ANNIHILATOR VARIETIES, ADDUCED REPRESENTATIONS, WHITTAKER FUNCTIONALS, AND RANK FOR UNITARY REPRESENTATIONS OF $\mathrm{GL}(n)$

DMITRY GOUREVITCH AND SIDDHARTHA SAHI

Abstract. In this paper we study irreducible unitary representations of $GL_n(\mathbb{R})$ and prove a number of results. Our first result establishes a precise connection between the annihilator of a representation and the existence of degenerate Whittaker functionals, for both smooth and K-finite vectors, thereby generalizing results of Kostant, Matumoto and others.

Our second result relates the annihilator to the sequence of adduced representations, as defined in this setting by one of the authors. Based on those results, we suggest a new notion of rank of a smooth admissible representation of $GL_n(\mathbb{R})$, which for unitarizable representations refines Howe's notion of rank.

Our third result computes the adduced representations for (almost) all irreducible unitary representations in terms of the Vogan classification.

We also indicate briefly the analogous results over complex and p-adic fields.

Contents

1. Introduction

Many important problems in harmonic analysis require one to decompose a unitary representation of a real reductive group G on a Hilbert space $\mathcal{H}$, e.g. $\mathcal{H} = L^2(X)$ for some G-space X equipped with an invariant measure. In order to solve such problems one would like to know the irreducible unitary representations of G as explicitly as possible. The starting point is of course the determination of the unitary dual $\widehat{G}$, but then it is helpful to have additional knowledge about invariants of unitary representations, such as their annihilator varieties, existence of Whittaker functionals, etc.

The unitary dual of $G_n = GL(n,\mathbb{R})$ has been determined by Vogan [Vog86] and in this paper we consider the following invariants of $\pi \in \widehat{G_n}$, whose precise definitions are given below in section 1.2:

(1) the annihilator variety $\mathcal{V}(\pi) \subset \mathfrak{gl}(n,\mathbb{C})^*$,
(2) the space $Wh^*_\alpha(\pi)$ of (degenerate) Whittaker functionals of type α,
(3) the depth composition $DC(\pi)$ for iterated adduced representations of π,
(4) the Howe rank $HR(\pi)$ of π.

Our main theorem generalizes a number of existing results in the literature.

Theorem A. *Let $\pi \in \widehat{G_n}$ and let λ be the partition of n such that $\mathcal{V}(\pi) = \overline{\mathcal{O}_\lambda}$. Then*

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(1) $Wh^*_\lambda(\pi) \neq 0$.
(2) $DC(\pi) = \lambda$.

In particular, this implies that the depth composition is non-increasing. By Matumoto's theorem (see Corollary 1.4.1 below) our result implies also that λ is the biggest partition with $Wh^*_\lambda(\pi) \neq 0$. By [He08, Theorems 0.1, 0.2, 4.3], λ connects to Howe's notion of rank by

$$HR(\pi) = \min\left(\lfloor n/2 \rfloor, n - length(\lambda)\right).$$

We will give an independent proof of this result (see Remark 4.2.3).

Before giving the precise definitions of our invariants, we need to fix some notation regarding partitions, compositions, nilpotent orbits, and parabolic subgroups.

1.1. Notation.

Definition 1.1.1. *A* composition *of n of* length *k is a sequence $\alpha = (\alpha_1, ..., \alpha_k)$ of natural numbers (i.e. strictly positive integers) such that $\Sigma\alpha_i = n$; a* partition *is a nondecreasing composition. For a composition α we denote by $\alpha^{\geq}/\alpha^{\leq}$ the nondecreasing/nonincreasing reordering of α.*

Remark 1.1.2. *The Young diagram of a partition is a left alligned array of boxes with λ_i boxes in row i. The rows of the diagram for λ^t are the columns of the diagram for λ.*

We sometimes use "exponential" notation for partitions; thus $4^2 2^1 1^3$ denotes $(4,4,2,1,1,1)$.

We denote $\mathfrak{g}_n := \mathfrak{gl}_n(\mathbb{C})$.

Definition 1.1.3. *If α is a composition of n, we define $J_\alpha \in \mathfrak{g}_n$ to be the Jordan matrix with diagonal Jordan blocks of size $\alpha_1, ..., \alpha_k$; explicitly*

$$(J_\alpha)_{ij} = \begin{cases} 1 & \text{if } j = i+1 \text{ and } i \neq \alpha_1 + ... + \alpha_l \text{ for any } l \\ 0 & \text{else} \end{cases}$$

We define $\mathcal{O}_\alpha$ to be the orbit of J_α under the adjoint action of $GL(n,\mathbb{C})$ on $\mathfrak{g}_n$.

If $\lambda = \alpha^{\geq}$ then we have $\mathcal{O}_\alpha = \mathcal{O}_\lambda$. Moreover by the theorem of Jordan canonical form, for each nilpotent matrix X in $\mathfrak{g}_n$ there is a unique partition λ such that $X \in \mathcal{O}_\lambda$.

Let λ, μ be partitions of n and let $\overline{\mathcal{O}_\lambda}$ denote the Zariski closure of $\mathcal{O}_\lambda$ then we have

$$\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\lambda} \text{ if } \mu_1 + \cdots \mu_k \leq \lambda_1 + \cdots + \lambda_k \text{ for all } k;$$

If λ, μ satisfy this condition, will simply write $\mu \subseteq \lambda$.

Remark 1.1.4. *For X, Y in $\mathfrak{g}_n$ the trace pairing is defined to be*

$$\langle X, Y \rangle = trace\,(XY) \tag{1}$$

Then $\langle \cdot, \cdot \rangle$ is a nondegenerate symmetric $GL(n,\mathbb{C})$-invariant bilinear form (trace form) on $\mathfrak{g}_n$. The trace form gives rise to an isomorphism $\mathfrak{g}_n \approx \mathfrak{g}_n^*$ that intertwines the adjoint and coadjoint actions of $GL(n,\mathbb{C})$. This allows us to identify adjoint orbits and coadjoint orbits.

We next fix our conventions regarding parabolic subgroups of G_n.

If $\alpha = (\alpha_1, ..., \alpha_k)$ is a composition of n then we define

$$S_i = S_i(\alpha) := \{\alpha_1 + \cdots + \alpha_{i-1} + j : 1 \leq j \leq \alpha_i\}$$

For $g \in G_n$ we let g^{ij}_α denote the $\alpha_i \times \alpha_j$ submatrix of g with rows from S_i and columns from S_j.

Definition 1.1.5. *For a composition α we define subgroups $P_\alpha, L_\alpha, N_\alpha$ of G_n as follows*

$$P_\alpha = \left\{g \mid g^{ij}_\alpha = 0 \text{ if } i < j\right\}, L_\alpha = \left\{g \mid g^{ij}_\alpha = 0 \text{ if } i \neq j\right\}, N_\alpha = \left\{g \mid g^{ij}_\alpha = \delta^{ij} \text{ if } i \leq j\right\}.$$

Here δ^{ij} is Kronecker's δ, while 0 and 1 denote zero and identity matrices of appropriate size.

Thus $B = P_{1^n}$ is the standard Borel subgroup of upper triangular matrices, and each P_α is a standard parabolic subgroup containing B. N_α is the nilradical of P_α and $P_\alpha = L_\alpha N_\alpha$ is a Levi decomposition with $L_\alpha \approx G_{\alpha_1} \times \cdots \times G_{\alpha_k}$.

We now introduce the Bernstein-Zelevinsky product notation for parabolic induction.

Definition 1.1.6. *If* $\alpha = (\alpha_1, ..., \alpha_k)$ *is a composition of* n *and* $\pi_i \in \widehat{G_{a_i}}$ *then* $\pi_1 \otimes \cdots \otimes \pi_k$ *is an irreducible unitary representation of* $L_\alpha \approx G_{\alpha_1} \times \cdots \times G_{\alpha_k}$. *We extend this to* P_α *trivially on* N_α *and define*

$$\pi_1 \times \cdots \times \pi_k = Ind_{P_\alpha}^{G_n} (\pi_1 \otimes \cdots \otimes \pi_k),$$

where Ind *denotes normalized induction (see §§2.2).*

Remark 1.1.7. *It follows from* [Vog86] *or from (*[Sah89] *and* [Bar03]*) that if* $\pi_i \in \widehat{G_{a_i}}$ *then* $\pi_1 \times \cdots \times \pi_k \in \widehat{G_n}$. *In this case* $\pi_1 \times \cdots \times \pi_k$ *is unchanged under permutation of the* π_i.

Remark 1.1.8. *Since* G_n/P_α *is compact, one can define* $\pi_1 \times \cdots \times \pi_k$ *analogously in the* C^∞ *category. We refer the reader to* [Vog81] *for details. We will occasionally need to consider this case especially in connection with complementary series construction in the next section and elsewhere.*

1.2. **Invariants of unitary representations.** For a representation π of a Lie group G in a Hilbert space, we denote by π^∞ the space of smooth vectors and by π^ω the space of analytic vectors (see §§2.3). If G is a real reductive group with maximal compact subgroup K we also consider the Harish-Chandra module π^{HC} consisting of K-finite vectors.

By [Wall88, Theorem 3.4.12] if π is an irreducible unitary representation of a reductive group G with Lie algebra $\mathfrak{g}$ and maximal compact subgroup K, then π^{HC} is an irreducible $(\mathfrak{g}, K)$-module and thus π^∞ is a topologically irreducible representation of G.

1.2.1. *The annihilator variety and associated partition.* For an associative algebra A the annihilator of a module (σ, V) is

$$Ann(\sigma) = \{a \in A \colon \sigma(a)v = 0 \text{ for all } v \in V\}$$

If A is abelian then we define the annihilator variety of σ to be the variety corresponding to the ideal $Ann(\sigma)$, i.e. $\mathcal{V}(\sigma) = \text{Zeroes}(Ann(\sigma))$.

If (σ, V) is a module for a Lie algebra $\mathfrak{g}$, then one can apply the above considerations to the enveloping algebra $U(\mathfrak{g})$. While $U(\mathfrak{g})$ is not abelian it admits a natural filtration such that $gr(U(\mathfrak{g}))$ is the symmetric algebra $S(\mathfrak{g})$, and one has a "symbol" map gr from $U(\mathfrak{g})$ to $S(\mathfrak{g})$. We let $gr(Ann(\sigma))$ be the ideal in $S(\mathfrak{g})$ generated by the symbols $\{gr(a) \mid a \in Ann(\sigma)\}$ and define the annihilator variety of σ to be

$$\mathcal{V}(\sigma) = \text{Zeroes}(gr(Ann(\sigma))) \subset \mathfrak{g}^*$$

If $\mathfrak{g}$ is a complex reductive Lie algebra and M is an irreducible $\mathfrak{g}$-module, then it was shown by Joseph (see [Jos85]) that $\mathcal{V}(M)$ is the closure $\overline{\mathcal{O}}$ of a single nilpotent coadjoint orbit $\mathcal{O}$.

If π is a Hilbert space representation of a Lie group G then we define $\mathcal{V}(\pi) := \mathcal{V}(\pi^\omega)$. If G is reductive and π is an admissible (e.g. irreducible unitary) representation then $\pi^{HC} \subset \pi^\omega \subset \pi^\infty$, π^{HC} is dense in π^∞ and the action of $U(\mathfrak{g})$ is continuous. Thus, $\mathcal{V}(\pi) = \mathcal{V}(\pi^{HC})$.

If π is an irreducible unitary representation then π^{HC} is an irreducible $(\mathfrak{g}, K)$-module and thus is a finite direct sum of algebraically irreducible representations of $\mathfrak{g}$. These representations are K-conjugate and thus have the same annihilator variety. Thus [Jos85] implies that $\mathcal{V}(\pi)$ consists of a single nilpotent coadjoint orbit, that we call the *associated orbit.*

Definition 1.2.2. *If* λ *is a partition of* n *such that* $\mathcal{V}(\pi) = \overline{\mathcal{O}_\lambda}$ *we call* λ associated partition *and denote* $\lambda = AP(\pi)$.

