Representation theory - how to get from zero characteristic to positive one

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May 21, 2010

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Non-Archimedean local fields

Definition

We will discuss two types of local fields:

- zero characteristic – $\mathbb{Q}_p$ and its finite extensions.
- positive characteristic – $\mathbb{F}_p((t))$ and its finite extensions.

Those fields satisfy:

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Notation

For a local field F we denote

- $\mathcal{O}_F := \{x \in F \,|\, |x| \leq 1\}$
- $\mathcal{P}_F := \{x \in F \,|\, |x| < 1\}$

The ring of integers

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Example

- $\mathcal{O}_{\mathbb{Q}_p} = \mathbb{Z}_p$.
- $\mathcal{P}_{\mathbb{Q}_p} = p\mathbb{Z}_p$.
- $\mathcal{O}_{\mathbb{F}_p((t))} = \mathbb{F}_p[[t]]$.
- $\mathcal{P}_{\mathbb{F}_p((t))} = t\mathbb{F}_p[[t]]$.

Close local fields

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Example

$\mathbb{F}_p((t))$ is n-close to $\mathbb{Q}_p(\sqrt[n]{p})$.

Example (of reductive groups)

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Remark

If F and F' are n-close then $K_0(F)/K_n(F) \simeq K_0(F')/K_n(F')$.

Cartan decomposition

Let $\pi \in \mathcal{P}_F - \mathcal{P}_F^2$. For simplicity let $G = GL_n$. Let

$$\Lambda(G) := \left\{ \begin{pmatrix} \pi^{i_1} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \pi^{i_n} \end{pmatrix} \mid i_1 \geq ... \geq i_n \right\}.$$

Theorem

$K_0(F) \setminus G(F)/K_0(F) = \Lambda(G)$.

Definition

- A smooth representation of $G(F)$ is a linear action of $G(F)$ on a complex linear vector space V such that any $v \in V$ has an open stabilizer.
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Definition

The Hecke algebra, denoted by $\mathcal{H}_K(G(F))$, is the algebra of double K-invariant measures on $G(F)$. The algebra structure on $\mathcal{H}_K(G(F))$ is defined by convolution.

Theorem (Bernstein 1980)

- *$\mathcal{M}_{K_n}(G(F))$ is a direct summand of $\mathcal{M}(G(F))$.*
- *$\mathcal{M}_{K_n}(G(F))$ is equivalent to the category of $\mathcal{H}_{K_n}(G(F))-$modules*
- *The algebra $\mathcal{H}_{K_n}(G(F))$ is finite over its center which is finitely generated. Therefore, $\mathcal{H}_{K_n}(G(F))$ is Noetherian and finitely presented.*

Kazhdan's Theorem

Theorem (Kazhdan - 1984)

For any natural n there exists N s.t. if F and F' are $N-$close then

$$\mathcal{H}_{K_n(F)}(G(F)) \simeq \mathcal{H}_{K_n(F')}(G(F')).$$

Ingredients of the proof

- A construction of a linear isomorphism

$$\phi : \mathcal{H}_{K_n(F)}(G(F)) \to \mathcal{H}_{K_n(F')}(G(F'))$$

whenever F and F' are $n-$close, using Cartan decomposition.

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$$K_n(G,F) \setminus G(F)/K_n(G,F) =$$
$$= K_n(G,F) \setminus K_0(G,F)\Lambda(G)K_0(G,F)/K_n(G,F) =$$
$$= G(\mathcal{O}_F/\mathcal{P}_F^n)\Lambda(G)G(\mathcal{O}_F/\mathcal{P}_F^n)$$

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- The algebra $\mathcal{H}_{K_n}(G(F))$ is finitely presented.

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Conclusion

Let $H \subset G$ be a spherical pair (i.e. the Borel subgroup of G acts on G/H with finite number of orbits). One can consider harmonic analysis over G/H as a generalization of representation theory.

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$G(F), H(F)$ is a Gelfand pair if and only if
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Proof.

$\mathrm{Hom}_{H(F)}(\pi|_{H(F)}, \mathbb{C}) \cong \mathrm{Hom}_{G(F)}(\pi, C^\infty(G(F)/H(F))) \cong$
$\mathrm{Hom}_{G(F)}(\mathcal{S}(G(F)/H(F)), \tilde{\pi})$

Relative version of Kazhdan's theorem

Theorem (Aizenbud, Avni, Gourevitch - 2009)

Under certain conditions on the pair (G, H), *for any natural* n *the exist* N *s.t. if* F *and* F' *are* $N-$*close then*

$$\mathcal{S}(G(F)/H(F))^{K_n(F)} \simeq \mathcal{S}(G(F')/H(F'))^{K_n(F')}$$

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$$\mathcal{H}_{K_n(F)}(G(F)) \simeq \mathcal{H}_{K_n(F')}(G(F')) - \textit{module}$$

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$$\psi : \mathcal{S}(G(F)/H(F))^{K_n(F)} \to \mathcal{S}(G(F')/H(F'))^{K_n(F')}$$

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- The module $\mathcal{S}(G(F)/H(F))^{K_n(F)}$ is finitely generated over the algebra $\mathcal{H}_{K_n}(G(F))$.

Proof of the 3-rd ingredient

Observation

If the module $\mathcal{S}(G(F)/H(F))^{K_n(F)}$ is finitely generated over the algebra $\mathcal{H}_{K_n}(G(F))$ for all n then

$$\forall \pi \in irr(G) : \dim Hom_H(\pi|_H, \mathbb{C}) < \infty.$$

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The following are equivalent

- $\forall \pi \in irr(G) : \dim Hom_H(\pi|_H, \mathbb{C}) < \infty$
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(2) $\Rightarrow$ (3) - follows from the theory of Bernstein center.
The hard part, (1) $\Rightarrow$ (2), follows from estimation of co-homologies. □

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Let F be a local field of characteristic 0. Then the pair $(G(F), H(F))$ is a Gelfand pair i.e. for any irreducible representation π of $G(F)$ we have

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Theorem (Aizenbud, Gourevitch, Rallis, Schiffmann - 2007)

Let F be a local field of characteristic 0. Then the pair $(GL_{n+1}(F), GL_n(F))$ is a strong Gelfand pair i.e. for any irreducible representations π of $GL_{n+1}(F)$ and τ of $GL_n(F)$ we have

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$\mathrm{Hom}_H(\pi|_H, \tau) \cong \mathrm{Hom}_H(\pi|_H, \tilde{\tilde{\tau}}) \cong \mathrm{Hom}_H(\pi|_H, \tilde{\tau}^*) \cong \mathrm{Hom}_{\Delta H}(\pi \otimes \tilde{\tau}, \mathbb{C})$ □

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- What spherical pairs are Gelfand pairs?

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- Is $\mathrm{Hom}_H(\pi|_H, \mathbb{C})$ finite for any irreducible smooth representation π of G?
  Conjecture: Yes.
  Proven for large classes of spherical pairs, including symmetric pairs, by Delorme and by Sakellaridis-Venkatesh.
- If $\mathrm{Hom}_H(\pi|_H, \mathbb{C})$ is finite, is it bounded as a function of π?
  Our theorem implies that it is locally bounded, i.e. bounded on $\mathcal{M}_K(G(F))$ for every K.
- What spherical pairs are Gelfand pairs? Conjecture: (G, H) is a Gelfand pair iff $\dim \mathrm{Hom}_H(\pi|_H, \mathbb{C}) \leq 1$ for any irreducible representation π of principle series.