

Invariant Distributions and Gelfand Pairs

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Definition

A pair of compact topological groups ($G \supset H$) is called a **Gelfand pair** if the following equivalent conditions hold:

- $L^2(G/H)$ decomposes to direct sum of **distinct** irreducible representations of G .
- for any irreducible representation ρ of G $\dim \rho^H \leq 1$.
- for any irreducible representation ρ of G $\dim \text{Hom}_H(\rho, \mathbb{C}) \leq 1$.
- the algebra of bi- H -invariant functions on G , $C(H \backslash G/H)$, is commutative w.r.t. convolution.

Definition

A pair of compact topological groups $(G \supset H)$ is called a **strong Gelfand pair** if one of the following equivalent conditions is satisfied:

- the pair $(G \times H \supset \Delta H)$ is a Gelfand pair
- for any irreducible representations ρ of G and τ of H

$$\dim \text{Hom}_H(\rho|_H, \tau) \leq 1.$$

- the algebra of $\text{Ad}(H)$ -invariant functions on G , $C(G//H)$, is commutative w.r.t. convolution.

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($GL(n, \mathbb{R}), O(n, \mathbb{R})$) is a Gelfand pair - the irreducible representations of $GL(n, \mathbb{R})$ which have an $O(n, \mathbb{R})$ -invariant vector are the same as characters of the algebra $C(O(n, \mathbb{R}) \backslash GL(n, \mathbb{R}) / O(n, \mathbb{R}))$.

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The same for the pair ($GL(n, \mathbb{C}), U(n)$).



Proposition (Gelfand)

Let σ be an involutive anti-automorphism of G (i.e. $\sigma(g_1 g_2) = \sigma(g_2) \sigma(g_1)$ and $\sigma^2 = \text{Id}$) and assume $\sigma(H) = H$. Suppose that $\sigma(f) = f$ for all bi H -invariant functions $f \in C(H \backslash G/H)$. Then (G, H) is a Gelfand pair.



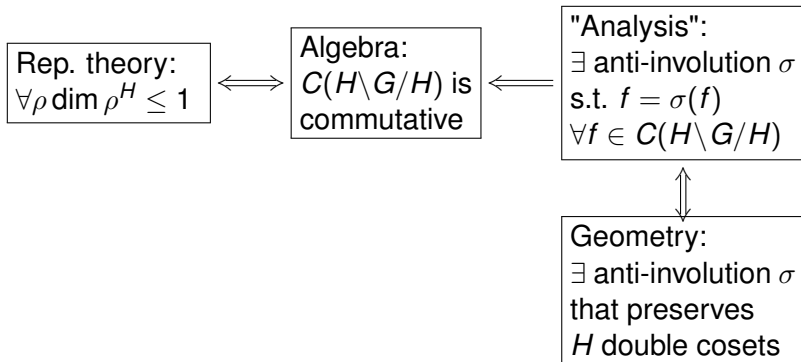
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Sum up



Classical examples

| Pair | Anti-involution |
|---|-----------------------------------|
| $(G \times G, \Delta G)$ | $(g, h) \mapsto (h^{-1}, g^{-1})$ |
| $(O(n+k), O(n) \times O(k))$ | $g \mapsto g^{-1}$ |
| $(U(n+k), U(n) \times U(k))$ | |
| $(GL(n, \mathbb{R}), O(n))$ | $g \mapsto g^t$ |
| (G, G^θ) , where G - Lie group, θ - involution, G^θ is compact | $g \mapsto \theta(g^{-1})$ |
| (G, K) , where G - is a reductive group, K - maximal compact subgroup | Cartan anti-involution |

Non compact setting

Setting

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Definition

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- non-Archimedean: \mathbb{Q}_p and their finite extensions

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Definition

A linear algebraic group is a subgroup of GL_n defined by polynomial equations.

Examples

GL_n , semisimple groups, O_n , U_n , Sp_{2n}, \dots

Reductive groups

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Reductive groups are unimodular.

Definition

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Smooth representations

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Definition

Over non-Archimedean F , by smooth representation V we mean a complex linear representation V such that for any $v \in V$ there exists an open compact subgroup $K < G$ such that $Kv = v$.

Notation

Let M be a smooth manifold. We denote by $C_c^\infty(M)$ the space of smooth compactly supported functions on M . We will consider the space $(C_c^\infty(M))^$ of distributions on M . Sometimes we will also consider the space $\mathcal{S}^*(M)$ of Schwartz distributions on M .*

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Definition

An ℓ -space is a Hausdorff locally compact totally disconnected topological space. For an ℓ -space X we denote by $\mathcal{S}(X)$ the space of compactly supported locally constant functions on X . We let $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ be the space of distributions on X .

Definition

A pair of reductive groups $(G \supset H)$ is called a **Gelfand pair** if for any irreducible admissible representation ρ of G

$$\dim \text{Hom}_H(\rho, \mathbb{C}) \cdot \dim \text{Hom}_H(\tilde{\rho}, \mathbb{C}) \leq 1$$

usually, this implies that

$$\dim \text{Hom}_H(\rho, \mathbb{C}) \leq 1.$$

Gelfand-Kazhdan distributional criterion



Theorem (Gelfand-Kazhdan,...)

Let σ be an involutive anti-automorphism of G and assume $\sigma(H) = H$.