For example, if π is finite-dimensional then $\mathcal{V}(\pi) = \{0\}$ and $AP(\pi) = 1^n$ and if π is generic then, by a result of Kostant (see §§1.4), $\mathcal{V}(\pi)$ is the nilpotent cone of $\mathfrak{g}_n^*$ and $AP(\pi) = n^1$.

For an admissible representation π of a real reductive group G with Lie algebra $\mathfrak{g}_\mathbb{R}$ and complexified Lie algebra $\mathfrak{g}$ one can define more refined invariants such as

(1) the asymptotic support $AS(\pi) \subset \mathfrak{g}_\mathbb{R}^*$
(2) the wave front set $WF(\pi) \subset \mathfrak{g}_\mathbb{R}^*$ (see e.g. [SV00]
(3) the associated variety $AV(\pi) \subset \mathfrak{k}^\perp \subset \mathfrak{g}^*$ (see e.g. [BV78]). Here $\mathfrak{k}$ denotes the complexified Lie algebra of the maximal compact subgroup.

By [Ros95] and [SV00], these three invariants determine each other and each of them determines $\mathcal{V}(\pi)$. For $GL(n, \mathbb{R})$ the converse is true as well, and since we are primarily interested in this case we will mostly ignore the refined invariants in this paper.

1.2.3. *Degenerate Whittaker functionals.* In this section we fix n, and write N for $N_{(1^n)}$; thus N is the subgroup of G_n consisting of unipotent upper triangular matrices. Let $\mathfrak{n}_{\mathbb{R}}$ be the Lie algebra of N and let $\mathfrak{n}$ be the complexification of $\mathfrak{n}_{\mathbb{R}}$.

Let Ψ denote the set of multiplicative unitary characters of N. Then Ψ can be identified with a subset of $\mathfrak{n}^*$ via the exponential map. More precisely we have

$$\Psi \approx \{\psi \in \mathfrak{n}^* \mid \psi([\mathfrak{n},\mathfrak{n}]) = 0,\, \psi(\mathfrak{n}_{\mathbb{R}}) \subset i\mathbb{R}\}$$

where an element ψ of the right side is regarded as character of N via the formula

$$\psi(\exp X) = e^{\psi(X)} \text{ for } X \in \mathfrak{n}_{\mathbb{R}}.$$

We will write $\mathbb{C}_\psi$ for the one-dimensional space regarded as a module for N or $\mathfrak{n}$ via ψ.

Definition 1.2.4. *If $\pi \in \widehat{G_n}$ and $\psi \in \Psi$ we define*

$$Wh'_\psi(\pi) = \mathrm{Hom}_{\mathfrak{n}}\left(\pi^{HC}, \mathbb{C}_\psi\right),\ Wh^*_\psi(\pi) = \mathrm{Hom}_N^{cont}(\pi^\infty, \mathbb{C}_\psi)$$

where Hom_N^{cont} *denotes the space of* continuous *N-homomorphisms.*

It is well known that π^{HC} is dense in π^∞, hence by restriction we get an inclusion

$$Wh^*_\psi(\pi) \subseteq Wh'_\psi(\pi) \tag{2}$$

Moreover π^{HC} is finitely generated as an $\mathfrak{n}$-module and hence we get

$$\dim Wh^*_\psi(\pi) \le \dim Wh'_\psi(\pi) < \infty$$

We refer to elements of $Wh^*_\psi(\pi)$ and $Wh'_\psi(\pi)$ as (degenerate) Whittaker functionals of type ψ.

Remark. *Let $\bar{\mathfrak{n}}$ be the space of strictly lower triangular matrices. Then the trace form restricts to a nondegenerate pairing of $\bar{\mathfrak{n}}$ with $\mathfrak{n}$, allowing one to identify $\bar{\mathfrak{n}} \approx \mathfrak{n}^*$. Under this identification elements of Ψ correspond to imaginary "subdiagonal" matrices, i.e to matrices $X \in \mathfrak{g}_n$ satisfying*

$$X_{pq} \in i\mathbb{R} \text{ if } p = q+1 \text{ and } X_{pq} = 0 \text{ if } p \neq q+1$$

Since we have identified $\mathfrak{g}_n^* \approx \mathfrak{g}_n$ via the trace form, the above remark also allows us to regard $\mathfrak{n}^*$ as a subspace of $\mathfrak{g}_n^*$. Hence we may also regard Ψ as a subspace of $\mathfrak{g}_n^*$.

Definition 1.2.5. *Let α be a composition of n, J_α be the corresponding Jordan matrix as in Definition 1.1.3, and w be the longest element of the Weyl group. Then $wJ_\alpha w^{-1}$ is subdiagonal and $iwJ_\alpha w^{-1}$ can be regarded as an element of Ψ by the above remark. We denote this character by ψ_α. Note that $\psi_\alpha \in \mathcal{O}_\alpha$.*

For $\pi \in \widehat{G_n}$ denote also

$$Wh'_\alpha(\pi) = Wh'_{\psi_\alpha}(\pi),\quad Wh^*_\alpha(\pi) = Wh^*_{\psi_\alpha}(\pi) \tag{3}$$

1.2.6. *The adduced representation, "derivatives "and the depth composition.* In [BZ77, Ber84] Bernstein and Zelevinsky introduced the important notion of "derivative" for representations of $GL(n, \mathbb{Q}_p)$. An Archimedean analog of the "highest" derivative for $\pi \in \widehat{G_n}$ was defined in [Sah89], where it was called the adduced representation and denoted $A\pi$. This definition which we now recall, involves two ingredients.

Let $P_n \subset G_n$ be the "mirabolic" subgroup consisting of matrices with last row $(0, 0, ..., 0, 1)$. To forestall confusion we note that P_n is *not* a parabolic subgroup, it has codimension 1 in $P_{(n-1,1)}$ and is completely different from $P_{(n)} = G_n$.

The first ingredient in the definition of $A\pi$ is the following result that was conjectured by Kirillov.

Theorem 1.2.7. *Let $\pi \in \widehat{G_n}$, then $\pi|_{P_n}$ is irreducible.*

This was first proven in the p-adic case in [Ber84], then in the complex case in [Sah89], and finally in the real case in [Bar03]. Three new proofs have been obtained recently in [AG09, SZ, GaLa].

The second ingredient is Mackey theory that describes the unitary dual of Lie groups, such as P_n, which are of the form $G = H \ltimes Z$ with Z abelian. In this case $\widehat{Z}$ consists of unitary characters and H acts on $\widehat{Z}$; for $\chi \in \widehat{Z}$ let S_χ denote its stabilizer in H. If $\sigma \in \widehat{S_\chi}$ then $\sigma \otimes \chi$ is a unitary representation of $S_\chi \ltimes Z$, and we define

$$I_\chi(\sigma) = Ind^G_{S_\chi \ltimes Z}(\sigma \otimes \chi)$$

The main results of Mackey theory are as follows:

(a) $I_\chi(\sigma)$ is irreducible for all $\sigma \in \widehat{S_\chi}$;

(b) $\widehat{G}$ is the disjoint union of $I_\chi\left(\widehat{S_\chi}\right)$ as χ ranges over representatives of distinct H-conjugacy classes in $\widehat{Z}$.

Since $P_n \approx G_{n-1} \ltimes \mathbb{R}^{n-1}$ we may analyze $\widehat{P_n}$ by Mackey theory. There are two G_{n-1}-conjugacy classes in $\widehat{\mathbb{R}^{n-1}}$; one class consists of the trivial character χ_0 alone, while the other class contains all other characters. As a representative of the second class we pick the character χ_1 defined by

$$\chi_1(a_1, ..., a_{n-1}) = \exp(ia_{n-1})$$

The stabilizers in G_{n-1} are $S_{\chi_0} = G_{n-1}$ and $S_{\chi_1} = P_{n-1}$ and therefore we get

$$\widehat{P_n} = I_{\chi_0}\left(\widehat{G_{n-1}}\right) \coprod I_{\chi_1}\left(\widehat{P_{n-1}}\right) \tag{4}$$

We may iterate (4) until we arrive at the trivial group $P_1 = G_0$

$$\widehat{P_n} = \coprod_{k=1}^{n} I_{\chi_1}^{k-1} I_{\chi_0}\left(\widehat{G_{n-k}}\right)$$

Combining this with Theorem 1.2.7 we deduce that for $\pi \in \widehat{G_n}$ with $n > 0$, there exists a unique natural number d and a unique $\pi' \in \widehat{G_{n-d}}$ such that

$$\pi|_{P_n} = I_{\chi_1}^{d-1} I_{\chi_0}(\pi') \tag{5}$$

Definition 1.2.8. [Sah89] *If $\pi \in \widehat{G_n}$ and $\pi' \in \widehat{G_{n-d}}$ satisfy (5) we say that π has depth d and that π' is the adduced representation (or highest derivative) of π, and we write $\pi' = A\pi$.*

Let $\star$ denote the trivial representation of the trivial group G_0. The procedure of taking the adduced representation can be iterated until we arrive at $\star$. Thus we obtain a sequence of unitary representations

$$(\pi_0 = \pi, \pi_1, \ldots, \pi_{l-1}, \pi_l = \star) \text{ satisfying } \pi_j = A(\pi_{j-1}) \text{ for } 1 \le j \le l. \tag{6}$$

and we write $d_i = d_i(\pi)$ for the depth of π_{i-1}. Note that $d_1, d_2, \ldots, d_l$ are natural numbers, and their sum is precisely n.

Definition 1.2.9. *The composition $(d_1, d_2, \ldots, d_l)$ is called the* depth composition *of $\pi \in \widehat{G_n}$ and is denoted $DC(\pi)$.*

1.2.10. *Howe Rank.* P_α is called a maximal parabolic if α has length 2 so that $\alpha = (a, b)$ with $a+b = n$. In this case N_α is abelian and isomorphic to $M_{a\times b}$, the additive group of $a\times b$ real matrices. The unitary dual of N_α consists of unitary characters and can also be identified with $M_{a\times b}$ via $\chi_y(x) = \exp(iTr(xy^t))$. The group $L_\alpha = G_a \times G_b$ acts on $M_{a\times b}$ in the usual manner, and the orbits are $\mathcal{R}_0, \ldots, \mathcal{R}_{\min(a,b)}$ where $\mathcal{R}_k$ denotes the set of matrices of rank k. Note that $\min(a, b) \le \lfloor n/2 \rfloor$ and that equality holds for $a = \lfloor n/2 \rfloor, b = n - \lfloor n/2 \rfloor$.

We now briefly describe the theory of Howe rank for G_n; let us fix $\alpha = (a, n-a)$ as above. If $\pi \in \widehat{G_n}$ then by Stone's theorem the restriction $\pi|_{N_\alpha}$ corresponds to a projection-valued Borel measure μ^π on $\widehat{N_\alpha} \approx M_{a\times(n-a)}$. Since π is a representation of P_α, μ^π is P_α-invariant and decomposes as a direct sum

$$\mu^\pi = \mu_0^\pi + \cdots + \mu_{\min(a,n-a)}^\pi \text{ with } \mu_k^\pi(E) = \mu^\pi(E \cap \mathcal{R}_k)$$

Building on the work of Howe [How82], Scaramuzzi [Sca90] proved that μ_π has "pure rank", i.e. there is some integer $k = HR(\pi, a)$ such that $\mu^\pi = \mu_k^\pi$. Moreover if we define $HR(\pi) = HR(\pi, \lfloor n/2 \rfloor)$ then

$$HR(\pi, a) = \min(HR(\pi), a, n-a) \text{ for all } a \le n$$

Definition 1.2.11. *For $\pi \in \widehat{G_n}$ the integer $HR(\pi)$ is called the Howe rank of π.*

1.3. **Results over other local fields and a uniform formulation of Theorem A.** Theorem A also holds in the complex case, and the proof is very similar. We comment on that in section 5.

