

# Multiplicity One Theorems

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It has the following corollary in representation theory.

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Similar theorems hold for orthogonal and unitary groups.

## Notation

*Let  $M$  be a smooth manifold. We denote by  $C_c^\infty(M)$  the space of smooth compactly supported functions on  $M$ . We will consider the space  $(C_c^\infty(M))^*$  of distributions on  $M$ . Sometimes we will also consider the space  $\mathcal{S}^*(M)$  of Schwartz distributions on  $M$ .*

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## Definition

An  $\ell$ -space is a Hausdorff locally compact totally disconnected topological space. For an  $\ell$ -space  $X$  we denote by  $\mathcal{S}(X)$  the space of compactly supported locally constant functions on  $X$ . We let  $\mathcal{S}^*(X) := \mathcal{S}(X)^*$  be the space of distributions on  $X$ .

- $\tilde{G} := GL_n(F) \rtimes \{1, \sigma\}$
- Define a character  $\chi$  of  $\tilde{G}$  by  $\chi(GL_n(F)) = \{1\}$ ,  
 $\chi(\tilde{G} - GL_n(F)) = \{-1\}$ .

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$$g \begin{pmatrix} A_{n \times n} & v_{n \times 1} \\ \phi_{1 \times n} & \lambda \end{pmatrix} g^{-1} = \begin{pmatrix} gAg^{-1} & gv \\ (g^*)^{-1}\phi & \lambda \end{pmatrix} \text{ and } \begin{pmatrix} A & v \\ \phi & \lambda \end{pmatrix}^t = \begin{pmatrix} A^t & \phi^t \\ v^t & \lambda \end{pmatrix}$$

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- $\tilde{G}$  acts on  $X$  by
$$g(A, v, \phi) = (gAg^{-1}, gv, (g^*)^{-1}\phi)$$
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$$\mathcal{S}^*(X)^{\tilde{G}, \chi} = 0.$$

# First tool: Stratification

## Setting

*A group  $G$  acts on a space  $X$ , and  $\chi$  is a character of  $G$ . We want to show  $S^*(X)^{G,\chi} = 0$ .*

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Let  $U \subset X$  be an open  $G$ -invariant subset and  $Z := X - U$ . Suppose that  $S^*(U)^{G,\chi} = 0$  and  $S_X^*(Z)^{G,\chi} = 0$ . Then  $S^*(X)^{G,\chi} = 0$ .

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## Proof.

$$0 \rightarrow S_X^*(Z)^{G,\chi} \rightarrow S^*(X)^{G,\chi} \rightarrow S^*(U)^{G,\chi} \quad \square$$



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For smooth manifolds, there is a slightly more complicated statement which takes into account transversal derivatives.

# Frobenius descent

$$\begin{array}{ccc} X_Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ z & \longrightarrow & Z \end{array}$$

## Theorem (Bernstein, Baruch, ...)

Let  $\psi : X \rightarrow Z$  be a map.

Let  $G$  act on  $X$  and  $Z$  such that  $\psi(gx) = g\psi(x)$ .

Suppose that the action of  $G$  on  $Z$  is transitive.

Suppose that both  $G$  and  $\text{Stab}_G(z)$  are unimodular. Then

$$S^*(X)^{G,X} \cong S^*(X_Z)^{\text{Stab}_G(z),X}.$$

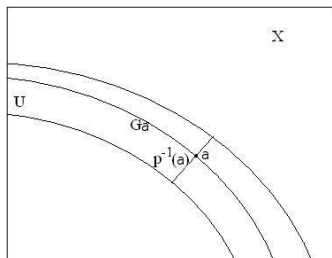
# Generalized Harish-Chandra descent

## Theorem

Let a reductive group  $G$  act on a smooth affine algebraic variety  $X$ . Let  $\chi$  be a character of  $G$ . Suppose that for any  $a \in X$  s.t. the orbit  $Ga$  is closed we have

$$S^*(N_{Ga,a}^X)^{G_a, \chi} = 0.$$

Then  $S^*(X)^{G, \chi} = 0$ .