*Suppose that $\sigma(\xi) = \xi$ for all bi H -invariant distributions ξ on G .
Then (G, H) is a Gelfand pair.*

Strong Gelfand Pairs

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Corollary

Let σ be an involutive anti-automorphism of G s.t. $\sigma(H) = H$. Suppose $\sigma(\xi) = \xi$ for all distributions ξ on G invariant with respect to conjugation by H . Then (G, H) is a strong Gelfand pair.

Rep. theory:
 $\forall \rho \dim \rho^H \leq 1$



Algebra:
 $C(H \backslash G/H)$
is commutative

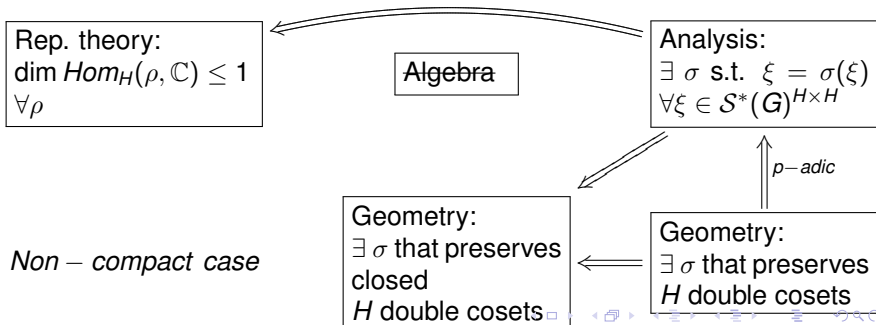
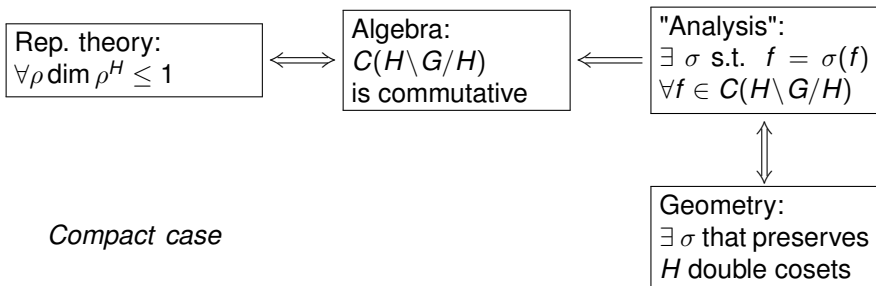


"Analysis":
 $\exists \sigma$ s.t. $f = \sigma(f)$
 $\forall f \in C(H \backslash G/H)$



Geometry:
 $\exists \sigma$ that preserves
 H double cosets

Compact case



Results on Gelfand pairs

| Pair | p-adic case | real case |
|--|----------------|-----------------------------|
| $(GL_n(E), GL_n(F))$ | Flicker | Aizenbud-Gourevitch |
| $(GL_{n+k}, GL_n \times GL_k)$ | Jacquet-Rallis | |
| $(O_{n+k}, O_n \times O_k)$ over \mathbb{C} | _____ | |
| (GL_n, O_n) over \mathbb{C} | | |
| (GL_{2n}, Sp_{2n}) | Heumos-Rallis | Aizenbud-Sayag |
| $(GL_{2n}, \left\{ \begin{pmatrix} g & u \\ 0 & g \end{pmatrix} \right\}, \psi)$ | Jacquet-Rallis | Aizenbud-Gourevitch-Jacquet |

Results on strong Gelfand pairs

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| (GL_{n+1}, GL_n) | Aizenbud-Gourevitch- | Aizenbud-Gourevitch |
| $(O(V \oplus F), O(V))$ | Rallis- | Sun-Zhu |
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- real: \mathbb{R} and \mathbb{C}
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Remark

The results from the last two slides are used to prove splitting of periods of automorphic forms.

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- We call (G, H, θ) **connected** if G/H is Zariski connected.
- Define an antiinvolution $\sigma : G \rightarrow G$ by $\sigma(g) := \theta(g^{-1})$.

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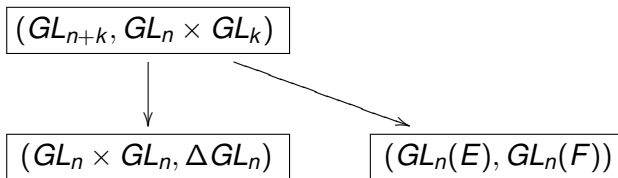
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- 3 Compute all the "descendants" of the pair and prove (2) for them.

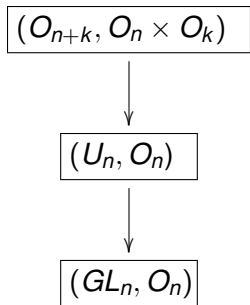
To check that a symmetric pair is a Gelfand pair

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- 2 Prove that any H -invariant distribution on \mathfrak{g}^σ is σ -invariant provided that this holds outside the cone of nilpotent elements.
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We call the property (2) regularity. We conjecture that all symmetric pairs are regular. This will imply that any good symmetric pair is a Gelfand pair.

Descendants





Regular symmetric pairs

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| (Sp_{2m}, GL_m) | Aizenbud | Sayag (based on work of Sekiguchi) |
| (E_6, Sp_8) | | |
| $(E_6, SL_6 \times SL_2)$ | | |
| (E_7, SL_8) | | |
| (E_8, SO_{16}) | | |
| $(F_4, Sp_6 \times SL_2)$ | | |
| $(G_2, SL_2 \times SL_2)$ | | |