In section 6 we prove a p-adic analog of Theorem A, using [BZ77, MW87, Zel80]. In the p-adic case one cannot define annihilator variety, but one can consider the wave front set $WF(\pi)$. It is not known to be the closure of a single nilpotent orbit for general reductive group but for irreducible smooth representations of $GL_n(F)$ this is proven to be the case in [MW87].

Since in the p-adic case the notion of derivative is defined for all smooth representations, this analog does not require the representation to be unitary. This gives us the following uniform formulation of the main theorem.

Theorem B. *Let F be a local field of characteristic zero. Let π be an irreducible smooth admissible representation of* $\mathrm{GL}(n,F))$ *and let $WF(\pi) \subset \mathfrak{gl}_n(F)$ denote the wave front set of π. Let λ be the partition of n such that $WF(\pi) = \overline{\mathcal{O}_\lambda}$. Suppose that either F is non-Archimedean or π is unitarizable. Then*

(1) $Wh^*_\lambda(\pi) \neq 0$, and for any composition α with $Wh^*_\alpha(\pi) \neq 0$ we have $\mathcal{O}_\alpha \subset \overline{\mathcal{O}_\lambda}$.
(2) $DC(\pi) = \lambda$; in particular $DC(\pi)$ is a non-increasing sequence.
(3) λ is the transpose of the classification partition of π (see Remark 1.3.1 below).

Moreover, if π is unitarizable then $HR(\pi) = \min(\lfloor n/2 \rfloor, n - length(\lambda))$.

If F is non-Archimedean we also have $\dim Wh^*_{DC(\pi)}(\pi) = 1$.

In the p-adic case, Wh^*_α denotes the space of all linear equivariant functionals, since all representations are considered in discrete topology.

Remark 1.3.1. *The classification partition is defined in* [Ven05] *for all irreducible unitary representations, through the Tadic-Vogan classification, and in* [OS09] *for all irreducible smooth representations through the Zelevinsky classification. For representations of Arthur type, this partition describes the $SL(2)$-type of the representation of the Weil-Deligne group corresponding to π by local Langlands correspondence.*

In the Archimedean case, Theorem B follows from Theorem A, Corollary 1.4.1, Theorem 4.2.1 and Remark 4.2.3 below.

1.4. **Earlier works.** It was proven by Casselman-Zuckerman for G_n (unpublished) and by Kostant ([Kos78]) for all quasi-split reductive groups that for a generic character ψ of N, $Wh^*_\psi(\pi) \neq 0$ if and only if $Wh'_\psi(\pi) \neq 0$ if and only if $\mathcal{V}(\pi)$ is maximal possible, i.e. equal to the nilpotent cone. Matumoto ([Mat87, Theorem 1]) proved a generalization of one direction of this statement. For the case of G_n his theorem implies

Corollary 1.4.1. *If α is a composition of n, and M is a $\mathfrak{g}_n$-module such that* $\mathrm{Hom}_{\mathfrak{n}}(M, \psi_\alpha) \neq 0$ *then $\mathcal{O}_\alpha \subset \mathcal{V}(M)$.*

Over p-adic fields, a connection between wave front set and generalized Whittaker functionals was investigated in [MW87] for smooth (not necessary unitarizable) representations of any reductive group. However, the main theorem of [MW87] involves a lot of choices and for the case of GL_n can be made much more concrete, using derivatives and Zelevinsky classification, following [BZ77, Zel80].

Several works (e.g. [GW80, Mat90, Mat88]) give a partial Archimedean analog of the results of [MW87], by considering non-degenerate ("admissible") characters of the (smaller) nilradicals of bigger parabolic subgroups. However, much less is known in the Archimedean case. Our work establishes a different type of analog: instead of considering non-degenerate characters of smaller nilradicals, we consider degenerate characters of the nilradical of the standard Borel subgroup. Following Zelevinsky [Zel80, §§8.3], we call functionals equivariant with respect to such characters *degenerate Whittaker functionals.*

1.5. **Generalizations in future works.** In [AGS], we define the notion of highest derivative for all admissible smooth Fréchet representations of G_n. We show that this notion extends the notion of adduced representation discussed in the current paper, and establish several properties of highest derivative analogous to the ones proven in the p-adic case in [BZ77]. We apply those properties to questions raised in the current paper. Namely, we complete the computation of adduced representations for all Speh complementary series, and prove that $\dim Wh^*_{DC(\pi)}(\pi) = 1$ for all $\pi \in \widehat{G_n}$.

In our work in progress [GS] we prove that

$$Wh^*_\alpha(\pi) \neq 0 \Leftrightarrow Wh'_\alpha\left(\pi^{HC}\right) \neq 0 \Leftrightarrow \mathcal{O}_\alpha \subset \mathcal{V}(\pi).$$

for any irreducible admissible smooth Fréchet representation π of G_n and any composition α of n.

Furthermore, we prove the following generalization for any quasi-split real reductive group G. Let N be the nilradical of a Borel subgroup of G, and K be maximal compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{n}$ be the complexified Lie algebras of G and N. Let π be an irreducible admissible smooth representation of G. Denote

$$\Psi(\pi) := \{\psi \in \Psi \text{ s.t. } Wh^*_\psi(\pi) \neq 0\}, \text{ and } \Psi(\pi^{HC}) := \{\psi \in \Psi \text{ s.t. } Wh'_\psi(\pi) \neq 0\}.$$

In [GS] we prove

(1) $\Psi(\pi^{HC}) = pr_{\mathfrak{n}^*}(AV(\pi^{HC}))\cap\Psi$
(2) $\Psi(\pi) \subset \Psi(\pi^{HC}) \cap i\Psi(\mathbb{R})$, and under certain condition on G equality holds.

1.6. **The proposed notion of rank.** We suggest the following definition of rank.

Definition 1.6.1. *If π is a smooth Fréchet representation of G_n, we define the rank of π, written* $\mathrm{rk}(\pi)$*, to be the maximum rank of a matrix in $\mathcal{V}(\pi)$.*

If $\mathcal{V}(\pi)$ is a closure of a single orbit given by a partition of length k then $\mathrm{rk}(\pi) = n - k$. Theorem B implies that for $\pi \in \widehat{G_n}$ our notion of rank agrees with the notion proposed in [Sah89], and extends Howe's notion of rank. In subsection 4.2 we compute (extended) ranks of all irreducible unitary representations of G_n in terms of the Vogan classification.

Our definition extends literally to all classical groups, and connects to Howe's notion of rank by [He08, Theorem 0.4].

For G_n we can give another interpretation of rank, in terms of parabolic induction. Let λ be a partition of n, $\alpha := \lambda^{\leq}$ be the inverse reordering of λ, P_α the corresponding standard parabolic subgroup, L_α its Levi component, N_α its nilradical, and $\mathfrak{n}_\alpha$ its Lie algebra. Consider the "naive" (non-exact) Jacquet restriction functor r_α that maps π to $\pi/\mathfrak{n}_\alpha\pi$. This functor is adjoint to parabolic induction from P_α. Note that $Wh^*_\alpha(\pi)$ is equal to the space $Wh^*(r_\alpha(\pi))$ of generic (classical) Whittaker functionals on $r_\alpha(\pi)$. Therefore, Theorem A implies that for $\pi \in \widehat{G_n}$, $\mathrm{rk}(\pi) \geq k$ if and only if the Jacquet restriction of π to some Levi subgroup of semi-simple rank k is generic.

The $\mathrm{rk}(\pi)$ is also equal to the (real) dimension of the variety consisting of unitary characters ψ of N such that $Wh^*_\psi(\pi) \neq 0$.

For $G \neq G_n$ we do not have equivalent descriptions of the rank using Jacquet functor or the space of functionals.

1.7. **Application to construction of Klyachko models.** Our technique can be applied to construction of another family of models for unitary representations of G_n, called Klyachko models. Let F be a local field. For any decomposition $n = r + 2k$ define a subgroup $H_{r,2k} \subset G_n$, which is a semi-direct product of $Sp_{2k}(F)$ and the nilradical $N_{(1,\ldots,1,2k)}$ of the standard parabolic $P_{(1,\ldots,1,2k)}$, and let $\phi_{2k,r}$ denote the generic character $\psi_{(n)}$ restricted to $N_{(1,\ldots,1,2k)}$ and then extended trivially to $H_{r,2k}$. Then the r-th Klyachko model of $\pi \in \widehat{G_{r+2k}}$ is defined to be $\mathrm{Hom}_{H_{r,2k}}(\pi^\infty, \phi_{r,2k})$.

Offen and Sayag proved existence, uniqueness and disjointness of Klyachko models over p-adic fields. They defined the appropriate r in terms of Tadic classification, used derivatives to reduce the statement to the case $r = 0$, in which an Sp-invariant functional is constructed using a global (automorphic) argument. Theorem B allows one to define r in terms of the partition corresponding to π, and then to extend the construction of Klyachko models given in [OS07, §3] to the Archimedean case. This is done in [GOSS].

1.8. **Structure of the paper.** In section 2 we give several necessary definitions and preliminary results on geometry of coadjoint orbits, analytic and smooth vectors, and annihilator varieties.

In section 3 we prove the main theorem by induction. First we note that $\pi|_{P_n}$ is induced from $A\pi$. This gives us a map $\pi^\infty \to (A\pi)^\infty$ which has dense image and satisfies a certain equivariant condition with respect to a subgroup of P_n that includes N. This also enables us to compute the dimension of $\mathcal{V}(\pi|_{P_n})$ in terms of $\dim \mathcal{V}(A\pi)$. Using the induction hypothesis and the map $\pi^\infty \to (A\pi)^\infty$, we show that $Wh^*_{DC(\pi)}(\pi) \neq 0$. By Matumoto's theorem this implies $\overline{\mathcal{O}_{DC(\pi)}} \subset \mathcal{V}(\pi)$. To show equality we prove that the dimension of the annihilator variety does not drop when we restrict π to P_n, and then use the induction hypothesis.

In section 4 we compute the adduced representation for representations of Speh complementary series (except one special case), thus almost finishing the computation of adduced representations of all unitary irreducible representations. We use the restriction on the annihilator variety of the adduced representation given by Theorem A and a restriction on the infinitesimal character of the adduced representation that we deduce from Casselman - Osborne lemma. Those two restrictions allow us to determine the adduced representation uniquely, except in one case where we present two possibilities.

In section 5 we explain how our proofs adapt to the complex case.

In section 6 we deal with the p-adic case. Most of the arguments here are simply sketched since the components of various proofs appear already in [Zel80] and [MW87].

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2. Preliminaries

2.1. **Induction and dimensions of nilpotent orbits.** We now recall Lusztig-Spaltenstein induction of nilpotent orbits. Let $\mathfrak{g}$ be a complex reductive Lie algebra. Let $\mathfrak{p}$ be a parabolic subalgebra with nilradical $\mathfrak{n}$ and Levi quotient $\mathfrak{l} := \mathfrak{p}/\mathfrak{n}$. Then we have a natural projection $pr\colon \mathfrak{g}^* \twoheadrightarrow \mathfrak{p}^*$ and a natural embedding $\mathfrak{l}^* \subset \mathfrak{p}^*$.

