Multiplicity one Theorems

DMITRY GOUREVITCH
(joint work with Avraham Aizenbud, Steve Rallis, and Gerard Schiffmann)

arXiv:0709.4215 [math.RT]

Let $\mathbb{F}$ be a local field non archimedean and of characteristic 0. Let W be a vector space over $\mathbb{F}$ of finite dimension $n+1 \geqslant 1$ and let $W = V \oplus U$ be a direct sum decomposition with $\dim V = n$. Then we have an imbedding of $GL(V)$ into $GL(W)$. Our goal is to prove the following Theorem:

Theorem 1: *If π (resp. ρ) is an irreducible admissible representation of $GL(W)$ (resp. of $GL(V)$) then*

$$\dim \left(\mathrm{Hom}_{GL(V)}(\pi_{|GL(V)}\,,\,\rho)\right) \leqslant 1.$$

We choose a basis of V and a non zero vector in U thus getting a basis of W. We can identify $GL(W)$ with $GL(n+1,\mathbb{F})$ and $GL(V)$ with $GL(n,\mathbb{F})$. The transposition map is an involutive anti-automorphism of $GL(n+1,\mathbb{F})$ which leaves $GL(n,\mathbb{F})$ stable. It acts on the space of distributions on $GL(n+1,\mathbb{F})$.

Theorem 1 is a Corollary of :

Theorem 2: *A distribution on $GL(W)$ which is invariant under the adjoint action of $GL(V)$ is invariant by transposition.*

One can raise a similar question for orthogonal and unitary groups. Let $\mathbb{D}$ be either $\mathbb{F}$ or a quadratic extension of $\mathbb{F}$. If $x \in \mathbb{D}$ then $\overline{x}$ is the conjugate of x if $\mathbb{D} \neq \mathbb{F}$ and is equal to x if $\mathbb{D} = \mathbb{F}$.

Let W be a vector space over $\mathbb{D}$ of finite dimension $n+1 \geqslant 1$. Let $\langle .,. \rangle$ be a non degenerate hermitian form on W. This form is bi-additive and

$$\langle dw, d'w' \rangle = d\,\overline{d'} \langle w, w' \rangle, \quad \langle w', w \rangle = \overline{\langle w, w' \rangle}.$$

Given a $\mathbb{D}-$linear map u from W into itself, its adjoint u^* is defined by the usual formula

$$\langle u(w), w' \rangle = \langle w, u^*(w') \rangle.$$

Choose a vector e in W such that $\langle e, e \rangle \neq 0$; let $U = \mathbb{D}e$ and $V = U^\perp$ the orthogonal complement. Then V has dimension n and the restriction of the hermitian form to V is non degenerate.

Let M be the unitary group of W, that is to say the group of all $\mathbb{D}-$linear maps m of W into itself which preserve the hermitian form or equivalently such that $mm^* = 1$. Let G be the unitary group of V. With the p-adic topology both groups are l-groups(see [**B-Z**]).

The group G is naturally imbedded into M.

Theorem 1′: *If π (resp ρ) is an irreducible admissible representation of M (resp of G) then*

$$\dim\left(\mathrm{Hom}_G(\pi_{|G},\rho)\right)\leq 1.$$

Choose a basis $e_1,\dots e_n$ of V such that $\langle e_i,e_j\rangle\in\mathbb{F}$. For

$$w=x_0e+\sum_1^n x_ie_i$$

put

$$\overline{w}=\overline{x}_0e+\sum_1^n\overline{x_i}\,e_i.$$

If u is a $\mathbb{D}-$linear map from W into itself, let $\overline{u}$ be defined by

$$\overline{u}(w)=\overline{u(\overline{w})}.$$

Let σ be the anti-involution $\sigma(m)=\overline{m}^{-1}$ of M; Theorem 1' is a consequence of

Theorem 2′: *A distribution on M which is invariant under the adjoint action of G is invariant under σ.*

Let us describe briefly our proof for $\mathrm{GL}(n)$. The proof is by induction on n; the case $n=0$ is trivial. In general we first linearize the problem by replacing the action of G on $\mathrm{GL}(W)$ by the action on the Lie algebra of $\mathrm{GL}(W)$. As a $G-$module this Lie algebra is isomorphic to a direct sum $\mathfrak{g}\oplus V\oplus V^*\oplus\mathbb{F}$ with $\mathfrak{g}$ the Lie algebra of G, V^* the dual space of V. The group $G=\mathrm{GL}(V)$ acts trivially on $\mathbb{F}$, by the adjoint action on its Lie algebra and the natural actions on V and V^*. The component $\mathbb{F}$ plays no role. Let u be a linear bijection of V onto V^* which transforms some basis of V into its dual basis. The involution may be taken as

$$(X,v,v^*)\mapsto(u^{-1}\,{}^tX\,u,u^{-1}(v^*),u(v)).$$

We have to show that a distribution T on $\mathfrak{g}\oplus V\oplus V^*$ which is invariant under G and skew relative to the involution is 0.