Let  $V$  be a finite dimensional vector space over  $F$  and  $Q$  be a non-degenerate quadratic form on  $V$ . Let  $\widehat{\xi}$  denote the Fourier transform of  $\xi$  defined using  $Q$ .

## Proposition

*Let  $G$  act on  $V$  linearly and preserving  $Q$ . Let  $\xi \in \mathcal{S}^*(V)^{G,\chi}$ . Then  $\widehat{\xi} \in \mathcal{S}^*(V)^{G,\chi}$ .*

# Fourier transform and homogeneity

- We call a distribution  $\xi \in \mathcal{S}^*(V)$  **abs-homogeneous of degree  $d$**  if for any  $t \in F^\times$ ,

$$h_t(\xi) = u(t)|t|^d \xi,$$

where  $h_t$  denotes the homothety action on distributions and  $u$  is some unitary character of  $F^\times$ .

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Theorem (Jacquet, Rallis, Schiffmann,...)

Assume  $F$  is **non-archimedean**. Let  $\xi \in \mathcal{S}_V^*(Z(Q))$  be s.t.  $\widehat{\xi} \in \mathcal{S}_V^*(Z(Q))$ . Then  $\xi$  is abs-homogeneous of degree  $\frac{1}{2} \dim V$ .

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## Theorem (archimedean homogeneity)

Let  $F$  be any local field. Let  $L \subset \mathcal{S}_V^*(Z(Q))$  be a non-zero linear subspace s. t.  $\forall \xi \in L$  we have  $\widehat{\xi} \in L$  and  $Q\xi \in L$ .

Then there exists a non-zero distribution  $\xi \in L$  which is abs-homogeneous of degree  $\frac{1}{2} \dim V$  or of degree  $\frac{1}{2} \dim V + 1$ .



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In the non-Archimedean case we define the singular support to be the Zariski closure of the wave front set.

# Properties and the Integrability Theorem

Let  $X$  be a smooth algebraic variety.

- Let  $\xi \in \mathcal{S}^*(X)$ . Then  $\overline{\text{Supp}(\xi)}_{\text{Zar}} = p_X(\text{SS}(\xi))$ , where  $p_X : T^*X \rightarrow X$  is the projection.

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- Let an algebraic group  $G$  act on  $X$ . Let  $\xi \in \mathcal{S}^*(X)^{G, \chi}$ . Then

$$\text{SS}(\xi) \subset \{(x, \phi) \in T^*X \mid \forall \alpha \in \mathfrak{g} \quad \phi(\alpha(x)) = 0\}.$$

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- Let  $V$  be a linear space. Let  $Z \subset V^*$  be a closed subvariety, invariant with respect to homotheties in  $V$ . Let  $\xi \in \mathcal{S}^*(V)$ . Suppose that  $\text{Supp}(\hat{\xi}) \subset Z$ . Then  $\text{SS}(\xi) \subset V \times Z$ .

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- Integrability theorem:  
Let  $\xi \in \mathcal{S}^*(X)$ . Then  $SS(\xi)$  is (weakly) coisotropic.

## Definition

Let  $M$  be a smooth algebraic variety and  $\omega$  be a symplectic form on it. Let  $Z \subset M$  be an algebraic subvariety. We call it  **$M$ -coisotropic** if the following equivalent conditions holds.

- At every smooth point  $z \in Z$  we have  $T_z Z \supset (T_z Z)^\perp$ . Here,  $(T_z Z)^\perp$  denotes the orthogonal complement to  $T_z Z$  in  $T_z M$  with respect to  $\omega$ .
- The ideal sheaf of regular functions that vanish on  $\bar{Z}$  is closed under Poisson bracket.

If there is no ambiguity, we will call  $Z$  a coisotropic variety.



