# EXERCISE 3 IN INTRODUCTION TO REPRESENTATION THEORY

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(1) Let $V$ be a vector space. Define a symmetric bilinear form on $\mathrm{End}(V)$ by $\langle A, B\rangle := \mathrm{Tr}(AB)$. Show that it is non-degenerate. Show that if $V$ is a representation of $G$ then this form is invariant with respect to the conjugation action of $G$ on $\mathrm{End}(V)$.

(2) Let $\rho \in Rep(G)$. Show that the natural map $\rho : \mathcal{A}(G) \to \mathrm{End}_F(\rho)$ given by $f \mapsto \sum_{g\in G} f(g)\rho(g)$ is a morphism of algebras and of representations of $G \times G$.

(3) Define a bilinear form on $\mathcal{A}(G)$ by

$$\langle f, h\rangle := \sum_{g\in G} f(g)h(g^{-1})$$

Show that this form is bilinear, symmetric and non-degenerate.

(4) (P) Let $(\pi, V)$ be an irreducible complex representation of a finite group $G$. Show that it has an invariant Hermitian form H and that any two such forms are proportional.

(5) (P) Let $G$ be a finite group and let $(\pi, V)$ be a finite dimensional representation of $G$ over the field of real numbers $R$.
   (a) Show that $(\pi, V)$ is isomorphic to the dual representation $(\pi^*, V)$.
   (b) Give an example of irreducible representations $(\pi, G, V)$ and $(\tau, H, L)$ over the field $\mathbb{R}$ such that the tensor product representation $(\pi \otimes \tau, G \times H, V \otimes L)$ is reducible.

(6) (P) Show that if $X, Y$ are finite $G$-sets and $\chi$ ia a character of $G$ then the intertwining number $\langle \pi_X, \chi\pi_Y\rangle$ equals to the number of $G$-orbits $O$ in the set $X \times Y$ such that for any point $z \in O$, the restriction $\chi|_{G_z}$ of $\chi$ to the stabilizer $G_z$ of $z$ is trivial.

(7) Let $L, V$ be finite-dimensional linear spaces and let $X \in \mathrm{End}\,V, Y \in \mathrm{End}\,L$. Define $\Psi_{X,Y} : \mathrm{Hom}(L, V) \to \mathrm{Hom}(L, V)$ by $\Psi_{X,Y}(A) := XAY$. Then $\mathrm{Tr}\,\Psi_{X,Y} = \mathrm{Tr}\,X\,\mathrm{Tr}\,Y$.

*URL*: http://www.wisdom.weizmann.ac.il/~dimagur/IntRepTheo4.html

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