

Invariant Distributions and Gelfand Pairs

D. Gourevitch

<http://www.math.ias.edu/~dimagur>

Definition

A pair of compact topological groups $(G \supset H)$ is called a **Gelfand pair** if the following equivalent conditions hold:

- $L^2(G/H)$ decomposes to direct sum of **distinct** irreducible representations of G .
- for any irreducible representation ρ of G $\dim \rho^H \leq 1$.
- for any irreducible representation ρ of G $\dim \text{Hom}_H(\rho, \mathbb{C}) \leq 1$.
- the algebra of bi- H -invariant functions on G , $C(H \backslash G/H)$, is commutative w.r.t. convolution.

Definition

A pair of compact topological groups $(G \supset H)$ is called a **strong Gelfand pair** if one of the following equivalent conditions is satisfied:

- the pair $(G \times H \supset \Delta H)$ is a Gelfand pair
- for any irreducible representations ρ of G and τ of H

$$\dim \text{Hom}_H(\rho|_H, \tau) \leq 1.$$

- the algebra of $\text{Ad}(H)$ -invariant functions on G , $C(G//H)$, is commutative w.r.t. convolution.

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($GL(n, \mathbb{R}), O(n, \mathbb{R})$) is a Gelfand pair - the irreducible representations of $GL(n, \mathbb{R})$ which have an $O(n, \mathbb{R})$ -invariant vector are the same as characters of the algebra $C(O(n, \mathbb{R}) \backslash GL(n, \mathbb{R}) / O(n, \mathbb{R}))$.

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The same for the pair ($GL(n, \mathbb{C}), U(n)$).



Proposition (Gelfand)

Let σ be an involutive anti-automorphism of G (i.e. $\sigma(g_1 g_2) = \sigma(g_2) \sigma(g_1)$ and $\sigma^2 = \text{Id}$) and assume $\sigma(H) = H$. Suppose that $\sigma(f) = f$ for all bi H -invariant functions $f \in C(H \backslash G/H)$. Then (G, H) is a Gelfand pair.



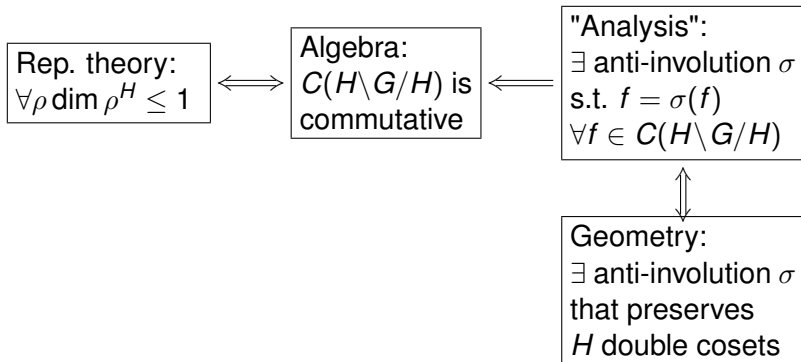
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Sum up



Classical examples

Pair	Anti-involution
$(G \times G, \Delta G)$	$(g, h) \mapsto (h^{-1}, g^{-1})$
$(O(n+k), O(n) \times O(k))$	$g \mapsto g^{-1}$
$(U(n+k), U(n) \times U(k))$	
$(GL(n, \mathbb{R}), O(n))$	$g \mapsto g^t$
(G, G^θ) , where G - Lie group, θ - involution, G^θ is compact	$g \mapsto \theta(g^{-1})$
(G, K) , where G - is a reductive group, K - maximal compact subgroup	Cartan anti-involution

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Setting

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Definition

A local field is a locally compact non-discrete topological field.
There are 3 types of local fields:

- Archimedean: \mathbb{R} and \mathbb{C}
- p-adic: \mathbb{Q}_p and their finite extensions
- positive characteristic: $\mathbb{F}_q((t))$

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Definition

A linear algebraic group is a subgroup of GL_n defined by polynomial equations.

Examples

GL_n , semisimple groups, O_n , U_n , Sp_{2n}, \dots

Reductive groups

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Reductive groups are unimodular.

Definition

Over Archimedean F , by smooth representation V we mean a complex Fréchet representation V such that for any $v \in V$ the map $G \rightarrow V$ defined by v is smooth.

Smooth representations

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Over non-Archimedean F , by smooth representation V we mean a complex linear representation V such that any $v \in V$ has open stabilizer.

Notation

Let M be a smooth manifold. We denote by $C_c^\infty(M)$ the space of smooth compactly supported functions on M . We will consider the space $(C_c^\infty(M))^$ of distributions on M . Sometimes we will also consider the space $\mathcal{S}^*(M)$ of Schwartz distributions on M .*

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Definition

An ℓ -space is a Hausdorff locally compact totally disconnected topological space. For an ℓ -space X we denote by $\mathcal{S}(X)$ the space of compactly supported locally constant functions on X . We let $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ be the space of distributions on X .