Theorem 2.1.1 ([CoMG93], Theorem 7.1.1)**.** *In the above situation let $\mathcal{O}_\mathfrak{l} \subset \mathfrak{l}^*$ be a nilpotent orbit. Then there exists a unique nilpotent orbit $\mathcal{O}_\mathfrak{g}$ that meets $pr^{-1}(\mathcal{O}_\mathfrak{l})$ in an open dense subset. We have*

$$\dim \mathcal{O}_\mathfrak{g} = \dim \mathcal{O}_\mathfrak{l} + 2 \dim \mathfrak{n}.$$

Definition 2.1.2. *The orbit $\mathcal{O}_\mathfrak{g}$ is denoted $Ind_\mathfrak{l}^\mathfrak{g}(\mathcal{O}_\mathfrak{l})$ and called the induced orbit of $\mathcal{O}_\mathfrak{l}$.*

Theorem 2.1.3 ([BB89], Theorem 2)**.** *Let G be a real reductive Lie group. If π is an irreducible representation of G that is parabolically induced from a representation σ of a Levi subgroup L, then $\mathcal{O}(\pi) = Ind_\mathfrak{l}^\mathfrak{g}[\mathcal{O}(\sigma)]$.*

We now specialize this discussion to $\mathfrak{g} = \mathfrak{g}_n$. In this case nilpotent obits are described by partitions.

Definition 2.1.4. *If λ, μ are partitions of p, q of lengths k, l respectively, we define a partition $\lambda + \mu$ of $p + q$ of length $m = \max(k, l)$ as follows:*

$$(\lambda + \mu)_i = \lambda_i + \mu_i \text{ for } i \leq m$$

where the missing λ_i, μ_i are treated as 0.

Remark 2.1.5. *We can describe $\lambda+\mu$ using transposed partitions: $(\lambda + \mu)^t$ is the partition rearrangement of the composition (λ^t, μ^t). On the level of Young diagrams: $\lambda+\mu$ is obtained by concatenating the columns of λ and μ and reordering them in descending order.*

Proposition 2.1.6 ([CoMG93], Lemma 7.2.5)**.** *Let λ be a partition of l, and μ be a partition of m. Then*

$$Ind_{\mathfrak{g}_l \times \mathfrak{g}_m}^{\mathfrak{g}_{l+m}}(\mathcal{O}_\lambda \times \mathcal{O}_\mu) = \mathcal{O}_{\lambda+\mu}.$$

Proposition 2.1.7 ([CoMG93], Corollary 7.2.4.). *If λ is a partition of n and $\nu = \lambda^t$ then*

$$\dim \mathcal{O}_\lambda = n^2 - \sum_j \nu_j^2.$$

We recall the BZ-product notation for parabolic induction as in Definition 1.1.6 and the associated partition $AP(\pi)$ as in Definition 1.2.2. From Proposition 2.1.6 we obtain

Corollary 2.1.8. *If $\sigma \times \tau$ is irreducible then*

$$AP(\sigma \times \tau) = AP(\sigma) + AP(\tau).$$

2.2. **Unitary induction.** Let G be a Lie group and let dx denote a *right* invariant Haar measure then we have

$$\int \delta_G(g) f(gx)\, dx = \int f(x)\, dx$$

where $\delta_G(g) = |\det\ Ad(g)|$ is the modular function of G.

Let H be a closed subgroup of G and write $\delta_{H\backslash G}(h) = \delta_H(h)\,\delta_G(h)^{-1}$. We define $C_c\left(H\backslash G, \delta_{H\backslash G}\right)$ to be the space of continuous functions $f \in C(G)$ that satisfy

$$f(hx) = \delta_{H\backslash G}(h) f(x) \text{ for all } h \in H \tag{7}$$

and for which $\mathrm{supp}(f) \subseteq HS$ for some compact set S. Then G acts on $C_c\left(H\backslash G, \delta_{H\backslash G}\right)$ by right translations and there is a unique continuous G-invariant functional $\int_{H\backslash G}$ on $C_c\left(H\backslash G, \delta_{H\backslash G}\right)$ satisfying

$$\int_{H\backslash G}\left[\int_H \delta_{H\backslash G}(h)^{-1} f(xh)\, dh\right] = \int_G f(x)\, dx \text{ for all } f \in C_c(G)$$

One shows that $f \mapsto \int_H \delta_{H\backslash G}(h)^{-1} f(xh)\, dh$ is a continuous surjection from $C_c(G)$ to $C_c(H\backslash G)$ whose kernel is densely spanned by functions $f(x) - \delta_G(h) f(hx)$. Consequently $\int_G dx$ vanishes on the kernel and descends to a functional on $C_c\left(H\backslash G, \delta_{H\backslash G}\right)$.

If (σ, V) is a unitary representation of H, we define $C_c\left(H\backslash G, \delta_{H\backslash G}^{1/2}\sigma\right)$ to be the set of continuous functions $f\colon G \to V$ that satisfy

$$f(hg) = \delta_{H\backslash G}^{1/2}(h)\,\sigma(h) f(g) \tag{8}$$

and for which $\mathrm{supp}(f) \subseteq HS$ for some compact set S. For such f we have $\|f(x)\|_V^2 \in C_c\left(H\backslash G, \delta_{H\backslash G}\right)$ and we define W to be the closure of $C_c\left(H\backslash G, \delta_{H\backslash G}^{1/2}\sigma\right)$ with respect to the norm $\int_{H\backslash G}\|f(x)\|_V^2$. Then W is a Hilbert space under $\langle f_1, f_2\rangle = \int_{H\backslash G}\langle f_1(g), f_2(g)\rangle_V$ and the action of G by right translations defines a unitary representation (π, W) of G.

Definition 2.2.1. *In the above situation (π, W) is called the unitarily induced representation and denoted by $Ind_H^G(\sigma)$.*

2.3. **Analytic and smooth vectors.**

Definition 2.3.1. *Let M be an analytic manifold and B be a Banach space. A map $M \to B$ is said to be* smooth *if it is infinitely differentiable, and* analytic *if its Taylor series at every point has a positive radius of convergence.*

Remark 2.3.2. *A map $f\colon M \to B$ is smooth/analytic if and only if the composition $\phi \circ f$ is smooth/analytic for every continuous linear functional $\phi\colon B \to \mathbb{C}$.*

Definition 2.3.3. *Let (σ, B) be a continuous Banach representation of a Lie group G. A vector $v \in B$ is called smooth/analytic if the action map $G \to B$ defined by $g \mapsto \sigma(g)v$ is smooth/analytic. Both G and its Lie algebra $\mathfrak{g}$ act on the spaces of smooth and analytic vectors and we denote the corresponding representations by $(\sigma^\infty, B^\infty)$ and $(\sigma^\omega, B^\omega)$ respectively. $(\sigma^\infty, B^\infty)$ is naturally a Fréchet representation of G.*

Remark 2.3.4. *By* [Nel59, Theorem 4]*, B^ω is dense in B.*

The following theorem by Poulsen can be interpreted as a representation-theoretic version of Sobolev's embedding theorem.

Theorem 2.3.5 ([Pou72], Theorem 5.1 and Corollary 5.2)**.** *Let $(\pi, W) = Ind_H^G(\sigma)$; if $f \in W^\infty$ (W^ω) then f is a smooth (analytic) function from G to V.*

Corollary 2.3.6. *Let $(\pi, W) = Ind_H^G(\sigma)$ be as above; if $f \in W^\infty$ (W^ω) then f takes values in V^∞ (V^ω).*

Proof. Let $f \in W^\infty$ (W^ω) and let $v := f(g)$, then the action map of H on v is

$$h \mapsto \sigma(h)\, v = \sigma(h)\, f(g) = \delta_{H\backslash G}^{-1/2}(h)\, f(hg)$$

This is a smooth (analytic) map by Poulsen's theorem; hence v is in V^∞ (V^ω). □

Corollary 2.3.7. *Let $(\pi, W) = Ind_H^G(\sigma)$; then $f \mapsto f(e)$ defines a continuous H-equivariant morphism $(\pi^\infty, W^\infty) \twoheadrightarrow (\widetilde{\sigma}^\infty, V^\infty)$, where $\widetilde{\sigma}(h) = \delta_{H\backslash G}^{1/2}(h)\, \sigma(h)$ as in (8) above.*

The continuity of the evaluation morphism follows from [Pou72, Lemma 5.2].

Definition 2.3.8. *Let V_1, V_2 be two modules for a Lie algebra $\mathfrak{g}$; we say V_1, V_2 are non-degenerately $\mathfrak{g}$-paired if there is a non-degenerate bilinear pairing $\langle \cdot, \cdot \rangle$ that is invariant in the Lie algebra sense,* i.e.

$$\langle Xv_1, v_2 \rangle = -\langle v_1, Xv_2 \rangle \text{ for all } v_1 \in V_1, v_2 \in V_2, X \in \mathfrak{g}$$

Lemma 2.3.9. *Let $(\pi, W) = Ind_H^G(\sigma)$; suppose G/H is connected and $\widetilde{\sigma}^\omega$ is non-degenerately $\mathfrak{h}$-paired with an $\mathfrak{h}$-module τ. Then π^ω is non-degenerately $\mathfrak{g}$-paired with a quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \tau$.*

Here and elsewhere $\mathfrak{g}$, $\mathfrak{h}$ denote the Lie algebras of G, H and $U(\mathfrak{g})$, $U(\mathfrak{h})$ their enveloping algebras.

Proof. As noted above $\mathfrak{g}$ and hence $U(\mathfrak{g})$ acts on W^ω. Let $u \mapsto u'$ be the principal anti-automorphism of $U(\mathfrak{g})$ extending $X \mapsto -X$ on $\mathfrak{g}$, and define a pairing between π^ω and $U(\mathfrak{g}) \otimes \tau$ by $\langle f, u \otimes v \rangle := \langle (u'f)(e), v \rangle$. The pairing descends to $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \tau$ in the second variable since if X is in $\mathfrak{h}$ then

$$\langle f, uX \otimes v \rangle = -\langle [\pi^\omega(X)\, u'f](e), v \rangle = -\langle \widetilde{\sigma}^\omega(X)\, [u'f(e)], v \rangle = \langle u'f(e), \tau(X)\, v \rangle = \langle f, u \otimes Xv \rangle$$

Let us check that the pairing is non-degenerate in the first variable. If f lies in the left kernel, then $\langle (u'f)(e), v \rangle = 0$ for any u, v and hence f vanishes at e together with all its derivatives. By Theorem 2.3.5, f is an analytic function and therefore f vanishes in the connected component of e. Since the support of f is H-invariant and G/H is connected this implies $f = 0$.

Now if the pairing is degenerate in the second variable, we quotient $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \tau$ by the right kernel. □

2.4. **Gelfand-Kirillov dimension.** Let A be a finitely generated associative algebra over $\mathbb{C}$ with increasing filtration $\{F^i A : i \geq 0\}$, and let M be a finitely generated A-module. Choose a finite dimensional generating subspace S, define a filtration of M by $F^i M := (F^i A)\, S$. It is known that there exists a polynomial p such that $p(i) := \dim F^i M$ for large enough i, and that the degree of p does not depend on the choice of the finite dimensional generating subspace S. This degree is called the Gelfand-Kirillov dimension of M and denoted by $\mathrm{GKdim}(M)$.

We will apply this in particular to the case when A is the universal enveloping algebra $U(\mathfrak{g})$ for some complex Lie algebra $\mathfrak{g}$, equipped with the usual filtration inherited from the tensor algebra $T(\mathfrak{g})$.

Lemma 2.4.1 ([Vog78], Lemma 2.3)**.** *Let $\mathfrak{h} \subset \mathfrak{g}$ be Lie algebras, and let M be a finitely generated $U(\mathfrak{h})$ module. Then $N = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} M$ is a finitely generated $U(\mathfrak{g})$-module, and we have*

$$\mathrm{GKdim}(N) = \mathrm{GKdim}(M) + \dim(\mathfrak{g}/\mathfrak{h})$$

If (σ, M) is a finitely generated $\mathfrak{g}$-module, we will sometimes write $\mathrm{GKdim}(\sigma)$ for $\mathrm{GKdim}(M)$ etc. Recall the annihilator variety $\mathcal{V}(\sigma) \subset \mathfrak{g}^*$ as defined in §§§1.2.1. It is easy to show that

$$\mathrm{GKdim}(\sigma) \leq \dim \mathcal{V}(\sigma).$$

There is also a bound in the other direction for the Lie algebras we consider in this paper.

