SOME REGULAR SYMMETRIC PAIRS

AVRAHAM AIZENBUD AND DMITRY GOUREVITCH

ABSTRACT. In [AG2] we explored the question what symmetric pairs are Gelfand pairs. We introduced the notion of regular symmetric pair and conjectured that all symmetric pairs are regular. This conjecture would imply that many symmetric pairs are Gelfand pairs, including all connected symmetric pair over $\mathbb{C}$.

In this paper we show that the pairs

$$(GL(V), O(V)), (GL(V), U(V)), (U(V), O(V)), (O(V \oplus W), O(V) \times O(W)), (U(V \oplus W), U(V) \times U(W))$$

are regular where V and W are quadratic or hermitian spaces over arbitrary local field of characteristic zero. We deduce from this that the pairs $(GL_n(\mathbb{C}), O_n(\mathbb{C}))$ and $(O_{n+m}(\mathbb{C}), O_n(\mathbb{C}) \times O_m(\mathbb{C}))$ are Gelfand pairs.

CONTENTS

1. INTRODUCTION

In [AG2] we explored the question what symmetric pairs are Gelfand pairs. We introduced the notion of regular symmetric pair and conjectured that all symmetric pairs are regular. This conjecture would imply that many symmetric pairs are Gelfand pairs, including all connected symmetric pair over $\mathbb{C}$.

In this paper we show that the pairs

$$(GL(V), O(V)), (GL(V), U(V)), (U(V), O(V)), (O(V \oplus W), O(V) \times O(W)), (U(V \oplus W), U(V) \times U(W))$$

are regular where V and W are quadratic or hermitian spaces over arbitrary local field of characteristic zero. We deduce from this that the pairs $(GL_n(\mathbb{C}), O_n(\mathbb{C}))$ and $(O_{n+m}(\mathbb{C}), O_n(\mathbb{C}) \times O_m(\mathbb{C}))$ are Gelfand pairs.

In general, if we would know that all symmetric pairs are regular, then in order to show that a given symmetric pair (G, H) is a Gelfand pair it would be enough to check the following condition that we called "goodness":

(*) Every closed H-double coset in G is invariant with respect to σ. Here, σ is the anti-involution defined by $\sigma(g) := \theta(g^{-1})$ and θ is an involution (i.e. automorphism of order 2) of G such that $H = G^\theta$.

This condition always holds for connected symmetric pairs over $\mathbb{C}$.

Meanwhile, before the conjecture is proven, in order to show that a given symmetric pair is a Gelfand pair one has to verify that the pair is good, to prove that it is regular and also to compute its "descendants" and show that they are also regular. The "descendants" are certain symmetric pairs related to centralizers of semisimple elements.

In this paper we develop further the tools from [AG2] for proving regularity of symmetric pairs. We also introduce a systematic way to compute descendants of classical symmetric pairs.

Based on that we show that all the descendants of the above symmetric pairs are regular.

1.1. **Structure of the paper.**

In section 2 we introduce the notions that we discuss in this paper. In subsection 2.1 we discuss the notion of Gelfand pair and review a classical technique for proving Gelfand property due to Gelfand and Kazhdan. In subsection 2.2 we review the results of [AG2], introduce the notions of symmetric pair, descendants of a symmetric pair, good symmetric pair and regular symmetric pair mentioned above and discuss their relations to Gelfand property.

In section 3 we formulate the main results of the paper. We also explain how they follow from the rest of the paper.

In section 4 we introduce terminology that enables us to prove regularity for symmetric pairs in question.

In section 5 we prove regularity for symmetric pairs in question.

In section 6 we compute the descendants of those symmetric pairs.

2. Preliminaries and notations

- Throughout the paper we fix an arbitrary local field F of characteristic zero.
- All the algebraic varieties and algebraic groups that we will consider will be defined over F.
- For a group G acting on a set X and an element $x \in X$ we denote by G_x the stabilizer of x.
- By a reductive group we mean an algebraic reductive group.

