Holonomicity of spherical characters and applications to multiplicity bounds

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Holonomic D-modules and distributions

A D-module over a smooth affine algebraic variety X is a module over the ring $D(X)$ of differential operators on X.
A D-module M given by generators and relations can be thought of as a system of PDE. A solution of M is a D-module homomorphism of M to an appropriate space of functions.

Definition

Let M be a D-module over X with generators $m_1 \ldots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree i and $F_i(M) := F_i(D(X))(m_1 \ldots m_k)$. Define

$$SS(M) := supp(gr_F(M)) \subset T^*X.$$

For a distribution ξ on $X(\mathbb{R})$ define

$$SS(\xi) := SS(D(X)\xi) = \bigcap_{d\xi=0} Zeros(symbol(d)).$$

A distribution (or a D-module) ξ is called holonomic if

$$\dim(SS(\xi)) = \dim X.$$

Main results

Theorem (Aizenbud, G., Minchenko 2015)

Let G be an algebraic reductive group, $H_1, H_2 \subset G$ be spherical subgroups (i.e. $H_i \setminus G/B$ is finite). The following system of equations on a distribution ξ on G is holonomic:

- *ξ is left H_1 invariant*
- *ξ is right H_2 invariant*
- *ξ is eigen w.r.t. the center $\mathfrak{z}(u(\mathfrak{g}))$ of the universal enveloping algebra of the Lie algebra of G.*

Corollary

Let (π, V) be an admissible representation of $G(\mathbb{R})$ and $v_1 \in (V^*)^{H_1}$, $v_2 \in (\tilde{V}^*)^{H_2}$. Let ξ be the corresponding spherical character:

$$\langle \xi, f \rangle := \langle \pi^*(f) v_1, v_2 \rangle.$$

Then ξ is a holonomic distribution.

Applications to the spherical character

Corollary (Aizenbud, G., Minchenko, Sayag)

Let F be a local field of characteristic zero. Then the wave front set of any spherical character of an admissible representation of $G(F)$ is included in a conic subvariety of T^*G of middle dimension. If $F = \mathbb{R}$ then the subvariety is Lagrangian.

Corollary

Any spherical character of an admissible representation of $G(F)$ is smooth in a Zariski open dense set.

Bernstein-Kashiwara theorem

Theorem (Bernstein, Kashiwara ∼1974)

Let X be a real algebraic manifold. Let M be a holonomic right D_X-module. Then $\dim Hom(M, \mathcal{S}^*(X)) < \infty$.

Theorem (Bernstein, Kashiwara, Aizenbud, G., Minchenko)

Let X, Y be smooth algebraic varieties and $\mathcal{M}$ be a family of D_X-modules parameterized by Y. Suppose that $\mathcal{M}_y$ is holonomic. Then $\dim Hom(\mathcal{M}_y, \mathcal{S}^*(X))$ *is bounded when y ranges over Y.*

Corollary (Aizenbud, G., Minchenko)

Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let $\mathcal{E}$ be an algebraic G-equivariant bundle on X and χ be a character of $\mathfrak{g}$. Then,

$$\dim \mathcal{S}^*(X, \mathcal{E})^{\mathfrak{g},\chi} < \infty.$$

Moreover, it remains bounded when we change χ or tensor $\mathcal{E}$ with a representation of $\mathfrak{g}$ of a fixed dimension.

Applications to multiplicities

We reprove the following theorem

Theorem (Kobayashi, Krötz, Oshima, Schlichtkrull 2013)

Let G be a real reductive group, H be a Zariski closed subgroup, and $\mathfrak{h}$ be the Lie algebra of H.

1. *If H is a spherical subgroup then there exists $C \in \mathbb{N}$ such that $\dim(\pi^*)^{\mathfrak{h},\chi} \leq C$ for any $\pi \in Irr(G)$ and any character χ of $\mathfrak{h}$.*
2. *If H is a real spherical subgroup then, for every irreducible admissible representation $\pi \in Irr(G)$, and natural number $n \in \mathbb{N}$ there exists $C_n \in \mathbb{N}$ such that for every n-dimensional representation τ of $\mathfrak{h}$ we have*

$$\dim Hom_{\mathfrak{h}}(\pi, \tau) \leq C_n.$$

Corollaries for homologies

Theorem (Aizenbud, G., Krötz, Liu)

Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let $\mathcal{E}$ be an algebraic G-equivariant bundle on X and χ be a tempered character of G. Then, the homology $\mathrm{H}_0(\mathfrak{g}, \mathcal{S}(X, \mathcal{E}) \otimes \chi))$ is separated and is non-degenerately paired with $\mathcal{S}^*(X, \mathcal{E})^{\mathfrak{g}, -\chi}$. I.e.

$$\mathfrak{g}\mathcal{S}(X, \mathcal{E}) \otimes \chi \subset \mathcal{S}(X, \mathcal{E}) \otimes \chi$$

is closed and has finite codimension.

Corollary

Let G be a real reductive group, H be a real spherical subgroup, and $\mathfrak{h}$ be the Lie algebra of H. Let χ be a tempered character of H. Then for any admissible representation π of G, $\mathrm{H}_0(\mathfrak{h}, \pi \otimes \chi)$ is separated and is non-degenerately paired with $(\pi^*)^{\mathfrak{h}, -\chi}$.

Geometric formulation

Theorem (A., Gourevitch, Minchenko 2015)

Let

$$S = \{g \in G, x \in \mathfrak{g}^* | x \in \mathfrak{h}_1^\perp, ad(g)(x) \in \mathfrak{h}_2^\perp, x \text{ is nilpotent}\} = \\ = G \times \mathcal{N} \cap \bigcup_{g \in G} CN^G_{H_1 g H_2, g}$$

Then

$$\dim S = \dim G.$$

The group case

Assume $H_1 = H_2 = H$, diagonally embedded in $G = H \times H$.
Translating the problem to $H = G/H$ we obtain:

$$S' = \{g \in H, x \in \mathfrak{h}^* | Ad(g)(x) = x, x \in \mathcal{N}_H\} = H \times \mathcal{N}_H \cap \bigcup_{g \in H} CN^H_{ad(G)g,g}$$

passing to the Lie algebra

$$S' = \{g \in \mathfrak{h}, x \in \mathfrak{h} | [x, g] = 0, x \text{ is nilpotent}\}$$

So

$$S' \subset \bigcup_{x \in \mathcal{N}_H} CN^{\mathfrak{h}}_{ad(G)x,x}$$

Springer resolution and Steinberg theorem

Let $\mathcal{B}$ be the flag variety. $T^*\mathcal{B} \cong \{B \in \mathcal{B}, x \in \mathfrak{b}^\perp\}$. We have a natural map $\mu : T^*\mathcal{B} \to \mathcal{N}$. It is called the Springer resolution.

Theorem (Steinberg 1976)

$$\forall \eta \in \mathcal{N} \text{ we have } \dim G_\eta - 2 \dim \mu^{-1}(\eta) = \mathrm{rk} G.$$

Idea of the proof

$$
\begin{array}{ccccc}
T^*\mathcal{B} \times T^*\mathcal{B} & & & & G \times \mathcal{N} \\
& \searrow^{\mu\times\mu} & & \swarrow^{(g,x)\mapsto(x,\mathrm{Ad}^*(g^{-1})x)} & \\
& & \mathcal{N} \times \mathcal{N} & & \\
& & \downarrow{\scriptstyle res} & & \\
& & \mathfrak{h}_1^* \times \mathfrak{h}_2^* & &
\end{array}
$$

Passing to the fiber of $0 \in \mathfrak{h}_1^* \times \mathfrak{h}_2^*$ we get:

$$
\begin{array}{ccccc}
L_1 \times L_2 & & & & S \\
& \searrow & & \swarrow & \\
& & \mathcal{N}_{\mathfrak{h}_1} \times \mathcal{N}_{\mathfrak{h}_2} & &
\end{array}
$$

Where

$\mathcal{N}_{\mathfrak{h}_i} := \mathcal{N} \cap \mathfrak{h}_i^{\perp}$ and $L_i := \{(B, X) \in T^*\mathcal{B} \,|\, X \in \mathfrak{h}_i^{\perp}\} = \bigcup_{x\in\mathcal{B}} CN^{\mathcal{B}}_{H_i x, x}$.

The estimate on dim S follows from the Steinberg theorem and:

$$\dim L_i = \dim \mathcal{B}.$$