Theorem 2.4.2 (Gabber-Joseph)**.** *Let $\mathfrak{g}$ be the Lie algebra of an algebraic group over an algebraically closed field. Let M be a finitely generated $U(\mathfrak{g})$ module. Then*

$$\mathrm{GKdim}(M) \geq \frac{1}{2} \dim \mathcal{V}(M).$$

For proof see [Jos81, Proposition 6.1.4] or [KrLe99, Theorem 9.11]. This theorem does not hold for general Lie algebras. On the other hand a stronger result is true for reductive Lie algebras.

Theorem 2.4.3 ([Vog78], Theorem 1.1)**.** *Let G be a real reductive group, $\mathfrak{g}$ be its complexified Lie algebra and K be a maximal compact subgroup. Let (σ, M) be an irreducible Harish-Chandra module over $(\mathfrak{g}, K)$. Then*

$$\mathrm{GKdim}(\sigma) = \frac{1}{2} \dim \mathcal{V}(\sigma).$$

3. Proof of Theorem A

We will prove the theorem by induction, using the following three lemmas that will be proven in the next three subsections.

Lemma 3.0.1. *Let $\pi \in \widehat{G}_n$ and let $d := depth(\pi)$. Let $\alpha = (n_1, ..., n_k)$ be a composition of $n-d$ and $\beta = (d, \alpha) = (d, n_1, n_2, ..., n_k)$. Then we have a natural embedding $Wh^*_\alpha(A\pi) \hookrightarrow Wh^*_\beta(\pi)$.*

Lemma 3.0.2. *Let λ, μ be partitions of n and $n-d$ respectively, and suppose that*

(1) $\mathcal{O}_{(d,\mu)} \subseteq \overline{\mathcal{O}_\lambda}$.
(2) $\dim \mathcal{O}_\lambda \leq \dim \mathcal{O}_\mu + (2n-d)(d-1)$.

Then $\lambda = (d, \mu)$.

Lemma 3.0.3. *Let $\pi \in \widehat{G}_n$ and let $d := depth(\pi)$. Then*

$$\dim \mathcal{V}(\pi) \leq \dim \mathcal{V}(A\pi) + (2n-d)(d-1).$$

Proof of Theorem A. We prove the statement by induction on n. For $n = 0, 1$ there is nothing to prove. Now take $n > 1$ and suppose that the theorem holds true for all $r < n$. Let $\pi \in \widehat{G}_n$, $d := depth(\pi)$ and $\lambda := AP(\pi)$.

Let μ be the depth composition of $A\pi$. By the induction hypothesis we know that μ is a partition of $n-d$, $\mathcal{V}(A\pi) = \overline{\mathcal{O}_\mu}$ and $Wh^*_\mu(A\pi^\infty) \neq 0$.

Let $\beta = (d, \mu)$. Then β is the depth composition of π. From the induction hypothesis and Lemma 3.0.1 we obtain $Wh^*_\beta(\pi^\infty) \neq 0$. It suffices to show that $\beta = \lambda$.

By Corollary 1.4.1 we have $\mathcal{O}_\beta \subset \mathcal{V}(\pi) = \overline{\mathcal{O}_\lambda}$. By Lemma 3.0.3 we have $\dim \mathcal{O}_\lambda \leq \dim \mathcal{O}_\mu + (2n-d)(d-1)$. Thus, by Lemma 3.0.2, $\beta = \lambda$. □

3.1. Proof of Lemma 3.0.1.

Now we construct a functional on $\pi \in \widehat{G}_n$ from a functional on its adduced representation.

Let $P_{(n-d,d)} = (G_{n-d} \times G_d) \ltimes N_{(n-d,d)}$ be the maximal parabolic subgroup corresponding to the partition $(n-d, d)$ and define the subgroup $S_{n-d,d} := (G_{n-d} \times N_{1^d}) \ltimes N_{(n-d,d)}$ where N_{1^d} is the nilradical of the Borel subgroup of G_d.

Lemma 3.1.1. *Let $\pi \in \widehat{G_n}$, and let $d := depth(\pi)$. Then $\pi|_{P_n} = Ind^{P_n}_{S_{n-d,d}}(A\pi \otimes \psi \otimes 1)$ where ψ is a nondegenerate unitary character of N_{1^d}.*

Proof. This follows from the definition of $A\pi$ by a straightforward argument involving induction by stages. □

Proposition 3.1.2. *Let $\pi \in \widehat{G_n}$, and let $d := depth(\pi)$. Then there exists an $S_{n-d,d}$-equivariant map from π^∞ to $(A\pi)^\infty \otimes |\det|^{(d-1)/2}$ with dense image.*

Proof. Note that for $g \in S_{n-d,d}$, $\Delta_{P_n}(g) = |\det(g)|$ and $\Delta_{S_{n-d,d}}(g) = |\det(g)|^d$. By Corollary 2.3.7, the evaluation $f \mapsto f(e)$ is an $S_{n-d,d}$-equivariant map from $(\pi|_{P_n})^\infty$ to $(A\pi)^\infty \otimes |\det|^{(d-1)/2}$. Note that $A\pi$ is an irreducible unitary representation of G_d and thus, by [Wall88, Theorem 3.4.12], $(A\pi)^\infty$ is an irreducible Fréchet representation of G_d. Thus it is enough to show that this map does not vanish on π^∞. For that let $v \in \pi^\infty$ be a non-zero vector. Then $v \in \pi|_{P_n}^\infty = Ind_{S_{n-d,d}}^{P_n}(A\pi \otimes \psi \otimes 1)^\infty$ defines a smooth function on P_n that does not vanish at some point p. Then $\pi(p)v$ does not vanish at e. □

This proposition immediately implies Lemma 3.0.1.

3.2. **Proof of Lemma 3.0.2.** It will be useful to have a second formula for $\dim \mathcal{O}_\lambda$ directly in terms of λ. We will also consider reorderings of λ.

Lemma 3.2.1. *If α is a composition of n then we have*

$$\dim \mathcal{O}_\alpha \geq n^2 + n - 2\sum\nolimits_i i\alpha_i. \tag{9}$$

Moreover, equality holds if and only if α is a partition.

Proof. Let $\lambda = \alpha^{\geq}$ and let $\nu = \lambda^t$ be the transposed partition.
We first show that equality holds in (9) if α is a partition, i.e. $\alpha = \lambda$. By Proposition 2.1.7 it suffices to prove

$$\sum\nolimits_j \nu_j^2 = 2\sum\nolimits_i i\lambda_i - n.$$

Consider the Young diagram of λ and write the number $2i$ in every box in the i-th row. Compute the sum of these numbers in two ways: a) adding rows first, b) adding columns first. This gives

$$2\sum\nolimits_i i\lambda_i = 2\sum\nolimits_j (1 + \cdots + \nu_j) = \sum\nolimits_j \left(\nu_j^2 + \nu_j\right) = n + \sum\nolimits_j \nu_j^2,$$

as needed.

Now, for general α we have $\mathcal{O}_\alpha = \mathcal{O}_\lambda$. If α is not a partition then we have strict inequality in (9) since $\sum_i i\alpha_i > \sum_i i\lambda_i$. □

Corollary 3.2.2. *Let μ be a partition of $n-d$ and write $\alpha = (d, \mu)$ then we have*

$$\dim \mathcal{O}_\alpha \geq \dim \mathcal{O}_\mu + (2n-d)(d-1),$$

and equality holds if and only if $\alpha = (d, \mu)$ is a partition (i.e. $d \geq \mu_1$).

Proof. We observe that

$$\sum\nolimits_j j\alpha_j = d + \sum\nolimits_i (i+1)\mu_i = n + \sum\nolimits_i i\mu_i$$

Let $r = n - d$ and apply Lemma 3.2.1 to get the following inequality

$$\begin{aligned}\dim \mathcal{O}_\alpha - \dim \mathcal{O}_\mu &\geq \left(n^2 + n - 2\sum\nolimits_j j\alpha_j\right) - \left(r^2 + r - 2\sum\nolimits_i i\mu_i\right) \\ &= \left(n^2 - n\right) - \left(r^2 + r\right) = (n+r)(n-r-1) = (2n-d)(d-1)\end{aligned}$$

By Lemma 3.2.1 equality holds iff α is a partition. □

Proof of Lemma 3.0.2. Let us write $\alpha = (d, \mu)$. By assumptions (1) and (2) and Corollary 3.2.2 we get

$$\dim \mathcal{O}_\alpha \leq \dim \mathcal{O}_\lambda \leq \dim \mathcal{O}_\mu + (2n-d)(d-1) \leq \dim \mathcal{O}_\alpha.$$

Hence equality must hold throughout. This implies that $\dim \mathcal{O}_\alpha = \dim \mathcal{O}_\lambda$ and, by Corollary 3.2.2, that α is a partition. Now assumption (1) implies that $\alpha = \lambda$. □

3.3. **Proof of Lemma 3.0.3.** First we want to prove that $\dim \mathcal{V}(\pi|_{P_n}) \geq \dim \mathcal{V}(\pi)$. We will start with a geometric lemma.

Lemma 3.3.1. *Let $\mathcal{O} \subset \mathfrak{g}_n^*$ be a nilpotent coadjoint orbit. Then there exists an open dense subset $U \subset \mathcal{O}$ such that the restriction to U of the projection $pr := pr^{\mathfrak{g}_n^*}_{\mathfrak{p}_n^*}$ is injective.*

For the proof see §§§3.3.4.

Corollary 3.3.2. *Let $\mathcal{V} \subset \mathfrak{g}_n^*$ be the closure of a nilpotent coadjoint orbit. Then* $\dim pr^{\mathfrak{g}_n^*}_{\mathfrak{p}_n^*}(\mathcal{V}) = \dim \mathcal{V}$.

Corollary 3.3.3. *Let $\pi \in \widehat{G_n}$. Then* $\dim \mathcal{V}(\pi|_{P_n}) \geq \dim \mathcal{V}(\pi)$.

Proof. Let $I = Ann_{U(\mathfrak{g}_n)}\pi^\omega$ and $J = Ann_{U(\mathfrak{p}_n)}(\pi|_{P_n})^\omega$. Since $(\pi|_{P_n})^\omega \supset \pi^\omega$ we have $J \subset Ann_{U(\mathfrak{p}_n)}\pi^\omega = I \cap U(\mathfrak{p}_n)$ and hence we get

$$gr(J) \subset gr(I \cap U(\mathfrak{p}_n)) \subset gr(I) \cap S(\mathfrak{p}_n).$$

Since $\mathcal{V}(\pi)$ is the closure of a nilpotent coadjoint orbit, by the previous Corollary we conclude

$$\dim \mathcal{V}(\pi|_{P_n}) \geq \dim \mathrm{Zeroes}\,(gr(I) \cap S(\mathfrak{p}_n)) = \dim pr^{\mathfrak{g}_n^*}_{\mathfrak{p}_n^*}(\mathcal{V}(\pi)) = \dim \mathcal{V}(\pi).$$

□

Now we want to use Theorem 2.4.2 to bound $\dim \mathcal{V}(\pi|_{P_n})$. In order to do that we find a finitely generated $U(\mathfrak{p}_n)$-module which is non-degenerately paired with $(\pi|_{P_n})^\omega$ and therefore has the same annihilator.

Proof of Lemma 3.0.3. We will use Lemma 3.1.1. Let $\sigma := A\pi \otimes \psi \otimes 1$ and $\tau := |\det|^{(1-d)/2}(\overline{(A\pi)^{HC}} \otimes \psi \otimes 1)$ be representations of $S_{n-k,k}$ and $\mathfrak{s}_{n-k,k}$ respectively, where $\overline{(A\pi)^{HC}}$ denotes the complex conjugate representation to $(A\pi)^{HC}$. Then τ is equivariantly non-degenerately paired with $\widetilde{\sigma} = |\det|^{(d-1)/2}\sigma$.