In this paper we will refer to distributions on algebraic varieties over archimedean and non-archimedean fields. In the non-archimedean case we mean the notion of distributions on l-spaces from [BZ], that is linear functionals on the space of locally constant compactly supported functions. In the archimedean case one can consider the usual notion of distributions, that is continuous functionals on the space of smooth compactly supported functions, or the notion of Schwartz distributions (see e.g. [AG1]). It does not matter here which notion to choose since in the cases of consideration of this paper, if there are no

nonzero equivariant Schwartz distributions then there are no nonzero equivariant distributions at all (see Theorem 4.0.2 in [AG2]).

Notation 2.0.1. *Let E be an extension of F. Let G be an algebraic group defined over F. We denote by $G_{E/F}$ the canonical algebraic group defined over F such that $G_{E/F}(F) = G(E)$.*

2.1. **Gelfand pairs.**
In this section we recall a technique due to Gelfand and Kazhdan ([GK]) which allows to deduce statements in representation theory from statements on invariant distributions. For more detailed description see [AGS], section 2.

Definition 2.1.1. *Let G be a reductive group. By an **admissible representation of** G we mean an admissible representation of $G(F)$ if F is non-archimedean (see [BZ]) and admissible smooth Fréchet representation of $G(F)$ if F is archimedean.*

We now introduce three notions of Gelfand pair.

Definition 2.1.2. *Let $H \subset G$ be a pair of reductive groups.*
- *We say that (G, H) satisfy **GP1** if for any irreducible admissible representation (π, E) of G we have*
$$\dim Hom_{H(F)}(E, \mathbb{C}) \leq 1$$
- *We say that (G, H) satisfy **GP2** if for any irreducible admissible representation (π, E) of G we have*
$$\dim Hom_{H(F)}(E, \mathbb{C}) \cdot \dim Hom_H(\widetilde{E}, \mathbb{C}) \leq 1$$
- *We say that (G, H) satisfy **GP3** if for any irreducible **unitary** representation $(\pi, \mathcal{H})$ of $G(F)$ on a Hilbert space $\mathcal{H}$ we have*
$$\dim Hom_{H(F)}(\mathcal{H}^\infty, \mathbb{C}) \leq 1.$$

Property GP1 was established by Gelfand and Kazhdan in certain p-adic cases (see [GK]). Property GP2 was introduced in [Gro] in the p-adic setting. Property GP3 was studied extensively by various authors under the name **generalized Gelfand pair** both in the real and p-adic settings (see e.g. [vD, BvD]).

We have the following straightforward proposition.

Proposition 2.1.3. $GP1 \Rightarrow GP2 \Rightarrow GP3$.

We will use the following theorem from [AGS] which is a version of a classical theorem of Gelfand and Kazhdan.

Theorem 2.1.4. *Let $H \subset G$ be reductive groups and let τ be an involutive anti-automorphism of G and assume that $\tau(H) = H$. Suppose $\tau(\xi) = \xi$ for all bi $H(F)$-invariant distributions ξ on $G(F)$. Then (G, H) satisfies GP2.*

In the cases we consider in this paper GP2 is equivalent to GP1 by the following proposition.

Proposition 2.1.5.
(i) Let V be a quadratic space (i.e. a linear space with a non-degenerate quadratic form) and let $H \subset GL(V)$ be any transpose invariant subgroup. Then GP1 is equivalent to GP2 for the pair $(\mathrm{GL}(V), H)$.
(ii) Let V be a quadratic space and let $H \subset O(V)$ be any subgroup. Then GP1 is equivalent to GP2 for the pair $(O(V), H)$.

It follows from the following 2 propositions.

Proposition 2.1.6. *Let $H \subset G$ be reductive groups and let τ be an anti-automorphism of G such that*
(i) $\tau^2 \in Ad(G(F))$
(ii) τ preserves any closed conjugacy class in $G(F)$
(iii) $\tau(H) = H$.
Then GP1 is equivalent to GP2 for the pair (G, H).

For proof see [AG2], Corollary 8.2.3.

Proposition 2.1.7.
(i) Let V be a quadratic space and let $g \in GL(V)$. Then g is conjugate to g^t.
(ii) Let V be a quadratic space and let $g \in O(V)$. Then g is conjugate to g^{-1} inside $O(V)$.

