

Holonomicity of spherical characters and applications to multiplicity bounds

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Let M be a D -module over X with generators $m_1 \dots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree i and $F_i(M) := F_i(D(X))(m_1 \dots m_k)$. Define

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A distribution (or a D -module) ξ is called holonomic if

$$\dim(SS(\xi)) = \dim X.$$

Main results

Theorem (Aizenbud, G., Minchenko 2015)

Let G be an algebraic reductive group, $H_1, H_2 \subset G$ be spherical subgroups (i.e. $H_i \backslash G/B$ is finite). The following system of equations on a distribution ξ on G is holonomic:

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Corollary

Let (π, V) be an admissible representation of $G(\mathbb{R})$ and $v_1 \in (V^*)^{H_1}$, $v_2 \in (\tilde{V}^*)^{H_2}$. Let ξ be the corresponding spherical character:

$$\langle \xi, f \rangle := \langle \pi^*(f)v_1, v_2 \rangle.$$

Then ξ is a holonomic distribution.

Applications to the spherical character

Corollary (Aizenbud, G., Minchenko, Sayag)

*Let F be a local field of characteristic zero. Then the wave front set of any spherical character of an admissible representation of $G(F)$ is included in a conic subvariety of T^*G of middle dimension. If $F = \mathbb{R}$ then the subvariety is Lagrangian.*

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Corollary

Any spherical character of an admissible representation of $G(F)$ is smooth in a Zariski open dense set.

Bernstein-Kashiwara theorem

Theorem (Bernstein, Kashiwara ~1974)

Let X be a real algebraic manifold. Let M be a holonomic right D_X -module. Then $\dim \text{Hom}(M, \mathcal{S}^(X)) < \infty$.*

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Theorem (Bernstein, Kashiwara, Aizenbud, G., Minchenko)

Let X, Y be smooth algebraic varieties and \mathcal{M} be a family of D_X -modules parameterized by Y . Suppose that \mathcal{M}_y is holonomic. Then $\dim \operatorname{Hom}(\mathcal{M}_y, \mathcal{S}^(X))$ is bounded when y ranges over Y .*

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Corollary (Aizenbud, G., Minchenko)

Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let \mathcal{E} be an algebraic G -equivariant bundle on X and χ be a character of \mathfrak{g} . Then,

$$\dim \mathcal{S}^*(X, \mathcal{E})^{\mathfrak{g}, \chi} < \infty.$$

Moreover, it remains bounded when we change χ or tensor \mathcal{E} with a representation of \mathfrak{g} of a fixed dimension.

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- 1 *If H is a spherical subgroup then there exists $C \in \mathbb{N}$ such that $\dim(\pi^*)^{\mathfrak{h}, \chi} \leq C$ for any $\pi \in \text{Irr}(G)$ and any character χ of \mathfrak{h} .*

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- 2 If H is a real spherical subgroup then, for every irreducible admissible representation $\pi \in \text{Irr}(G)$, and natural number $n \in \mathbb{N}$ there exists $C_n \in \mathbb{N}$ such that for every n -dimensional representation τ of \mathfrak{h} we have*

$$\dim \text{Hom}_{\mathfrak{h}}(\pi, \tau) \leq C_n.$$

Theorem (Aizenbud, G., Krötz, Liu)

Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let \mathcal{E} be an algebraic G -equivariant bundle on X and χ be a tempered character of G . Then, the homology $H_0(\mathfrak{g}, \mathcal{S}(X, \mathcal{E}) \otimes \chi)$ is separated and is non-degenerately paired with $\mathcal{S}^(X, \mathcal{E})^{\mathfrak{g}, -\chi}$. I.e.*

$$\mathfrak{g}\mathcal{S}(X, \mathcal{E}) \otimes \chi \subset \mathcal{S}(X, \mathcal{E}) \otimes \chi$$

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Corollaries for homologies

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Corollary

Let G be a real reductive group, H be a real spherical subgroup, and \mathfrak{h} be the Lie algebra of H . Let χ be a tempered character of H . Then for any admissible representation π of G , $H_0(\mathfrak{h}, \pi \otimes \chi)$ is separated and is non-degenerately paired with $(\pi^)^{\mathfrak{h}, -\chi}$.*

Theorem (A., Gourevitch, Minchenko 2015)

Let

$$\begin{aligned} S &= \{g \in G, x \in \mathfrak{g}^* \mid x \in \mathfrak{h}_1^\perp, ad(g)(x) \in \mathfrak{h}_2^\perp, x \text{ is nilpotent}\} = \\ &= G \times \mathcal{N} \cap \bigcup_{g \in G} CN_{H_1 g H_2, g}^G \end{aligned}$$

