Invariant Distributions and Gelfand Pairs

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Gelfand Pairs

Example

Let X be a finite set. Let the symmetric group $Perm(X)$ act on X. Consider the space $F(X)$ of complex valued functions on X as a representation of $Perm(X)$. Then it decomposes to direct sum of **distinct** irreducible representations.

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Definition

A pair of compact topological groups $(G \supset H)$ is called a **Gelfand pair** if the following equivalent conditions hold:

- $L^2(G/H)$ decomposes to direct sum of **distinct** irreducible representations of G.
- for any irreducible representation ρ of G $dim \rho^H \leq 1$.
- for any irreducible representation ρ of G $dim Hom_H(\rho, \mathbb{C}) \leq 1$.
- the algebra of bi-H-invariant functions on G, $C(H \backslash G/H)$, is commutative w.r.t. convolution.

Strong Gelfand Pairs

Definition

A pair of compact topological groups $(G \supset H)$ is called a **strong Gelfand pair** if one of the following equivalent conditions is satisfied:

- the pair $(G \times H \supset \Delta H)$ is a Gelfand pair
- for any irreducible representations ρ of G and τ of H

$$dimHom_H(\rho|_H, \tau) \leq 1.$$

- the algebra of Ad(H)-invariant functions on G, $C(G//H)$, is commutative w.r.t. convolution.

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- Classification of representations:
  $(GL(n,\mathbb{R}), O(n,\mathbb{R}))$ is a Gelfand pair -
  the irreducible representations of $GL(n,\mathbb{R})$ which have an $O(n,\mathbb{R})$-invariant vector are the same as characters of the algebra $C(O(n,\mathbb{R})\backslash GL(n,\mathbb{R})/O(n,\mathbb{R}))$.

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  The same for the pair $(GL(n,\mathbb{C}), U(n))$.

Gelfand trick

Proposition (Gelfand)

Let σ be an involutive anti-automorphism of G (i.e. $\sigma(g_1 g_2) = \sigma(g_2)\sigma(g_1)$ and $\sigma^2 = Id$) and assume $\sigma(H) = H$. Suppose that $\sigma(f) = f$ for all bi H-invariant functions $f \in C(H\backslash G/H)$. Then (G, H) is a Gelfand pair.

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Sum up

Rep. theory: $\forall \rho \dim \rho^H \leq 1$ $\Longleftrightarrow$ Algebra: $C(H\backslash G/H)$ is commutative $\Longleftarrow$ "Analysis": $\exists$ anti-involution σ s.t. $f = \sigma(f)$ $\forall f \in C(H\backslash G/H)$

$\Updownarrow$

Geometry: $\exists$ anti-involution σ that preserves H double cosets

Classical examples

Pair	Anti-involution
$(G \times G, \Delta G)$	$(g, h) \mapsto (h^{-1}, g^{-1})$
$(O(n+k), O(n) \times O(k))$	$g \mapsto g^{-1}$
$(U(n+k), U(n) \times U(k))$	
$(GL(n, \mathbb{R}), O(n))$	$g \mapsto g^t$
(G, G^θ), where G - Lie group, θ- involution, G^θ is compact	$g \mapsto \theta(g^{-1})$
(G, K), where G - is a reductive group, K - maximal compact subgroup	Cartan anti-involution

Non compact setting

Setting

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Definition

A local field is a locally compact non-discrete topological field.
There are 2 types of local fields of characteristic zero:

- Archimedean: $\mathbb{R}$ and $\mathbb{C}$
- non-Archimedean: $\mathbb{Q}_p$ and their finite extensions

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Definition

A linear algebraic group is a subgroup of GL_n defined by polynomial equations.

Examples

GL_n, semisimple groups, O_n, U_n, Sp_{2n},...

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Reductive groups

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Fact

Reductive groups are unimodular.

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Over Archimedean F, by smooth representation V we mean a complex Fréchet representation V such that for any $v \in V$ the map $G \to V$ defined by v is smooth.

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Over non-Archimedean F, by smooth representation V we mean a complex linear representation V such that for any $v \in V$ there exists an open compact subgroup $K < G$ such that $Kv = v$.

Distributions

Notation

Let M be a smooth manifold. We denote by $C_c^\infty(M)$ the space of smooth compactly supported functions on M. We will consider the space $(C_c^\infty(M))^*$ of distributions on M. Sometimes we will also consider the space $\mathcal{S}^*(M)$ of Schwartz distributions on M.

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Definition

An ℓ-space is a Hausdorff locally compact totally disconnected topological space. For an ℓ-space X we denote by $\mathcal{S}(X)$ the space of compactly supported locally constant functions on X. We let $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ be the space of distributions on X.