Part (i) is well known. For the proof of (ii) see [MVW], Proposition I.2 in chapter 4.

2.2. Symmetric pairs.

In this subsection we review some tools developed in [AG2] that enable to prove that a symmetric pair is a Gelfand pair. The main results discussed in this subsection are Theorem 2.2.16, Theorem 2.2.24 and Proposition 2.2.19.

Definition 2.2.1. *A **symmetric pair** is a triple (G, H, θ) where $H \subset G$ are reductive groups, and θ is an involution of G such that $H = G^\theta$. We call a symmetric pair **connected** if G/H is connected.*

*For a symmetric pair (G, H, θ) we define an antiinvolution $\sigma : G \to G$ by $\sigma(g) := \theta(g^{-1})$, denote $\mathfrak{g} := LieG$, $\mathfrak{h} := LieH$, $\mathfrak{g}^\sigma := \{a \in \mathfrak{g} | \theta(a) = -a\}$. Note that H acts on $\mathfrak{g}^\sigma$ by the adjoint action. Denote also $G^\sigma := \{g \in G | \sigma(g) = g\}$ and define a **symmetrization map** $s : G \to G^\sigma$ by $s(g) := g\sigma(g)$.*

In case when the involution is obvious we will omit it.

Remark 2.2.2. *Let (G, H, θ) be a symmetric pair. Then $\mathfrak{g}$ has a $\mathbb{Z}/2\mathbb{Z}$ grading given by θ.*

Definition 2.2.3. *Let (G_1, H_1, θ_1) and (G_2, H_2, θ_2) be symmetric pairs. We define their **product** to be the symmetric pair $(G_1 \times G_2, H_1 \times H_2, \theta_1 \times \theta_2)$.*

Definition 2.2.4. *We call a symmetric pair (G, H, θ) **good** if for any closed $H(F) \times H(F)$ orbit $O \subset G(F)$, we have $\sigma(O) = O$.*

Proposition 2.2.5. *Every connected symmetric pair over $\mathbb{C}$ is good.*

For proof see e.g. [AG2], Corollary 7.1.7.

Definition 2.2.6. *We say that a symmetric pair (G, H, θ) is a **GK pair** if any $H(F) \times H(F)$ - invariant distribution on $G(F)$ is σ - invariant.*

Remark 2.2.7. *Theorem 2.1.4 implies that any GK pair satisfies GP2.*

2.2.1. *Descendants of symmetric pairs.*

Proposition 2.2.8. *Let (G, H, θ) be a symmetric pair. Let $g \in G(F)$ such that HgH is closed. Let $x = s(g)$. Then x is a semisimple element of G.*

For proof see e.g. [AG2], Proposition 7.2.1.

Definition 2.2.9. *In the notations of the previous proposition we will say that the pair $(G_x, H_x, \theta|_{G_x})$ is a **descendant** of (G, H, θ).*

2.2.2. *Tame symmetric pairs.*

Definition 2.2.10. *Let π be an action of a reductive group G on a smooth affine variety X. We say that an algebraic automorphism τ of X is G-**admissible** if*
(i) $\pi(G(F))$ is of index at most 2 in the group of automorphisms of X generated by $\pi(G(F))$ and τ.
(ii) For any closed $G(F)$ orbit $O \subset X(F)$, we have $\tau(O) = O$.

Definition 2.2.11. *We call an action of a reductive group G on a smooth affine variety X **tame** if for any G-admissible $\tau : X \to X$, every $G(F)$-invariant distribution on $X(F)$ is τ-invariant.*
*We call a symmetric pair (G, H, θ) **tame** if the action of $H \times H$ on G is tame.*

Remark 2.2.12. *Evidently, any good tame symmetric pair is a GK pair.*

Notation 2.2.13. *Let V be an algebraic finite dimensional representation over F of a reductive group G. Denote $Q(V) := V/V^G$. Since G is reductive, there is a canonical embedding $Q(V) \hookrightarrow V$.*

Notation 2.2.14. *Let (G, H, θ) be a symmetric pair. We denote by $\mathcal{N}_{G,H}$ the subset of all the nilpotent elements in $Q(\mathfrak{g}^\sigma)$. Denote $R_{G,H} := Q(\mathfrak{g}^\sigma) - \mathcal{N}_{G,H}$.*

Note that our notion of $R_{G,H}$ coincides with the notion $R(\mathfrak{g}^\sigma)$ used in [AG2], Notation 2.3.10. This follows from Lemma 7.1.11 in [AG2].