Then

$$\dim S = \dim G.$$

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So

$$S' \subset \bigcup_{x \in \mathcal{N}_H} CN_{ad(G)x,x}^{\mathfrak{h}}$$

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Springer resolution and Steinberg theorem

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Springer resolution and Steinberg theorem

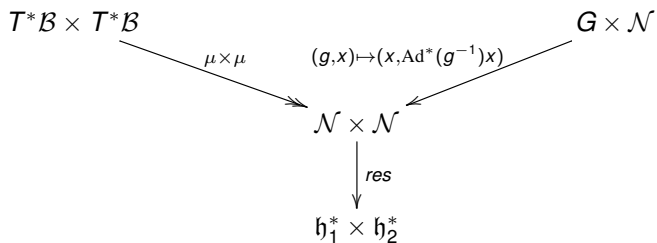
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Theorem (Steinberg 1976)

$\forall \eta \in \mathcal{N}$ we have $\dim G_\eta - 2 \dim \mu^{-1}(\eta) = \text{rk} G$.

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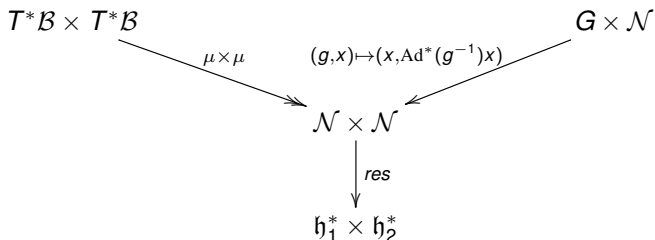


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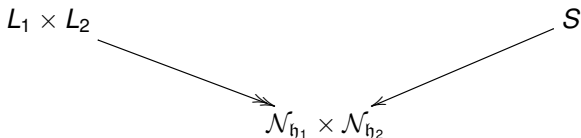
$$\begin{array}{ccc} T^*\mathcal{B} \times T^*\mathcal{B} & & G \times \mathcal{N} \\ & \begin{array}{c} \xrightarrow{\mu \times \mu} \\ \xrightarrow{(g,x) \mapsto (x, \text{Ad}^*(g^{-1})x)} \end{array} & \\ & \mathcal{N} \times \mathcal{N} & \\ & \downarrow \text{res} & \\ & \mathfrak{h}_1^* \times \mathfrak{h}_2^* & \end{array}$$

Passing to the fiber of $0 \in \mathfrak{h}_1^* \times \mathfrak{h}_2^*$ we get:

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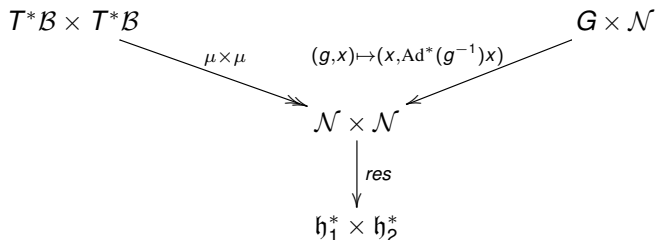
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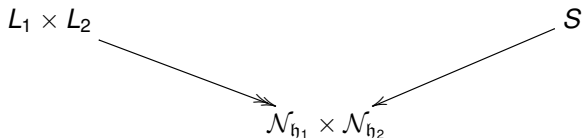
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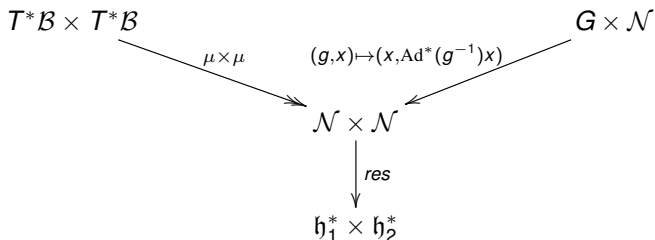
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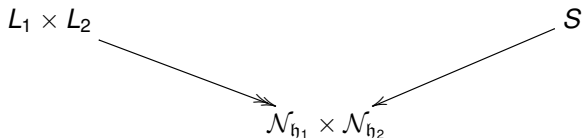
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The estimate on $\dim S$ follows from the Steinberg theorem and:

$$\dim L_i = \dim \mathcal{B}.$$