Gelfand Pairs

Definition

A pair of reductive groups ($G \supset H$) is called a **Gelfand pair** if for any irreducible admissible representation ρ of G

$$dimHom_H(\rho, \mathbb{C}) \cdot dimHom_H(\widetilde{\rho}, \mathbb{C}) \leq 1$$

usually, this implies that

$$dimHom_H(\rho, \mathbb{C}) \leq 1.$$

Gelfand-Kazhdan distributional criterion

Theorem (Gelfand-Kazhdan,...)

Let σ be an involutive anti-automorphism of G and assume $\sigma(H) = H$.
Suppose that $\sigma(\xi) = \xi$ for all bi H-invariant distributions ξ on G.
Then (G, H) is a Gelfand pair.

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The pair (G, H) is a strong Gelfand pair if and only if the pair $(G \times H, \Delta H)$ is a Gelfand pair.

Corollary

Let σ be an involutive anti-automorphism of G s.t. $\sigma(H) = H$. Suppose $\sigma(\xi) = \xi$ for all distributions ξ on G invariant with respect to conjugation by H. Then (G, H) is a strong Gelfand pair.

Rep. theory:
$\forall \rho \dim \rho^H \leq 1$

$\Longleftrightarrow$

Algebra:
$C(H \backslash G / H)$
is commutative

$\Longleftarrow$

"Analysis":
$\exists\ \sigma$ s.t. $f = \sigma(f)$
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Geometry:
$\exists\, \sigma$ that preserves
H double cosets

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$\dim Hom_H(\rho, \mathbb{C}) \leq 1$
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~~Algebra~~

Analysis:
$\exists\ \sigma$ s.t. $\xi = \sigma(\xi)$
$\forall \xi \in \mathcal{S}^*(G)^{H \times H}$

$p-adic$

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Non – compact case

Results on Gelfand pairs

Pair	p-adic case	real case
$(GL_n(E), GL_n(F))$	Flicker	Aizenbud-Gourevitch
$(GL_{n+k}, GL_n \times GL_k)$	Jacquet-Rallis	
$(O_{n+k}, O_n \times O_k)$ over $\mathbb{C}$	————	
(GL_n, O_n) over $\mathbb{C}$		
(GL_{2n}, Sp_{2n})	Heumos-Rallis	Aizenbud-Sayag
$\left(GL_{2n}, \left\{ \begin{pmatrix} g & u \\ 0 & g \end{pmatrix} \right\}, \psi\right)$	Jacquet-Rallis	Aizenbud-Gourevitch-Jacquet

Results on strong Gelfand pairs

<table>
<tr><th>Pair</th><th>p-adic case</th><th>real case</th></tr>
<tr><td>(GL_{n+1}, GL_n)</td><td rowspan="3">Aizenbud-Gourevitch-Rallis-Schiffmann</td><td>Aizenbud-Gourevitch Sun-Zhu</td></tr>
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- real: $\mathbb{R}$ and $\mathbb{C}$
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Remark

The results from the last two slides are used to prove splitting of periods of automorphic forms.

Symmetric pairs

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- We call (G, H, θ) **connected** if G/H is Zariski connected.
- Define an antiinvolution $\sigma : G \to G$ by $\sigma(g) := \theta(g^{-1})$.

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Definition

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To check that a symmetric pair is a Gelfand pair

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2. Prove that any H-invariant distribution on g^{σ} is σ-invariant provided that this holds outside the cone of nilpotent elements.
3. Compute all the "descendants" of the pair and prove (2) for them.

We call the property (2) regularity. We conjecture that all symmetric pairs are regular. This will imply that any good symmetric pair is a Gelfand pair.

Descendants

$$\begin{array}{ccc} \boxed{(GL_{n+k}, GL_n \times GL_k)} & & \\ \downarrow & \searrow & \\ \boxed{(GL_n \times GL_n, \Delta GL_n)} & & \boxed{(GL_n(E), GL_n(F))} \end{array}$$

Descendants

$$(O_{n+k}, O_n \times O_k)$$

$$\downarrow$$

$$(U_n, O_n)$$

$$\downarrow$$

$$(GL_n, O_n)$$

Regular symmetric pairs

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$(GL_{n+k}, GL_n \times GL_k)$	Jacquet-Rallis	
$(O_{n+k}, O_n \times O_k)$	Aizenbud-Gourevitch	
(GL_n, O_n)		
(GL_{2n}, Sp_{2n})	Heumos - Rallis	Aizenbud-Sayag
(Sp_{2m}, GL_m)	Aizenbud	Sayag (based on work of Sekiguchi)
(E_6, Sp_8)		
$(E_6, SL_6 \times SL_2)$		
(E_7, SL_8)		
(E_8, SO_{16})		
$(F_4, Sp_6 \times SL_2)$		
$(G_2, SL_2 \times SL_2)$		