

Holonomicity of spherical characters and applications to multiplicity bounds

Dmitry Gourevitch

Weizmann Institute of Science

J. w. Avraham Aizenbud and Andrey Minchenko

<http://www.wisdom.weizmann.ac.il/~dimagur/>

Holonomic D-modules and distributions

A D -module over a smooth affine algebraic variety X is a module over the ring $D(X)$ of differential operators on X . A D -module M given by generators and relations can be thought of as a system of PDE. A solution of M is a D -module homomorphism of M to an appropriate space of functions.

Definition

Let M be a D -module over X with generators $m_1 \dots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree i and $F_i(M) := F_i(D(X))(m_1 \dots m_k)$. Define

$$SS(M) := \text{supp}(gr_F(M)) \subset T^*X.$$

For a distribution ξ on $X(\mathbb{R})$ define

$$SS(\xi) := SS(D(X)\xi) = \bigcap_{d\xi=0} \text{Zeros}(\text{symbol}(d)).$$

A distribution (or a D -module) ξ is called holonomic if

$$\dim(SS(\xi)) = \dim X.$$

Theorem (Aizenbud, G., Minchenko 2015)

Let G be an algebraic reductive group, $H_1, H_2 \subset G$ be spherical subgroups (i.e. $H_i \backslash G/B$ is finite). The following system of equations on a distribution ξ on G is holonomic:

- ξ is left H_1 invariant
- ξ is right H_2 invariant
- ξ is eigen w.r.t. the center $\mathfrak{z}(u(\mathfrak{g}))$ of the universal enveloping algebra of the Lie algebra of G .

Corollary

Let (π, V) be an admissible representation of $G(\mathbb{R})$ and $v_1 \in (V^*)^{H_1}$, $v_2 \in (\tilde{V}^*)^{H_2}$. Let ξ be the corresponding spherical character:

$$\langle \xi, f \rangle := \langle \pi^*(f)v_1, v_2 \rangle.$$

Then ξ is a holonomic distribution.

Corollary (Aizenbud, G., Minchenko, Sayag)

*Let F be a local field of characteristic zero. Then the wave front set of any spherical character of an admissible representation of $G(F)$ is included in a conic subvariety of T^*G of middle dimension. If $F = \mathbb{R}$ then the subvariety is Lagrangian.*

Corollary

Any spherical character of an admissible representation of $G(F)$ is smooth in a Zariski open dense set.

Bernstein-Kashiwara theorem

Theorem (Bernstein, Kashiwara ~1974)

Let X be a real algebraic manifold. Let M be a holonomic right D_X -module. Then $\dim \text{Hom}(M, \mathcal{S}^(X)) < \infty$.*

Theorem (Bernstein, Kashiwara, Aizenbud, G., Minchenko)

Let X, Y be smooth algebraic varieties and \mathcal{M} be a family of D_X -modules parameterized by Y . Suppose that \mathcal{M}_y is holonomic. Then $\dim \text{Hom}(\mathcal{M}_y, \mathcal{S}^(X))$ is bounded when y ranges over Y .*

Corollary (Aizenbud, G., Minchenko)

Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let \mathcal{E} be an algebraic G -equivariant bundle on X and χ be a character of \mathfrak{g} . Then,

$$\dim \mathcal{S}^*(X, \mathcal{E})^{\mathfrak{g}, \chi} < \infty.$$

Moreover, it remains bounded when we change χ or tensor \mathcal{E} with a representation of \mathfrak{g} of a fixed dimension.

We reprove the following theorem

Theorem (Kobayashi, Krötz, Oshima, Schlichtkrull 2013)

Let G be a real reductive group, H be a Zariski closed subgroup, and \mathfrak{h} be the Lie algebra of H .