By Lemma 3.1.1, $\pi|_{P_n} = Ind^{P_n}_{S_{n-k,k}}(\sigma)$ and thus, by Lemma 2.3.9, $(\pi|_{P_n})^\omega$ is equivariantly non-degenerately paired with a quotient of $U(\mathfrak{p}_n) \otimes_{U(\mathfrak{s}_{n-k,k})} \tau$. Denote this quotient by L. Then $\mathcal{V}(\pi|_{P_n}) = \mathcal{V}(L)$. Note that twist by a character does not effect annihilator variety and Gelfand-Kirillov dimension. Neither does exterior tensor product with a character.

From Lemma 2.4.1 we obtain :

$$\mathrm{GKdim}(L) \leq \mathrm{GKdim}(\tau) + \dim \mathfrak{p}_n - \dim \mathfrak{s}_{n-k,k} = \mathrm{GKdim}(A\pi^{HC}) + (2n-d)(d-1)/2.$$

By Theorem 2.4.2 we have $\dim \mathcal{V}(L) \leq 2\,\mathrm{GKdim}(L)$. By Theorem 2.4.3 we have $2\,\mathrm{GKdim}(A\pi^{HC}) = \dim \mathcal{V}(A\pi)$. By Corollary 3.3.3 we have $\dim \mathcal{V}(\pi) \leq \dim \mathcal{V}(\pi|_{P_n})$. Altogether we have

$$\dim \mathcal{V}(\pi) \leq \dim \mathcal{V}(\pi|_{P_n}) = \dim \mathcal{V}(L) \leq 2\,\mathrm{GKdim}(L) \leq$$
$$\leq 2\,\mathrm{GKdim}(A\pi^{HC}) + (2n-d)(d-1) = \dim \mathcal{V}(A\pi) + (2n-d)(d-1).$$

□

3.3.4. *Proof of Lemma 3.3.1.* Identify $\mathfrak{g}_n^*$ with $\mathfrak{g}_n$ using the trace form; then $\mathfrak{p}_n^*$ consists of matrices whose last *column* is zero, and $pr\,(A)$ replaces the last column of A by zero. Now $\mathcal{O}$ corresponds to a nilpotent orbit in $\mathfrak{g}_n$; let k denote the size of the biggest Jordan block in $\mathcal{O}$ and define

$$U := \{A \in \mathcal{O} \mid e_n' A^{k-1} \neq 0\} \text{ where } e_n' = (0, \cdots, 0, 1)$$

Then U is an open dense subset of $\mathcal{O}$, and we will show $pr|_U$ is injective. Suppose $A, B \in U$ with $pr\,(A) = pr\,(B)$, then A and B differ only in the last column and so

$$B = A + ve_n'$$

where v is some column vector; it suffices to prove that $v = 0$.

We first prove by induction that $e_n' A^i v = 0$ for any $i \geq 0$. Since A and B are nilpotent we have

$$0 = \mathrm{Tr}\, B = \mathrm{Tr}\, A + \mathrm{Tr}\, ve_n' = e_n' v$$

which proves the claim for $i = 0$. Now suppose the claim holds for $i < l$, then

$$B^{l+1} = (A + ve_n')^{l+1} = A^{l+1} + \sum_{j=0}^{l} A^j\,(ve_n')\,A^{l-j} + \cdots \tag{10}$$

Each omitted term in (10) has at least two factors of the form ve_n', hence at least one factor of the form $e_n' A^i v$ for some $0 \leq i < l$, which vanishes by the induction hypothesis. Now taking trace in (10) we get

$$0 = \sum\nolimits_{j=0}^{l} \mathrm{Tr}\left(A^j v e_n' A^{l-j}\right) = (l+1)\, e_n' A^l v$$

which implies the claim for $i = l$, and by induction for all i.

Suppose now by way of contradiction that $v \neq 0$ and let $m \geq 0$ be the largest integer such that $A^m v \neq 0$. Substitute $l = k-1$ in (10); since $A^k = B^k = 0$ we get

$$0 = \sum\nolimits_{j=0}^{k-1} A^j v e_n' A^{k-1-j} = \sum\nolimits_{j=0}^{m} A^j v e_n' A^{k-1-j} \tag{11}$$

Note that $v, Av, ..., A^m v$ are linearly independent; indeed suppose $\sum_{j=i}^{m} c_j A^j v = 0$ with $c_i \neq 0$, then multiplying by A^{m-i} we deduce $c_i A^m v = 0$, which is a contradiction. Therefore we can choose a row vector ϕ such that $\phi A^m v = 1$ but $\phi A^j v = 0$ for any $j < m$. Multiplying (11) by ϕ on the left we get $0 = e_n' A^{k-1-m}$ which contradicts the assumption that $e_n' A^{k-1} \neq 0$. ☐

Remark 3.3.5. *Using* [Ber84, 3.1;4.1-4.2] *one can show that U is a single P_n-orbit.*

4. Computation of adduced representations for (almost) all unitary representations

It is an interesting and important problem to explicitly compute the adduced representations of all irreducible unitary representations of G_n. The answer has been conjectured in [SaSt90] and in the present paper we make substantial progress towards the proof of this conjecture.

4.1. **Vogan Classification.** By the Vogan classification [Vog86], irreducible unitary representations of G_n are Bernstein-Zelevinsky (BZ) products of the form

$$\pi = \pi_1 \times \cdots \times \pi_k$$

where each π_i is one of the following basic unitary representations:

(a) *A one-dimensional unitary character of some G_m.* Such a character is of the form

$$x \mapsto (\mathrm{sgn} \det x)^{\varepsilon} \left|\det x\right|^{z}, \varepsilon \in \{0,1\}, z \in \mathbb{C}$$

and we shall denote it by $\chi(m, \varepsilon, z)$. This character is unitary if z is imaginary, i.e.

$$z = it, t \in \mathbb{R}.$$

(b) *A Stein complementary series representation of some G_{2m}*, twisted by a unitary character. The Stein representations are complementary series of the form

$$\sigma(2m, s) = \chi(m, 0, s) \times \chi(m, 0, -s), s \in (0, 1/2)$$

and we write $\sigma(2m, s; \varepsilon, it)$ to denote its twist by$\chi(2m, \varepsilon, it)$.

(c) *A Speh representation of some G_{2m}*, twisted by a unitary character. As shown in [BSS90] and [SaSt90] the Speh representation $\delta(2m, k)$ is the unique irreducible submodule of

$$\chi(m, 0, k/2) \times \chi(m, \varepsilon_{k+1}, -k/2), k \in \mathbb{N}, \varepsilon_{k+1} \equiv k+1 \pmod 2 \tag{12}$$

and we write $\delta(2m, k; it)$ to denote its twist by $\chi(2m, 0, it)$.

(d) *A Speh complementary series representation of some G_{4m}*, twisted by a unitary character. The Speh complementary series representation is

$$\psi(4m, k, s) = \delta(2m, k; 0, s) \times \delta(2m, k; 0, -s), s \in (0, 1/2)$$

and we write $\psi(4m, k, s; it)$ to denote its twist by $\chi(4m, 0, it)$.

The reader might well ask why we do not consider twists in (c) and (d) for $\varepsilon = 1$. The reason is the following:

Lemma 4.1.1. *Speh representation and their complementary series are* unchanged *if we twist them by the sign character.*

Proof. If π is a representation of G_n we denote its sign twist by $\widetilde{\pi} = \pi \otimes \chi(n,1,0)$. This operation is compatible with parabolic induction in the sense that we have

$$\widetilde{\pi_1 \times \pi_2} = \widetilde{\pi_1} \times \widetilde{\pi_2}. \tag{13}$$

We leave the easy verification of (13) to the reader.

For the group $GL(2,\mathbb{R})$, the Speh representations $\delta(2,k)$ are precisely the discrete series. In this case the result $\delta(2,k) = \widetilde{\delta(2,k)}$ is well known (see 1.4.7 in [Vog81]). The general Speh representation $\delta(2m,k)$ is the unique irreducible quotient (Langlands quotient) of

$$\delta(2,k;s_1) \times \cdots \times \delta(2,k;s_m) \text{ where } s_i = \frac{m+1}{2} - i \tag{14}$$

see, e.g. [Vog86]. By (13) the induced representation (14) is unchanged under sign twist. Therefore so is its unique irreducible quotient.

The result for Speh complementary series now follows from (13). □

4.2. **Annihilator variety and rank.** In this subsection we compute the associated partition in terms of the Vogan classification. Let

$$\pi = \prod_{i=1}^{k} \chi_i \times \prod_{j=1}^{l} \delta_j \text{ with } n = \sum_{i=1}^{k} p_i + 2\sum_{j=1}^{l} q_j \tag{15}$$

where χ_i is a character of G_{p_i} and δ_j is a Speh representation of G_{2q_j} (perhaps twisted by a nonunitary character in order to include complementary series). By Corollary 2.1.8 , to compute $AP(\pi)$ it suffices to determine $AP(\chi_i)$ and $AP(\delta_j)$. Clearly $AP(\chi_i) = 1^i$. By [SaSt90, Theorem 3], the adduced of the Speh representation $\delta(2n,k) \in \widehat{G}_{2n}$ is the Speh representation $\delta(2(n-1),k) \in \widehat{G}_{2n-2}$. By induction, we obtain from Theorem A that $AP(\delta_j) = 2^j$.

Theorem 4.2.1. *For π as in (15), $AP(\pi)^t$ has one part of size p_i for each i and two parts of size q_j for each j. Consequently the rank (see Definition 1.6.1) and the Gelfand-Kirillov dimension of π are given by*

$$\operatorname{rk}(\pi) = n - \max(p_i, q_j), \quad \operatorname{GKdim}(\pi) = \frac{1}{2}\left(n^2 - \sum p_i^2 - 2\sum q_j^2\right) \tag{16}$$

Corollary 4.2.2. *Let $\pi \in \widehat{G}_n$ and let $k < n/2$. Then $\operatorname{rk}(\pi) = k$ if and only if there exist a representation $\sigma \in \widehat{G}_k$ and a character $\chi \in \widehat{G}_{n-k}$ such that $\pi = \sigma \times \chi$.*

Remark 4.2.3. *By* [Sca90, Part II, Corollary 3.2]*, an analogous statement holds for Howe rank. This gives an independent proof of the statement that for any $\pi \in \widehat{G}_n$,*

$$\textit{Howe rank } (\pi) = \min(\operatorname{rk}(\pi), \lfloor n/2 \rfloor).$$

The following theorem follows from [BSS90, SaSt90].

Theorem 4.2.4. *For the Speh representation $\delta(2n,k) \in \widehat{G}_{2n}$ there exist degenerate principal series representations π_k and π_{-k}, induced from certain characters of the standard parabolic subgroup given by the partition (n,n), an embedding $i\colon \delta(2n,k) \hookrightarrow \pi_k$ and an epimorphism $p\colon \pi_{-k} \twoheadrightarrow \delta(2n,k)$.*

Corollary 4.2.5. *Any irreducible unitarizable representation π of G_n can be presented both as a subrepresentation and as a quotient of a degenerate principal series representation with the same annihilator variety. Those degenerate principal series representations will be induced from characters of the standard parabolic described by the partition which is transposed to the partition describing $\mathcal{V}(\pi)$.*

Remark 4.2.6. *In* [AGS] *we show, using* [BSS90]*, that all other Jordan-Holder constituents of the degenerate principal series representations mentioned above will have smaller annihilator varieties.*

4.3. **Infinitesimal characters - general considerations.** In this subsection only, we let $\mathfrak{g}$ denote an arbitrary complex reductive Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, and let W denote the Weyl group of $\mathfrak{g}$. Let $Z(\mathfrak{g})$ denote the center of the universal enveloping algebra $U(\mathfrak{g})$. Then by the Harish-Chandra homomorphism we have $Z(\mathfrak{g}) \approx S(\mathfrak{h})^W$. Thus each $\lambda \in \mathfrak{h}^*$ determines a character χ_λ of $Z(\mathfrak{g})$ with $\chi_{w\lambda} = \chi_\lambda$ for all $w \in W$.