Definition 2.2.15. *We call a symmetric pair (G, H, θ) **weakly linearly tame** if for any H-admissible transformation τ of $\mathfrak{g}^\sigma$ such that every $H(F)$-invariant distribution on $R_{G,H}$ is also τ-invariant, we have*
(*) every $H(F)$-invariant distribution on $Q(\mathfrak{g}^\sigma)$ is also τ-invariant.

Theorem 2.2.16. *Let (G, H, θ) be a symmetric pair. Suppose that all its descendants (including itself) are weakly linearly tame. Then (G, H, θ) is tame.*

For proof see Theorem 7.3.3 in [AG2].

Now we would like to formulate a criterion for being weakly linearly tame. For it we will need the following lemma and notation.

Lemma 2.2.17. *Let (G, H, θ) be a symmetric pair. Then any nilpotent element $x \in \mathfrak{g}^\sigma$ can be extended to an sl_2 triple $(x, d(x), x_-)$ such that $d(x) \in \mathfrak{h}$ and $x_- \in \mathfrak{g}^\sigma$.*

For proof see e.g. [AG2], Lemma 7.1.11.

Notation 2.2.18. *We will use the notation $d(x)$ from the last lemma in the future. It is not uniquely defined but whenever we will use this notation nothing will depend on its choice.*

Proposition 2.2.19. *Let (G, H, θ) be a symmetric pair. Suppose that for any nilpotent $x \in \mathfrak{g}^\sigma$ we have*

$$\mathrm{Tr}(ad(d(x))|_{\mathfrak{h}_x}) < \dim Q(\mathfrak{g}^\sigma).$$

Then the pair (G, H, θ) is weakly linearly tame.

This proposition follows from [AG2] (Propositions 7.3.7 and 7.3.5).

2.2.3. *Regular symmetric pairs.*

Definition 2.2.20. *Let (G, H, θ) be a symmetric pair. We call an element $g \in G(F)$ **admissible** if*
(i) $Ad(g)$ commutes with θ (or, equivalently, $s(g) \in Z(G)$) and
(ii) $Ad(g)|_{\mathfrak{g}^\sigma}$ is H-admissible.

Definition 2.2.21. *We call a symmetric pair (G, H, θ) **regular** if for any admissible $g \in G(F)$ such that every $H(F)$-invariant distribution on $R_{G,H}$ is also $Ad(g)$-invariant, we have*
(*) every $H(F)$-invariant distribution on $Q(\mathfrak{g}^\sigma)$ is also $Ad(g)$-invariant.

The following two propositions are evident.

Proposition 2.2.22. *Let (G, H, θ) be symmetric pair. Suppose that any $g \in G(F)$ satisfying $\sigma(g)g \in Z(G(F))$ lies in $Z(G(F))H(F)$. Then (G, H, θ) is regular. In particular if the normalizer of $H(F)$ lies inside $Z(G(F))H(F)$ then (G, H, θ) is regular.*

Proposition 2.2.23.
(i) Any weakly linearly tame pair is regular.
(ii) A product of regular pairs is regular (see [AG2], Proposition 7.4.4).

In section 4 we will introduce terminology that will help to verify the condition of Proposition 2.2.19. The importance of the notion of regular pair is demonstrated by the following theorem.

Theorem 2.2.24. *Let (G, H, θ) be a good symmetric pair such that all its descendants (including itself) are regular. Then it is a GK pair.*

For proof see [AG2], Theorem 7.4.5.

3. Main Results

Here we formulate the main results of the paper and explain how they follow from the rest of the paper.

Definition 3.0.1. *A quadratic space is a linear space with a fixed non-degenerate quadratic form.*

Let F' be an extension of F and V be a quadratic space over it. We denote by $O(V)$ the canonical algebraic group such that its F-points form the group of orthogonal transformations of V.