- 1 If H is a spherical subgroup then there exists $C \in \mathbb{N}$ such that $\dim(\pi^*)^{\mathfrak{h}, \chi} \leq C$ for any $\pi \in \text{Irr}(G)$ and any character χ of \mathfrak{h} .*
- 2 If H is a real spherical subgroup then, for every irreducible admissible representation $\pi \in \text{Irr}(G)$, and natural number $n \in \mathbb{N}$ there exists $C_n \in \mathbb{N}$ such that for every n -dimensional representation τ of \mathfrak{h} we have*

$$\dim \text{Hom}_{\mathfrak{h}}(\pi, \tau) \leq C_n.$$

Theorem (Aizenbud, G., Krötz, Liu)

Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let \mathcal{E} be an algebraic G -equivariant bundle on X and χ be a tempered character of G . Then, the homology $H_0(\mathfrak{g}, \mathcal{S}(X, \mathcal{E}) \otimes \chi)$ is separated and is non-degenerately paired with $\mathcal{S}^(X, \mathcal{E})^{\mathfrak{g}, -\chi}$. I.e.*

$$\mathfrak{g}\mathcal{S}(X, \mathcal{E}) \otimes \chi \subset \mathcal{S}(X, \mathcal{E}) \otimes \chi$$

is closed and has finite codimension.

Corollary

Let G be a real reductive group, H be a real spherical subgroup, and \mathfrak{h} be the Lie algebra of H . Let χ be a tempered character of H . Then for any admissible representation π of G , $H_0(\mathfrak{h}, \pi \otimes \chi)$ is separated and is non-degenerately paired with $(\pi^)^{\mathfrak{h}, -\chi}$.*

Theorem (A., Gourevitch, Minchenko 2015)

Let

$$\begin{aligned} S &= \{g \in G, x \in \mathfrak{g}^* \mid x \in \mathfrak{h}_1^\perp, ad(g)(x) \in \mathfrak{h}_2^\perp, x \text{ is nilpotent}\} = \\ &= G \times \mathcal{N} \cap \bigcup_{g \in G} CN_{H_1 g H_2, g}^G \end{aligned}$$

Then

$$\dim S = \dim G.$$

The group case

Assume $H_1 = H_2 = H$, diagonally embedded in $G = H \times H$.
Translating the problem to $H = G/H$ we obtain:

$$S' = \{g \in H, x \in \mathfrak{h}^* | Ad(g)(x) = x, x \in \mathcal{N}_H\} = H \times \mathcal{N}_H \cap \bigcup_{g \in H} CN_{ad(G)g,g}^H$$

passing to the Lie algebra

$$S' = \{g \in \mathfrak{h}, x \in \mathfrak{h} | [x, g] = 0, x \text{ is nilpotent}\}$$

So

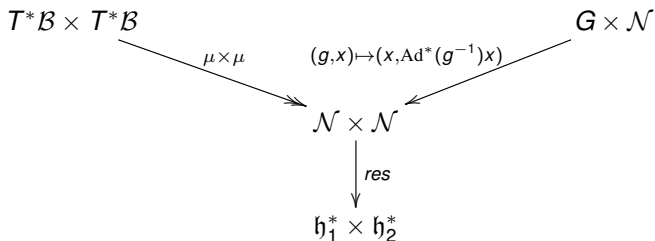
$$S' \subset \bigcup_{x \in \mathcal{N}_H} CN_{ad(G)x,x}^{\mathfrak{h}}$$

Let \mathcal{B} be the flag variety. $T^*\mathcal{B} \cong \{B \in \mathcal{B}, x \in \mathfrak{b}^\perp\}$. We have a natural map $\mu : T^*\mathcal{B} \rightarrow \mathcal{N}$. It is called the Springer resolution.

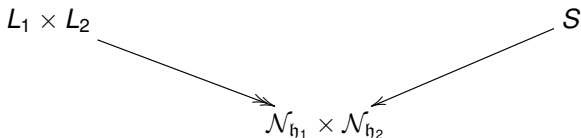
Theorem (Steinberg 1976)

$\forall \eta \in \mathcal{N}$ we have $\dim G_\eta - 2 \dim \mu^{-1}(\eta) = \text{rk} G$.

Idea of the proof



Passing to the fiber of $0 \in \mathfrak{h}_1^* \times \mathfrak{h}_2^*$ we get:



Where

$\mathcal{N}_{\mathfrak{h}_i} := \mathcal{N} \cap \mathfrak{h}_i^\perp$ and $L_i := \{(B, X) \in T^*\mathcal{B} \mid X \in \mathfrak{h}_i^\perp\} = \bigcup_{x \in \mathcal{B}} \text{CN}_{H_i x, x}^\mathcal{B}$.

The estimate on $\dim S$ follows from the Steinberg theorem and:

$$\dim L_i = \dim \mathcal{B}.$$