Definition

A pair of reductive groups $(G \supset H)$ is called a **Gelfand pair** if for any irreducible admissible representation ρ of G

$$\dim \text{Hom}_H(\rho, \mathbb{C}) \cdot \dim \text{Hom}_H(\tilde{\rho}, \mathbb{C}) \leq 1$$

usually, this implies that

$$\dim \text{Hom}_H(\rho, \mathbb{C}) \leq 1.$$

Gelfand-Kazhdan distributional criterion



Theorem (Gelfand-Kazhdan,...)

Let σ be an involutive anti-automorphism of G and assume $\sigma(H) = H$.

*Suppose that $\sigma(\xi) = \xi$ for all bi H -invariant distributions ξ on G .
Then (G, H) is a Gelfand pair.*

Strong Gelfand Pairs

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Proposition

The pair (G, H) is a strong Gelfand pair if and only if the pair $(G \times H, \Delta H)$ is a Gelfand pair.

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Proposition

The pair (G, H) is a strong Gelfand pair if and only if the pair $(G \times H, \Delta H)$ is a Gelfand pair.

Corollary

Let σ be an involutive anti-automorphism of G s.t. $\sigma(H) = H$. Suppose $\sigma(\xi) = \xi$ for all distributions ξ on G invariant with respect to conjugation by H . Then (G, H) is a strong Gelfand pair.

Rep. theory:
 $\forall \rho \dim \rho^H \leq 1$



Algebra:
 $C(H \backslash G/H)$
is commutative

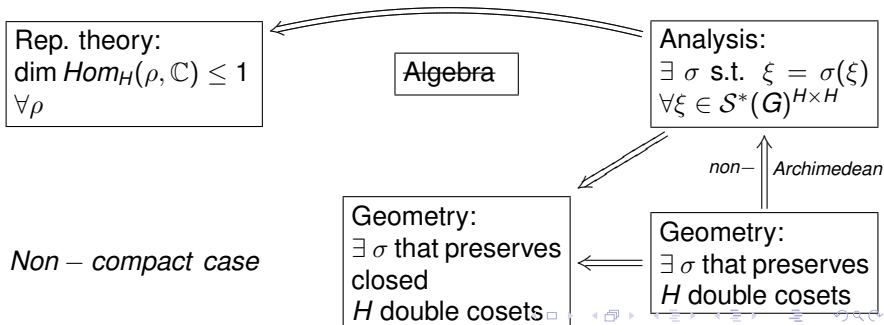
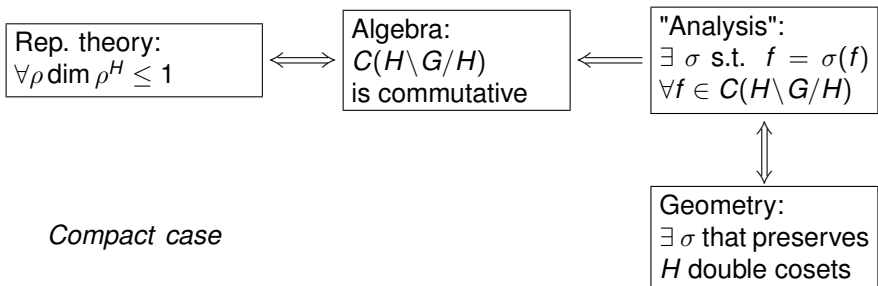


"Analysis":
 $\exists \sigma$ s.t. $f = \sigma(f)$
 $\forall f \in C(H \backslash G/H)$



Geometry:
 $\exists \sigma$ that preserves
 H double cosets

Compact case



Results on Gelfand pairs

Pair	p-adic	char $F > 0$	real
$(GL_n(E), GL_n(F))$	Flicker	Flicker	Aizenbud-G.
$(GL_{n+k}, GL_n \times GL_k)$	Jacquet-Rallis	Aizenbud-Avni-G.	
$(O_{n+k}, O_n \times O_k)$ over \mathbb{C}	_____	_____	
(GL_n, O_n) over \mathbb{C}			
(GL_{2n}, Sp_{2n})	Heumos-Rallis	Heumos-Rallis	Aizenbud-Sayag
$(GL_{2n}, \left\{ \begin{pmatrix} g & u \\ 0 & g \end{pmatrix} \right\}, \psi)$	Jacquet-Rallis		Aizenbud-G.-Jacquet

Results on strong Gelfand pairs

Pair	p-adic	char $F > 0$	real
(GL_{n+1}, GL_n)	Aizenbud-G.- Rallis-	Aizenbud- Avni-G.	Aizenbud-G. Sun-Zhu
$(O(V \oplus F), O(V))$	Schiffmann	?	
$(U(V \oplus F), U(V))$?	Sun-Zhu

- real: \mathbb{R} and \mathbb{C}
- p-adic: \mathbb{Q}_p and its finite extensions.
- char $F > 0$: $\mathbb{F}_q((t))$.

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- real: \mathbb{R} and \mathbb{C}
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Remark

The results from the last two slides are used to prove splitting of periods of automorphic forms.