(\mathcal{M}_y, \mathcal{S}^*(X))$ *is bounded when y ranges over Y.*

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Corollary (Aizenbud, G., Minchenko)

Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let $\mathcal{E}$ be an algebraic G-equivariant bundle on X and χ be a character of $\mathfrak{g}$. Then,

$$\dim \mathcal{S}^*(X, \mathcal{E})^{\mathfrak{g}, \chi} < \infty.$$

Moreover, it remains bounded when we change χ or tensor $\mathcal{E}$ with a representation of $\mathfrak{g}$ of a fixed dimension.

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1. *If H is a spherical subgroup then there exists $C \in \mathbb{N}$ such that $\dim(\pi^*)^{\mathfrak{h},\chi} \leq C$ for any $\pi \in Irr(G)$ and any character χ of $\mathfrak{h}$.*
2. *If H is a real spherical subgroup then, for every irreducible admissible representation $\pi \in Irr(G)$, and natural number $n \in \mathbb{N}$ there exists $C_n \in \mathbb{N}$ such that for every n-dimensional representation τ of $\mathfrak{h}$ we have*

$$\dim Hom_{\mathfrak{h}}(\pi, \tau) \leq C_n.$$

Corollaries for homologies

Theorem (Aizenbud, G., Krötz, Liu)

Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let $\mathcal{E}$ be an algebraic G-equivariant bundle on X and χ be a tempered character of G. Then, the homology $\mathrm{H}_0(\mathfrak{g}, \mathcal{S}(X, \mathcal{E}) \otimes \chi))$ is separated and is non-degenerately paired with $\mathcal{S}^*(X, \mathcal{E})^{\mathfrak{g}, -\chi}$. I.e.

$$\mathfrak{g}\mathcal{S}(X, \mathcal{E}) \otimes \chi \subset \mathcal{S}(X, \mathcal{E}) \otimes \chi$$

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Corollary

Let G be a real reductive group, H be a real spherical subgroup, and $\mathfrak{h}$ be the Lie algebra of H. Let χ be a tempered character of H. Then for any admissible representation π of G, $\mathrm{H}_0(\mathfrak{h}, \pi \otimes \chi)$ is separated and is non-degenerately paired with $(\pi^*)^{\mathfrak{h}, -\chi}$.

Geometric formulation

Theorem (A., Gourevitch, Minchenko 2015)

Let

$$S = \{g \in G, x \in \mathfrak{g}^* | x \in \mathfrak{h}_1^\perp, ad(g)(x) \in \mathfrak{h}_2^\perp, x \text{ is nilpotent}\} =$$
$$= G \times \mathcal{N} \cap \bigcup_{g \in G} CN^G_{H_1 g H_2, g}$$

Then

$$\dim S = \dim G.$$

The group case

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So

$$S' \subset \bigcup_{x \in \mathcal{N}_H} CN^{\mathfrak{h}}_{ad(G)x,x}$$

Springer resolution and Steinberg theorem

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Theorem (Steinberg 1976)

$$\forall \eta \in \mathcal{N} \textit{ we have } \dim G_\eta - 2 \dim \mu^{-1}(\eta) = \mathrm{rk} G.$$

Idea of the proof

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$$\begin{array}{ccccc} T^*\mathcal{B} \times T^*\mathcal{B} & & & & G \times \mathcal{N} \\ & \searrow^{\mu\times\mu} & & {}^{(g,x)\mapsto(x,\mathrm{Ad}^*(g^{-1})x)}\swarrow & \\ & & \mathcal{N} \times \mathcal{N} & & \\ & & \downarrow \textit{res} & & \\ & & \mathfrak{h}_1^* \times \mathfrak{h}_2^* & & \end{array}$$

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Where
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$$T^*\mathcal{B}\times T^*\mathcal{B} \xrightarrow{\mu\times\mu} \mathcal{N}\times\mathcal{N} \xleftarrow{(g,x)\mapsto(x,\mathrm{Ad}^*(g^{-1})x)} G\times\mathcal{N}$$

$$\mathcal{N}\times\mathcal{N} \xrightarrow{\;res\;} \mathfrak{h}_1^*\times\mathfrak{h}_2^*$$

Passing to the fiber of $0\in\mathfrak{h}_1^*\times\mathfrak{h}_2^*$ we get:

$$L_1\times L_2 \longrightarrow \mathcal{N}_{\mathfrak{h}_1}\times\mathcal{N}_{\mathfrak{h}_2} \longleftarrow S$$

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The estimate on dim S follows from the Steinberg theorem and:

$$\dim L_i = \dim\mathcal{B}.$$