First we prove that such a distribution must have "singular support". On the $\mathfrak{g}$ side, using Harish-Chandra descent we get that the support of T must be contained in $\mathfrak{z}\times\mathcal{N}\times(V\oplus V^*)$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $\mathcal{N}$ the cone of nilpotent elements in $\mathfrak{g}$. On the $V\oplus V^*$ side we show that the support must be contained in $\mathfrak{g}\times\Gamma$ where Γ is the cone $\langle v,v^*\rangle=0$ in $V\oplus V^*$. On $\mathfrak{z}$ the action is trivial so we are reduced to the case of a distribution on $\mathcal{N}\times\Gamma$.

The end of the proof is based on two remarks. First, viewing the distribution as a distribution on $\mathcal{N}\times(V\oplus V^*)$ its partial Fourier transform relative to $V\oplus V^*$ has the same invariance properties and hence must also be supported by $\mathcal{N}\times\Gamma$. This implies in particular a homogeneity condition on $V\oplus V^*$. The idea of using Fourier transform in this kind of situation goes back at least to Harish-Chandra

([**Ha**]) and is conveniently expressed using a particular case of the Weil or oscillator representation.

For $(v, v^*) \in \Gamma$, let X_{v,v^*} be the map $x \mapsto \langle x, v^* \rangle v$ of V into itself. The second remark is that the one parameter group of transformations

$$(X, v, v^*) \mapsto (X + \lambda X_{v,v^*}, v, v^*)$$

is a group of (non linear) homeomorphisms of $[\mathfrak{g}, \mathfrak{g}] \times \Gamma$ which commute with G and the involution. It follows that the image of the support of our distribution must also be singular. Precisely this allows us to replace the condition $\langle v, v^* \rangle = 0$ by the stricter condition $X_{v,v^*} \in \mathrm{Im}\ \mathrm{ad}X$.

Using the stratification of $\mathcal{N}$ we proceed one nilpotent orbit at a time, transferring the problem to $V \oplus V^*$ and a fixed nilpotent matrix X. The support condition turns out to be compatible with direct sum so that it is enough to consider the case of a principal nilpotent element. In this last situation the key is the homogeneity condition coupled with an easy induction.

The orthogonal and unitary cases are proved roughly in the same way. Here the main difference is that we use Harish-Chandra descent directly on the group. Note that some Levi subgroups have components of type GL so that theorem 2 has to be assumed.

We systematically use two classical results from [**Ber**]: Bernstein's localization principle and a variant of Frobenius reciprocity.

Similar theorems should be true in the archimedean case. A partial result is given by [**A-G-S**].

References

[**AGRS**] A. Aizenbud, D. Gourevitch, S. Rallis, G. Schifmann, *Multiplicity One Theorems*,

[**A-G-S**] Avraham Aizenbud, Dmitry Gourevitch, Eitan Sayag, *(GL(n+1,R),GL(n,R)) is a Gelfand pair*, arXiv:0709.1273.

[**Ber**] J. Bernstein, *P-invariant Distributions on* GL(N) *and the classification of unitary representations of* GL(N) *(non-archimedean case)*, Lie group representations, II (College Park, Md., 1982/1983), 50–102, Lecture Notes in Math., **1041**, Springer, Berlin (1984).

[**B-Z**] J. Bernstein, A.V. Zelevinsky, *Representations of the group GL(n, F), where F is a local non-Archimedean field*, Uspekhi Mat. Nauk **10**, No.3, 5-70 (1976).

[**G-K**] I.M. Gelfand, D. Kazhdan, *Representations of the group* GL(n, K) *where K is a local field*, Lie groups and their representations (Proc. Summer School, Bolyai Janos Math. Soc., Budapest, 1971), pp. 95–118. Halsted, New York (1975).

[**Ha**] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups* Preface and notes by Stephen DeBacker and Paul J. Sally, Jr. University Lecture Series, **16** American Mathematical Society, Providence, RI, 1999. xiv+97 pp. ISBN: 0-8218-2025-7