Invariant Distributions and Gelfand Pairs

D. Gourevitch

http://www.math.ias.edu/~dimagur

Gelfand Pairs

Definition

A pair of compact topological groups ($G \supset H$) is called a **Gelfand pair** if the following equivalent conditions hold:

- $L^2(G/H)$ decomposes to direct sum of **distinct** irreducible representations of G.
- for any irreducible representation ρ of G $dim \rho^H \leq 1$.
- for any irreducible representation ρ of G $dim Hom_H(\rho, \mathbb{C}) \leq 1$.
- the algebra of bi-H-invariant functions on G, $C(H \backslash G/H)$, is commutative w.r.t. convolution.

Strong Gelfand Pairs

Definition

A pair of compact topological groups $(G \supset H)$ is called a **strong Gelfand pair** if one of the following equivalent conditions is satisfied:

- the pair $(G \times H \supset \Delta H)$ is a Gelfand pair
- for any irreducible representations ρ of G and τ of H

$$dimHom_H(\rho|_H, \tau) \leq 1.$$

- the algebra of Ad(H)-invariant functions on G, $C(G//H)$, is commutative w.r.t. convolution.

Some classical aplications

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  $(GL(n,\mathbb{R}), O(n,\mathbb{R}))$ is a Gelfand pair -
  the irreducible representations of $GL(n,\mathbb{R})$ which have an $O(n,\mathbb{R})$-invariant vector are the same as characters of the algebra $C(O(n,\mathbb{R})\backslash GL(n,\mathbb{R})/O(n,\mathbb{R}))$.

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  The same for the pair $(GL(n,\mathbb{C}), U(n))$.

Gelfand trick

Proposition (Gelfand)

Let σ be an involutive anti-automorphism of G (i.e. $\sigma(g_1 g_2) = \sigma(g_2)\sigma(g_1)$ and $\sigma^2 = Id$) and assume $\sigma(H) = H$. Suppose that $\sigma(f) = f$ for all bi H-invariant functions $f \in C(H\backslash G/H)$. Then (G, H) is a Gelfand pair.

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Sum up

Rep. theory:
$\forall \rho \dim \rho^H \leq 1$

$\Longleftrightarrow$

Algebra:
$C(H\backslash G/H)$ is commutative

$\Longleftarrow$

"Analysis":
$\exists$ anti-involution σ s.t. $f = \sigma(f)$ $\forall f \in C(H\backslash G/H)$

$\Updownarrow$

Geometry:
$\exists$ anti-involution σ that preserves H double cosets

Classical examples

Pair	Anti-involution
$(G \times G, \Delta G)$	$(g,h) \mapsto (h^{-1}, g^{-1})$
$(O(n+k), O(n) \times O(k))$	$g \mapsto g^{-1}$
$(U(n+k), U(n) \times U(k))$	
$(GL(n,\mathbb{R}), O(n))$	$g \mapsto g^t$
(G, G^θ), where G - Lie group, θ- involution, G^θ is compact	$g \mapsto \theta(g^{-1})$
(G, K), where G - is a reductive group, K - maximal compact subgroup	Cartan anti-involution

Non compact setting

Setting

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Definition

A local field is a locally compact non-discrete topological field.

There are 3 types of local fields:

- Archimedean: $\mathbb{R}$ and $\mathbb{C}$
- p-adic: $\mathbb{Q}_p$ and their finite extensions
- positive characteristic: $\mathbb{F}_q((t))$

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Definition

A linear algebraic group is a subgroup of GL_n defined by polynomial equations.

Reductive groups

Examples

GL_n, semisimple groups, O_n, U_n, Sp_{2n},...

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Fact

Reductive groups are unimodular.

Smooth representations

Definition

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Over non-Archimedean F, by smooth representation V we mean a complex linear representation V such that any $v \in V$ has open stabilizer.

Notation

Let M be a smooth manifold. We denote by $C_c^\infty(M)$ the space of smooth compactly supported functions on M. We will consider the space $(C_c^\infty(M))^*$ of distributions on M. Sometimes we will also consider the space $\mathcal{S}^*(M)$ of Schwartz distributions on M.