The Harish-Chandra homomorphism is a special case of the following more general construction. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a standard parabolic subalgebra containing $\mathfrak{b}$ so that $\mathfrak{h} \subset \mathfrak{l}$ and $\mathfrak{u} \subset \mathfrak{n}$. Then we have a triangular decomposition $\mathfrak{g} = \mathfrak{u} + \mathfrak{l} + \overline{\mathfrak{u}}$, and by the PBW theorem the universal enveloping algebra of $\mathfrak{g}$ can be decomposed as follows:

$$U(\mathfrak{g}) = U(\mathfrak{u}) \otimes U(\mathfrak{l}) \otimes U(\overline{\mathfrak{u}}) = U(\mathfrak{l}) \oplus [\mathfrak{u}U(\mathfrak{g}) + U(\mathfrak{g})\overline{\mathfrak{u}}] \tag{17}$$

Let $\mathcal{P} = \mathcal{P}_{\mathfrak{l}}^{\mathfrak{g}}$ denote the corresponding projection from $U(\mathfrak{g})$ to $U(\mathfrak{l})$, and let $Z(\mathfrak{g})$ and $Z(\mathfrak{l})$ denote the centers of $U(\mathfrak{g})$ and $U(\mathfrak{l})$.

Lemma 4.3.1. *(1) $\mathcal{P}$ is $ad(\mathfrak{l})$-equivariant.*
(2) $\mathcal{P}$ maps $Z(\mathfrak{g})$ to $Z(\mathfrak{l})$.
(3) For $z \in Z(\mathfrak{g})$ we have $z - \mathcal{P}(z) \in \mathfrak{u}U(\mathfrak{g})$.

Proof. See e.g. [Vog81] P. 118. ☐

Lemma 4.3.2. *Let V be a $\mathfrak{g}$-module, then*
1. *The subspace $\mathfrak{u}V$ is $Z(\mathfrak{g})$-invariant and $U(\mathfrak{l})$-invariant.*
2. *The quotient space $V/\mathfrak{u}V$ is a $Z(\mathfrak{g})$-module and a $U(\mathfrak{l})$-module.*
3. *For $z \in Z(\mathfrak{g})$ the actions of z and $\mathcal{P}(z)$ agree on $V/\mathfrak{u}V$.*

Proof. Parts 1 and 2 are straightforward. Part 3 follows from the previous lemma. ☐

We say that a $\mathfrak{g}$-module V has infinitesimal character λ if each $z \in Z(\mathfrak{g})$ acts by the scalar $\chi_\lambda(z)$. We say that V has generalized infinitesimal character χ_λ if there is an integer n such that $(z - \chi_\lambda(z))^n$ acts by 0 for all $z \in Z(\mathfrak{g})$. We say that V is $Z(\mathfrak{g})$-finite if V is annihilated by an ideal of finite codimension in $Z(\mathfrak{g})$. If V is $Z(\mathfrak{g})$-finite then V decomposes as a finite direct sum

$$V = V_1 \oplus \cdots \oplus V_k$$

where each V_i has generalized infinitesimal character.

Corollary 4.3.3 (Casselman - Osborne)**.** *Let V be a $Z(\mathfrak{g})$-finite $\mathfrak{g}$-module, then $V/\mathfrak{u}V$ is a $Z(\mathfrak{l})$-finite $\mathfrak{l}$-module. Moreover if the generalized infinitesimal character χ_μ occurs in $V/\mathfrak{u}V$, then there exists $\lambda \in \mathfrak{h}^*$ such that*
1. *the generalized infinitesimal character χ_λ occurs in V;*
2. *$\mu = \lambda + \rho(\mathfrak{u})$ where $\rho(\mathfrak{u})$ is half the sum of the roots of $\mathfrak{h}$ in $\mathfrak{u}$.*

Proof. The proof is similar to that of Corollary 3.1.6 in [Vog81]. ☐

4.4. **Infinitesimal characters for G_n.** If π is an irreducible admissible representation of G_n, the infinitesimal character ξ_π of π can be regarded as a multiset (set with multiplicity) of n complex numbers $\{z_1, \ldots, z_n\}$. If all the z_i are real we say that π has real infinitesimal character. For convenience we write $\sqcup$ for the (disjoint) union of multisets; e.g. $\{1,2\} \sqcup \{2,3\} = \{1,2,2,3\}$.

For $m \in \mathbb{N}$ and $z \in \mathbb{C}$ we define the corresponding "segment" to be the set

$$S(m,z) = \{z_1, \cdots, z_m\} \text{ where } z_i = z + (m+1)/2 - k.$$

Thus $z_1, \cdots, z_m$ is an arithmetic progression of length m, mean z, and common difference 1.

The following lemma summarizes the main facts about infinitesimal characters for unitary representations of G_n.

Lemma 4.4.1. *(1) $\xi_{\pi_1 \times \cdots \times \pi_k} = \xi_{\pi_1} \sqcup \cdots \sqcup \xi_{\pi_k}$*
(2) For $\pi = \chi(m, \varepsilon, z)$ we have $\xi_\pi = S(m,z)$
(3) For $\pi = \sigma(2m, s; \varepsilon, it)$ we have $\xi_\pi = S(m, s+it) \sqcup S(m, -s+it)$

(4) *For* $\pi = \delta(2m, k; it)$ *we have* $\xi_\pi = S(m, (k/2) + it) \sqcup S(m, (-k/2) + it)$
(5) *For* $\pi = \psi(4m, k, s; it)$ *we have* $\xi_\pi = \bigsqcup S(m, \pm(k/2) \pm s + it)$

Proof. 1) and 2) are standard [Vog81] and together they imply 3). Similarly 1) and 4) imply 5). Part 4) follows from 1) and formula (12); alternatively one may deduce it from 1), formula (14), and the fact that the discrete series $\delta(2,k)$ of $GL(2,\mathbb{R})$ has infinitesimal character $\left\{\frac{k}{2}, -\frac{k}{2}\right\}$. □

Lemma 4.4.2. *If* $\pi \in \widehat{G_n}$ *then* ξ_π *is* symmetric *in the sense* z *and* $-\bar{z}$ *have the same multiplicity in* ξ_π *for all* $z \in \mathbb{C}$.

This lemma is in fact an easy elementary fact about Hermitian representations. For $\pi \in \widehat{G_n}$ it also follows easily from the previous lemma by checking it for basic representations.

4.5. **Adduced representations.**

Conjecture 1 ([SaSt90]). *Let* $\pi \in \widehat{G_n}$ *and write* $\pi = \pi_1 \times \cdots \times \pi_k$ *as in the Vogan classification with each* π_i *a basic unitary representation of type a-d listed above. Then we have*

$$A\pi = A\pi_1 \times \cdots \times A\pi_k \tag{18}$$

where

$$\begin{aligned} A(\chi(m,\varepsilon,it)) &= \chi(m-1,\varepsilon,it) \\ A(\sigma(2m,s;\varepsilon,it)) &= \sigma(2(m-1),s;\varepsilon,it) \\ A(\delta(2m,k;it)) &= \delta(2(m-1),k;it) \\ A(\psi(4m,k,s;it)) &= \psi(4(m-1),k,s;it) \end{aligned} \tag{19}$$

Most of this result is already known. The identity (18) is proved in [Sah89]. As for (19), part 1 is obvious, part 2 is proved in [Sah90], part 3 is proved in [SaSt90], where part 4 was conjectured. We show that the techniques of the present paper suffice to prove (19) part 4 for $k \neq m$.

Lemma 4.5.1. *Let* $\pi \in \widehat{G_n}$ *and for* $t \in \mathbb{R}$ *let* $\pi[it]$ *denote the unitary twist of* π *by the character* $|\det|^{it}$. *Then we have* $A(\pi[it]) = (A\pi)[it]$.

Proof. This is straightforward. □

Lemma 4.5.2. *The Speh complementary series representation* $\psi(4m,k,s;it)$ *is uniquely determined by its infinitesimal character and associated partition.*

Proof. By Lemma 4.4.1 $\pi = \psi(4m,k,s;it)$ has infinitesimal character $\xi_\pi = \bigsqcup S(m, \pm(k/2) \pm s + it)$, and by Theorem 4.2.1 its associated partition is 4^m. Since $0 < s < 1/2$ it follows that

(1) $2\,\mathrm{Re}(z)$ is not an integer for any $z \in \xi_\pi$,
(2) $\max\{2\,\mathrm{Re}(z) : z \in \xi_\pi\} = m - 1 + k + 2s$

Let π' be a unitary representation with the same infinitesimal character and associated partition as π. Write $\pi' = \pi_1 \times \cdots \times \pi_l$ as in the Vogan classification. Then condition 1 above implies that none of the π_i can be unitary characters or Speh representations, while condition 2 implies that π' cannot be the product of two Stein representations of G_{2m}, for then we would have $\max(2\,\mathrm{Re}(z)) < m$. Therefore we conclude that π' is a Speh complementary series representation of G_{4m}. Thus $\pi' = \psi(4m,k',s';it')$. By looking at the integral and fractional parts of $\max(2\,\mathrm{Re}(z))$ we deduce $k = k'$ and $s = s'$. By looking at the imaginary part of $z \in \xi_\pi = \xi_{\pi'}$ we conclude that $t = t'$. □

Remark 4.5.3. *Due to the previous lemma, Conjecture 1 becomes now equivalent to the statement that for any* $\pi \in \widehat{G_n}$, *the infinitesimal character parameter of* $A\pi$ *is obtained from the infinitesimal character parameter of* π *by the following procedure: consider the Young diagram* X *with the sizes of rows described by the partition corresponding to* π, *write the infinitesimal character parameter of* π *in the columns such that in each column we will have a segment, and make each of those segments shorter by one without changing its center. This is similar to the effect of highest derivatives on the Zelevinsky classification in the p-adic case (see §6). We cannot prove this statement here, but we deduce its weaker version from Corollary 4.3.3 and Proposition 3.1.2.*

Proposition 4.5.4. *Let $\pi \in \widehat{G_n}$ and let $d = depth(\pi)$. Let $S = \{z_1, \ldots, z_n\}$ and $S' = \{y_1, \ldots, y_{n-d}\}$ be the multisets corresponding to infinitesimal characters of π and $A\pi$ respectively. Then S' is obtained from S by deleting d of the z_i's and adding $1/2$ to each of the remaining z_i's.*

Proof. Let $\sigma := A\pi$ and let π^∞ and σ^∞ be the spaces of smooth vectors of π and σ, respectively, and let λ and μ be their infinitesimal character parameters.

By Proposition 3.1.2 there is an $S_{n-d,d}$-equivariant morphism $\varphi : \pi^\infty \to \sigma^\infty \otimes |\det|^{(d-1)/2}$ with dense image. Since $N_{n-d,d}$ acts trivially on $\sigma^\infty \otimes |\det|^{(d-1)/2}$, φ factors through the quotient $\pi^\infty / \mathfrak{u}\pi^\infty$, where $\mathfrak{u} = Lie\,(N_{n-d,d})$. By Corollary 4.3.3, the possible $G_{n-d} \times G_d$ infinitesimal characters in $\pi^\infty/\mathfrak{u}\pi^\infty$ are of the form $w\lambda + \rho\,(\mathfrak{u})$. Further restricting to G_{n-d} we conclude that these characters are of the form

$$\lambda^- + \frac{d}{2}\,(1, 1, \cdots, 1)$$

where $\lambda^- \subset \lambda$ is a subset of size $n - d$.

On the other hand, the G_{n-d} module $\sigma^\infty \otimes |\det|^{(d-1)/2}$ has infinitesimal character

$$\mu + \frac{d-1}{2}\,(1, 1, \cdots, 1)\,.$$

Comparing the two displayed expressions we get $\mu = \lambda^- + \frac{1}{2}\,(1, 1, \cdots, 1)$ and the result follows. $\square$

We are now ready to prove the main theorem of this section.