Definition 3.0.2. *Let D be a field with an involution τ. A hermitian space over (D, τ) is a linear space over D with a fixed non-degenerate hermitian form.*

Suppose that D is an extension of F and $F \subset D^\tau$. Let V be a hermitian space over (D, τ). We denote by $U(V)$ the canonical algebraic group such that its F-points form the group of unitary transformations of V.

Definition 3.0.3. *Let G be a reductive group and $\varepsilon \in G$ be an element of order 2. We denote by (G, G_ε) the symmetric pair defined by the involution $x \mapsto \varepsilon x \varepsilon$.*

The following lemma is straightforward.

Lemma 3.0.4. *Let V be a quadratic space.*
(i) Let $\varepsilon \in GL(V)$ be an element of order 2. Then $GL(V)_\varepsilon \cong GL(V_1) \times GL(V_2)$ for some decomposition $V = V_1 \oplus V_2$.
(ii) Let $\varepsilon \in O(V)$ be an element of order 2. Then $O(V)_\varepsilon \cong O(V_1) \times O(V_2)$ for some orthogonal decomposition $V = V_1 \oplus V_2$.
(iii) Let V be a hermitian space.
Let $\varepsilon \in U(V)$ be an element of order 2. Then $U(V)_\varepsilon \cong U(V_1) \times U(V_2)$ for some orthogonal decomposition $V = V_1 \oplus V_2$.

Theorem 3.0.5. *Let V be a quadratic space over F. Then all the descendants of the pair $(O(V), O(V)_\varepsilon)$ are regular.*

Proof. By Theorem 6.5.1 below, the descendants of the pair $(O(V), O(V)_\varepsilon)$ are products of pairs of the types
(i) $(GL(W), O(W))$ for some quadratic space W over some field F' that extends F
(ii) $(U(W_E), O(W))$ for some quadratic space W over some field F' that extends F, and some quadratic extension E of F'. Here, $W_E := W \otimes_{F'} E$ is the extension of scalars with the corresponding hermitian structure.
(iii) $(O(W), O(W)_\varepsilon)$ for some quadratic space W over F.
The pair (i) is regular by Theorem 5.1.1 below. The pair (ii) is regular by subsection 5.3 below. The pair (iii) is regular by subsection 5.2 below. □

Corollary 3.0.6. *Suppose that $F = \mathbb{C}$ and Let V be a quadratic space over it. Then the pair $(O(V), O(V)_\varepsilon)$ satisfies GP1.*

Proof. This pair is good by Proposition 2.2.5 and all its descendants are regular. Hence by Theorem 2.2.24 it is a GK pair. Therefore by Theorem 2.1.4 it satisfies GP2. Now, by Proposition 2.1.5, it satisfies GP1. □

Theorem 3.0.7. *Let D/F be a quadratic extension and $\tau \in Gal(D/F)$ be the non-trivial element. Let V be a hermitian space over (D, τ). Then all the descendants of the pair $(U(V), U(V)_\varepsilon)$ are regular.*

Proof. By theorem 6.6.1 below, the descendants of the pair $(U(V), U(V)_\varepsilon)$ are products of pairs of the types
(a) $(G \times G, \Delta G)$ for some reductive group G.
(b) $(GL(W), U(W))$ for some hermitian space W over some extension (D', τ') of (D, τ)
(c) $(G_{E/F}, G)$ for some reductive group G and some quadratic extension E/F.
(d) $(GL(W), GL(W)_\varepsilon)$ where W is a linear space over D and $\varepsilon \in GL(W)$ is an element of order ≤ 2.
(e) $(U(W), U(W)_\varepsilon)$ where W is a hermitian space over (D, τ).