Distributions

Notation

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Definition

An ℓ-space is a Hausdorff locally compact totally disconnected topological space. For an ℓ-space X we denote by $\mathcal{S}(X)$ the space of compactly supported locally constant functions on X. We let $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ be the space of distributions on X.

Gelfand Pairs

Definition

A pair of reductive groups ($G \supset H$) is called a **Gelfand pair** if for any irreducible admissible representation ρ of G

$$dimHom_H(\rho, \mathbb{C}) \cdot dimHom_H(\widetilde{\rho}, \mathbb{C}) \leq 1$$

usually, this implies that

$$dimHom_H(\rho, \mathbb{C}) \leq 1.$$

Gelfand-Kazhdan distributional criterion

Theorem (Gelfand-Kazhdan,...)

Let σ be an involutive anti-automorphism of G and assume $\sigma(H) = H$.
Suppose that $\sigma(\xi) = \xi$ for all bi H-invariant distributions ξ on G.
Then (G, H) is a Gelfand pair.

Strong Gelfand Pairs

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$$dimHom_H(\rho,\tau) \cdot dimHom_H(\widetilde{\rho},\widetilde{\tau}) \leq 1$$

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Proposition

The pair (G, H) is a strong Gelfand pair if and only if the pair $(G \times H, \Delta H)$ is a Gelfand pair.

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Proposition

The pair (G, H) is a strong Gelfand pair if and only if the pair $(G \times H, \Delta H)$ is a Gelfand pair.

Corollary

Let σ be an involutive anti-automorphism of G s.t. $\sigma(H) = H$. Suppose $\sigma(\xi) = \xi$ for all distributions ξ on G invariant with respect to conjugation by H. Then (G, H) is a strong Gelfand pair.

Rep. theory:
$\forall\rho \dim \rho^H \leq 1$

$\Longleftrightarrow$

Algebra:
$C(H\backslash G/H)$
is commutative

$\Longleftarrow$

"Analysis":
$\exists\ \sigma$ s.t. $f = \sigma(f)$
$\forall f \in C(H\backslash G/H)$

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Geometry:
$\exists\, \sigma$ that preserves
H double cosets

Compact case

Rep. theory:
$\forall\rho \dim \rho^H \leq 1$

Algebra:
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Rep. theory:
$\dim Hom_H(\rho, \mathbb{C}) \leq 1$
$\forall\rho$

~~Algebra~~

Analysis:
$\exists\ \sigma$ s.t. $\xi = \sigma(\xi)$
$\forall\xi \in \mathcal{S}^*(G)^{H\times H}$

non− *Archimedean*

Geometry:
$\exists\,\sigma$ that preserves
closed
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Non − compact case

Results on Gelfand pairs

Pair	p-adic	char$F > 0$	real
$(GL_n(E), GL_n(F))$	Flicker	Flicker	Aizenbud-G.
$(GL_{n+k}, GL_n \times GL_k)$	Jacquet-Rallis	Aizenbud-Avni-G.	
$(O_{n+k}, O_n \times O_k)$ over $\mathbb{C}$	——	——	
(GL_n, O_n) over $\mathbb{C}$			
(GL_{2n}, Sp_{2n})	Heumos-Rallis	Heumos-Rallis	Aizenbud-Sayag
$(GL_{2n}, \left\{\begin{pmatrix} g & u \\ 0 & g \end{pmatrix}\right\}, \psi)$	Jacquet-Rallis		Aizenbud-G.-Jacquet

Results on strong Gelfand pairs

<table>
<tr><th>Pair</th><th>p-adic</th><th>$\mathrm{char} F > 0$</th><th>real</th></tr>
<tr><td>(GL_{n+1}, GL_n)</td><td rowspan="3">Aizenbud-G.-Rallis-Schiffmann</td><td>Aizenbud-Avni-G.</td><td>Aizenbud-G. Sun-Zhu</td></tr>
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- real: $\mathbb{R}$ and $\mathbb{C}$
- p-adic: $\mathbb{Q}_p$ and its finite extensions.
- $\mathrm{char} F > 0$: $\mathbb{F}_q((t))$.

Results on strong Gelfand pairs

Pair	p-adic	$\mathrm{char} F > 0$	real
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Remark

The results from the last two slides are used to prove splitting of periods of automorphic forms.