Theorem 4.5.5. *Suppose $k \neq m$ then $A\,(\psi\,(4m, k, s; it)) = \psi\,(4\,(m-1)\,, k, s; it)$.*

Proof. Let $\pi = \psi\,(4m, k, s; it)\,, \pi' = \psi\,(4\,(m-1)\,, k, s; it)$ and let ξ, ξ', ξ'' be the infinitesimal characters of $\pi, \pi', A\pi$ respectively. By Theorem 4.2.1, $AP(\pi) = 4^m$ and $AP(\pi') = 4^{m-1}$. By Theorem A, $AP(A\pi) = 4^{m-1}$ as well. Therefore by Lemma 4.5.2 it suffices to show that $\xi' = \xi''$.

For simplicity we assume $t = 0$, the argument is the same if $t \neq 0$. Now by Lemma 4.4.1 we have

$$\xi = \bigsqcup S\left(m, \pm\frac{k}{2} \pm s\right) \text{ and } \xi' = \bigsqcup S\left(m-1, \pm\frac{k}{2} \pm s\right)$$

Now by 4.5.4 ξ'' is obtained from $B = \xi + \frac{1}{2}$ by deleting 4 elements. Indeed ξ' is obtained from B by deleting the following 4 elements

$$C = \{\frac{m}{2} \pm \frac{k}{2} \pm s\}.$$

and we need to show that *no* other infinitesimal character of a unitary representation can be so obtained.

In fact we note that ξ' is the only symmetric submultiset of $\xi + \frac{1}{2}$ with $|\xi'| = 4\,(m-1)$. Indeed any other symmetric subset of cardinality $|B| - 4$ is obtained from B by replacing a symmetric subset of ξ' with a symmetric subset of equal size contained in C, but if $k \neq m$ then C has no symmetric subsets. $\square$

Remark 4.5.6. *For $k = m$ the proof of Theorem 4.5.5 fails, since in this case we have*

$$C = \{m \pm s, \pm s\}$$

which admits the symmetric subset $\{\pm s\}$. Indeed it is easy to see that for $k = m$ the infinitesimal character of $\psi\left(4m-4, m-1, \frac{1}{2} - s\right)$ is also a subset of $\xi + \frac{1}{2}$. Therefore we may conclude only that

$$A\psi\,(4m, m, s) = \psi\,(4m-4, m, s) \;\text{ or }\; \psi\left(4m-4, m-1, \frac{1}{2} - s\right)$$

An additional argument is required to rule out the latter possibility.

5. The complex case

Let us discuss the setting and the geometry of nilpotent orbits. We have

$$G = G_n = \mathrm{GL}(n, \mathbb{C}), \quad \mathfrak{g} \approx \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C}), \quad \mathfrak{g}_{\mathbb{R}} = \mathfrak{gl}(n, \mathbb{C}) \oplus 0, \quad \mathfrak{k} = \{(x, x)\}, \quad \mathfrak{k}^{\perp} = \{(x, -x)\},$$

where $\mathfrak{k}$ denotes the complexified Lie algebra of maximal compact subgroup. The nilpotent orbits are parameterized by pairs of partitions and have the form $\mathcal{O}_\lambda \times \mathcal{O}_\mu$. However, associated orbits of Harish-Chandra modules intersect $\mathfrak{k}^\perp$ and are therefore of the form $\mathcal{O}_\lambda \times \mathcal{O}_\lambda$ and so are still parameterized by single partitions, rather than two partitions. Standard parabolic subgroups are also parameterized by single partitions. The degenerate Whittaker functionals are defined in the same way as in §§§1.2.3: using $J_\alpha \in \mathfrak{g}_{\mathbb{R}}$. Therefore they are also parameterized by single partitions. Therefore the formulation of Theorem A in the complex case stays the same. The proof of Theorem A is obtained from the proof in §3 by replacing the term "coadjoint nilpotent orbit" by "coadjoint nilpotent orbit that intersects $\mathfrak{k}^\perp$" and doubling all the expressions for the dimensions of such orbits.

The Vogan classification of $\widehat{GL(n, \mathbb{C})}$ is simpler: each $\pi \in \widehat{GL(n, \mathbb{C})}$ is a product of characters, where the non-unitary characters come in pairs. As shown in [Sah89, Sah90], the adduced representation of π is a product of the same form, where each character of G_n is restricted to G_{n-1}, and characters of G_1 are thrown away. The associated partition of π is determined in the obvious way, similar to Theorem 4.2.1.

6. The p-adic case

In this section we fix F to be a non-Archimedean local field of characteristic zero and let $G_n := GL(n, F)$.

6.1. Definition of derivatives.

In the p-adic case, there is an additional definition of highest derivative, using co-invariants. This definition works for all smooth admissible representations of G_n, not only unitarizable. Moreover, Bernstein and Zelevinsky define in [BZ77, §3] all derivatives and not only the highest ones in the following way.

Recall that $P_n = G_{n-1} \ltimes F^{n-1}$. Fix a non-trivial additive character χ of F and define a character θ_n of F^{n-1} by applying χ to the last coordinate. Denote by ν the determinant character $\nu(g) = |\det(g)|$.

Define two normalized coinvariants functors $\Psi^-\colon Rep(P_n) \to Rep(G_{n-1})$ and $\Phi^-\colon Rep(P_n) \to Rep(P_{n-1})$ by

$$\Psi^-(\tau) := \nu^{-1/2}\tau_{F^{n-1},1} \text{ and } \Phi^-(\tau) := \nu^{-1/2}\tau_{F^{n-1},\theta_n}.$$

Both functors are exact.

For a smooth representation τ of P_n they define $\tau^{(k)} := \Psi^-(\Phi^-)^{k-1}\tau$ and call it the k-th derivative of τ. For a smooth representation π of G_n, *k-th derivative* is defined by $D^k\pi := \pi^{(k)} := (\pi|_{P_n})^{(k)}$.

If $D^k\pi > 0$ but $D^{k+l}\pi = 0$ for any $l > 0$ then $D^k\pi$ is called the *highest derivative* of π and we denote it by $A(\pi)$, and k is called the *depth* of π and denoted $d(\pi)$.

For unitarizable representations, one can also define a *shifted highest derivative* of π using Mackey theory, in the same way as adduced representation is defined in the Archimedean case (see §§1.2.6). By [Ber84], this shifted highest derivative will be isomorphic to $\nu^{1/2}A(\pi)$.

For a composition $\alpha = (\alpha_1, ..., \alpha_k)$ we define $D^\alpha(\pi) := D^{\alpha_1}...D^{\alpha_k}(\pi)$. By [Zel80, §§8.3] we have $(D^{\alpha_k}D^{\alpha_{k-1}}...D^{\alpha_1}\pi)^* \simeq Wh^*_\alpha(\pi)$.

Denote by $\mathcal{M}(G_n)$ the category of smooth admissible representations of G_n and define the Grothendieck ring $R = \bigoplus_n \Gamma(\mathcal{M}(G_n))$. As an additive group it is the direct sum of Grothendieck groups of $\mathcal{M}(G_n)$ for all n, and the product is defined by parabolic induction. Define the total derivative $D\colon R \to R$ as the sum of all derivatives. By [BZ77, §§4.5], D is a homomorphism of rings.

6.2. Zelevinsky classification.

In [Zel80], Zelevinsky describes the generators of R in the following way. Denote by $C := \bigcup_n C_n$ the subset of all cuspidal irreducible representations of G_n for all n. For $\rho \in C_d$, a subset $\Delta \subset C_d$ of the form $(\rho, \nu\rho, \nu^2\rho, ..., \nu^{l-1}\rho)$ is called a *segment*. The representation $\nu^{(l-1)/2}\rho$ is called the center of Δ, the number l is called the length of Δ, and the number d is called the depth of Δ. We denote the set of all segments $\Delta \subset C$ by S. We define a segment Δ^- by $\Delta^- = (\rho, \nu\rho, \nu^2\rho, ..., \nu^{l-2}\rho)$.

Theorem 6.2.1 ([Zel80],§3 and §§7.5)**.** *Let $\Delta = (\rho, \nu\rho, \nu^2\rho, ..., \nu^{l-1}\rho) \subset C_d$ be a segment. Then the representation $\rho \times \nu\rho \times ...\nu^{l-1}\rho$ contains a unique irreducible constituent $\langle\Delta\rangle$ of the depth $d = depth(\Delta)$. Moreover,*

(1) $D(\langle\Delta\rangle) = \langle\Delta\rangle + \langle\Delta'\rangle$.
(2) R is a polynomial ring in indeterminates $\{\langle\Delta\rangle : \Delta \in S\}$

Zelevinsky furthermore describes all irreducible representations in terms of segments and shows, in [Zel80, §§8.1], that A maps irreducible representations to irreducible.

6.3. **The wave front set.** Let $\pi \in \mathcal{M}(G_n)$. Let χ_π be the character of π. Then χ_π defines a distribution ξ_π on a neighborhood of zero in $\mathfrak{g}_n$, by restriction to a neighborhood of $1 \in G$ and applying logarithm. This distribution is known to be a combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits ([HC77, p. 180]). This enables to define a wave front cycle, $WFC(\pi)$ as a linear combination of orbits. Clearly, WFC is additive on R. In [MW87, II.1] it is shown that it is also multiplicative, in the sense of Corollary 2.1.8. The wave front set, $WF(\pi)$, is defined to be the set of orbits that appear in $WFC(\pi)$ with non-zero coefficients. Denote by $WFmax(\pi)$ the set of maximal elements of $WF(\pi)$ (with respect to the Bruhat ordering). In [MW87, II.2], $WFC(\langle\Delta\rangle)$ is computed to be the orbit given by the partition λ_Δ that has $length(\Delta)$ parts of size $depth(\Delta)$, with multiplicity 1. Thus, for an arbitrary smooth irreducible representation π of G_n, $WFmax(\pi)$ consists of a single nilpotent orbit, given by the partition defined by the Zelevinsky classification (see [OS09]).

6.4. **Proof of Theorem B over** F**.** Since both D and in a sense WFC are homomorphisms, and since segment representations $\langle\Delta\rangle$ generate R, it is enough to prove Theorem B for them. The above information implies that $Wh_\alpha(\langle\Delta\rangle) = D^\alpha(\langle\Delta\rangle) = 0$ unless $\alpha = \lambda_\Delta$. Now, $Wh_{\lambda_\Delta}(\langle\Delta\rangle) = D^{\lambda_\Delta}(\Delta) = \langle\emptyset\rangle = \mathbb{C}$. Since $WFC(\langle\Delta\rangle) = \mathcal{O}_{\lambda_\Delta}$, this completes the proof of the main three parts of Theorem B.

We can prove the "moreover" part through an analog of Corollary 4.2.2: there is a Tadic classification of the unitary representations as products of certain building blocks (see [Tad86]), from which we see that $rank(\pi) = k < n/2$ if and only if $\pi = \chi \times \tau$, for some character χ of G_{n-k} and some $\tau \in \mathcal{M}(G_k)$. Since the same holds for Howe rank (by [Sca90, Part II, Corollary 3.2]), the "moreover" part follows. One can also prove the "moreover" part directly, as was done in [MW87, II.3] for the symplectic group.

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Dmitry Gourevitch, Faculty of Mathematics and Computer Science, Weizmann Institute of Science, POB 26, Rehovot 76100, Israel
E-mail address: dimagur@weizmann.ac.il
URL: http://www.wisdom.weizmann.ac.il/~dimagur

Siddhartha Sahi, Department of Mathematics, Rutgers University, Hill Center - Busch Campus, 110 Frelinghuysen Road Piscataway, NJ 08854-8019, USA
E-mail address: sahi@math.rugers.edu