The pairs (a) and (c) are regular by Theorem 4.2.12 below. The pairs (b) and (e) are regular by subsection 5.3 below. The pair (d) is regular by Theorem 4.2.13 below. □

Theorem 3.0.8. *Let V be a quadratic space over F. Then all the descendants of the pair $(GL(V), O(V))$ are weakly linearly tame. In particular, this pair is tame.*

Proof. By Theorem 6.2.1 below, the descendants of the pair $(GL(V), O(V))$ are products of pairs of the type $(GL(W), O(W))$ for some quadratic space W over some field F' that extends F. By Theorem 5.1.1 below, these pairs are weakly linearly tame. Now, the pair $(GL(V), O(V))$ is tame by Theorem 2.2.16. □

Corollary 3.0.9. *Suppose that $F = \mathbb{C}$ and Let V be a quadratic space over it. Then the pair $(GL(V), O(V))$ is GP1.*

Theorem 3.0.10. *Let D/F be a quadratic extension and $\tau \in Gal(D/F)$ be the non-trivial element. Let V be a hermitian space over (D, τ). Then all the descendants of the pair $(GL(V), U(V))$ are weakly linearly tame. In particular, this pair is tame.*

Proof. By Theorem 6.3.1 below, all the descendants of the pair $(GL(V), U(V))$ are products of pairs of the types
(i) $(GL(W) \times GL(W), \Delta GL(W))$ for some linear space W over some field D' that extends D
(ii) $(GL(W), U(W))$ for some hermitian space W over some (D', τ') that extends (D, τ).
The pair (i) is weakly linearly tame by Theorem 4.2.12 below and the pair (ii) is weakly linearly tame by subsection 5.3 below. Now, the pair $(GL(V), U(V))$ is tame by Theorem 2.2.16. □

Theorem 3.0.11. *Let V be a quadratic space over F. Let D/F be a quadratic extension and $\tau \in Gal(D/F)$ be the non-trivial element. Let $V_D := V \otimes_F D$ be its extension of scalars with the corresponding hermitian structure. Then all the descendants of the pair $(U(V_D), O(V))$ are weakly linearly tame. In particular, this pair is tame.*

Proof. By Theorem 6.4.1 below, all the descendants of the pair $(U(V_D), O(V))$ are products of pairs of the types
(i) $(GL(W), O(W))$ for some quadratic space W over some field F' that extends F.
(ii) $(U(W_{D'}), O(W))$ for some extension (D', τ') of (D, τ) and some quadratic space W over $D'^{\tau'}$.
The pair (i) is weakly linearly tame by Theorem 5.1.1 below and the pair (ii) is weakly linearly tame by subsection 5.3 below. Now, the pair $(GL(V), U(V))$ is tame by Theorem 2.2.16. □

4. $\mathbb{Z}/2\mathbb{Z}$ GRADED REPRESENTATIONS OF sl_2 AND THEIR DEFECTS

In this section we will introduce terminology that will help to verify the condition of Proposition 2.2.19.

4.1. **Graded representations of sl_2.**

Definition 4.1.1. *We fix standard basis e, h, f of $sl_2(F)$. We fix a grading on $sl_2(F)$ given by $h \in sl_2(F)_0$ and $e, f \in sl_2(F)_1$. A **graded representation of** sl_2 is a representation of sl_2 on a graded vector space $V = V_0 \oplus V_1$ such that $sl_2(F)_i(V_j) \subset V_{i+j}$ where $i, j \in \mathbb{Z}/2\mathbb{Z}$.*

The following lemma is standard.

Lemma 4.1.2.
(i) Every graded representation of sl_2 which is irreducible as a graded representation is irreducible just as a representation.
(ii) Every irreducible representation V of sl_2 admits exactly two gradings. In one highest weight vector lies in V_0 and in the other in V_1.

Definition 4.1.3. *We denote by V_λ^w the irreducible graded representation of sl_2 with highest weight λ and highest weight vector of parity p where $w = (-1)^p$.*

The following lemma is straightforward.

Lemma 4.1.4.