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- Every non-empty coisotropic subvariety of  $M$  has dimension at least  $\frac{\dim M}{2}$ .

## Definition

Let  $X$  be a smooth algebraic variety. Let  $Z \subset T^*X$  be an algebraic subvariety. We call it  $T^*X$ -**weakly coisotropic** if one of the following equivalent conditions holds.

- For a generic smooth point  $a \in p_X(Z)$  and for a generic smooth point  $y \in p_X^{-1}(a) \cap Z$  we have

$$CN_{p_X(Z), a}^X \subset T_y(p_X^{-1}(a) \cap Z).$$

- For any smooth point  $a \in p_X(Z)$  the fiber  $p_X^{-1}(a) \cap Z$  is locally invariant with respect to shifts by  $CN_{p_X(Z), a}^X$ .

# Weakly coisotropic varieties

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- For any smooth point  $a \in p_X(Z)$  the fiber  $p_X^{-1}(a) \cap Z$  is locally invariant with respect to shifts by  $CN_{p_X(Z),a}^X$ .

- Every non-empty weakly coisotropic subvariety of  $T^*X$  has dimension at least  $\dim X$ .

## Definition

Let  $X$  be a smooth algebraic variety. Let  $Z \subset X$  be a smooth subvariety and  $R \subset T^*X$  be any subvariety. We define **the restriction**  $R|_Z \subset T^*Z$  of  $R$  to  $Z$  by

$$R|_Z := q(p_X^{-1}(Z) \cap R),$$

where  $q : p_X^{-1}(Z) \rightarrow T^*Z$  is the projection.

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## Lemma

*Let  $X$  be a smooth algebraic variety. Let  $Z \subset X$  be a smooth subvariety. Let  $R \subset T^*X$  be a (weakly) coisotropic variety. Then, under some transversality assumption,  $R|_Z \subset T^*Z$  is a (weakly) coisotropic variety.*

# Harish-Chandra descent and homogeneity

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$$S := \{(A, v, \phi) \in X_n \mid A^n = 0 \text{ and } \phi(A^i v) = 0 \forall 0 \leq i \leq n\}.$$

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By the homogeneity theorem, the stratification method and Frobenius descent we get that any  $\xi \in \mathcal{S}^*(X)^{\tilde{G}, X}$  is supported in  $S'$ .

# Reduction to the geometric statement

## Notation

$$T' = \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in X \times X \mid \forall i, j \in \{1, 2\} \\ (A_i, v_j, \phi_j) \in S' \text{ and } [A_1, A_2] + v_1 \otimes \phi_2 - v_2 \otimes \phi_1 = 0\}.$$

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It is enough to show:

## Theorem (The geometric statement)

*There are no non-empty  $X \times X$ -weakly coisotropic subvarieties of  $T'$ .*

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## Notation

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## Lemma (Key Lemma)

*There are no non-empty  $V \times V^* \times V \times V^*$ -weakly coisotropic subvarieties of  $R_A$ .*

# Proof of the Key Lemma

## Notation

$$Q_A := S' \cap (\{A\} \times V \times V^*) = \bigcup_{i=1}^{n-1} (\text{Ker} A^i) \times (\text{Ker}(A^*)^{n-i})$$



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It is easy to see that  $R_A \subset Q_A \times Q_A$  and  $Q_A \times Q_A = \bigcup_{i,j=1}^{n-1} L_{ij}$ , where

$$L_{ij} := (\text{Ker}A^i) \times (\text{Ker}(A^*)^{n-i}) \times (\text{Ker}A^j) \times (\text{Ker}(A^*)^{n-j}).$$

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We know that there exists a nilpotent  $B$  satisfying  $[A, B] = M$ . Hence this  $B$  is upper nilpotent, which implies  $M_{i, i+1} = 0$  and hence  $f(v_1, \phi_1, v_2, \phi_2) = 0$ .

## Flowchart

$$sl(V) \times V \times V^*$$

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# Summary

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