(1)
$$(V_\lambda^w)^* = V_\lambda^{w(-1)^\lambda}$$

(2)
$$V_{\lambda_1}^{w_1} \otimes V_{\lambda_2}^{w_2} = \bigoplus_{i=0}^{\min(\lambda_1,\lambda_2)} V_{\lambda_1+\lambda_2-2i}^{w_1 w_2 (-1)^i}$$

(3)
$$\Lambda^2(V_\lambda^w) = \bigoplus_{i=0}^{\lfloor \frac{\lambda-1}{2} \rfloor} V_{2\lambda-4i-2}^{-1}.$$

4.2. Defects.

Definition 4.2.1. *Let π be a graded representation of sl_2. We define the **defect** of π to be*

$$def(\pi) = \mathrm{Tr}(h|_{(\pi^e)_0}) - \dim(\pi_1)$$

The following lemma is straightforward

Lemma 4.2.2.

(4)
$$def(\pi \oplus \tau) = def(\pi) + def(\tau)$$

(5)
$$def(V_\lambda^w) = \frac{1}{2}(\lambda w + w(\frac{1+(-1)^\lambda}{2}) - 1) = \frac{1}{2} \begin{cases} \lambda w + w - 1 & \lambda \text{ is even} \\ \lambda w - 1 & \lambda \text{ is odd} \end{cases}$$

Definition 4.2.3. *Let $\mathfrak{g}$ be a $(\mathbb{Z}/2\mathbb{Z})$ graded Lie algebra. We say that $\mathfrak{g}$ **is of negative defect** if for any graded homomorphism $\pi : sl_2 \to \mathfrak{g}$, the defect of $\mathfrak{g}$ with respect to the adjoint action of sl_2 is negative.*

*We say that $\mathfrak{g}$ **is of negative normalized defect** if the semi-simple part of $\mathfrak{g}$ (i.e. the quotient of $\mathfrak{g}$ by its center) is of negative defect.*

Remark 4.2.4. *Clearly, $\mathfrak{g}$ is of negative normalized defect if and only if for any graded homomorphism $\pi : sl_2 \to \mathfrak{g}$, the defect of $\mathfrak{g}$ with respect to the adjoint action of sl_2 is less than the dimension of the odd part of the center of $\mathfrak{g}$.*

Definition 4.2.5. *We say that a symmetric pair (G, H, θ) **is of negative normalized defect** if the Lie algebra $\mathfrak{g}$ with the grading defined by θ is of negative normalized defect.*

Lemma 4.2.6. *Let (G, H, θ) be a symmetric pair. Assume that $\mathfrak{g}$ is semi-simple. Then $Q(\mathfrak{g}^\sigma) = \mathfrak{g}^\sigma$.*

Proof.
Assume the contrary: there exist $0 \neq x \in \mathfrak{g}^\sigma$ such that $Hx = x$. Then $\dim(CN_{Hx,x}^{\mathfrak{g}^\sigma}) = \dim \mathfrak{g}^\sigma$, hence $CN_{Hx,x}^{\mathfrak{g}^\sigma} = \mathfrak{g}^\sigma$. On the other hand, $CN_{Hx,x}^{\mathfrak{g}^\sigma} \cong [\mathfrak{h}, x]^\perp = (\mathfrak{g}^\sigma)^x$ (here $(\cdot)^\perp$ means the ortogonal compliment w.r.t. the kiling form). Therefore $\mathfrak{g}^\sigma = (\mathfrak{g}^\sigma)^x$ and hence x lies in the center of $\mathfrak{g}$, which is impossible. $\square$

Proposition 2.2.19 can be rewritten now in the following form

Theorem 4.2.7. *A symmetric pair of negative normalized defect is weakly linearly tame.*

Evidently, a product of pairs of negative normalized defect is again of negative normalized defect. The following lemma is straightforward.

Lemma 4.2.8. *Let (G, H, θ) be a symmetric pair. Let F' be any field extending F. Let $(G_{F'}, H_{F'}, \theta)$ be the extension of (G, H, θ) to F'. Suppose that it is of negative normalized defect (as a pair over F'). Then (G, H, θ) and $(G_{F'/F}, H_{F'/F}, \theta)$ are of negative normalized defect (as pairs over F).*

In [AG2] we proved the following (easy) proposition (see [AG2], Lemma 7.6.6).

Proposition 4.2.9. *Let π be a representation of sl_2. Then $\mathrm{Tr}(h|_{(\pi^e)}) < \dim(\pi)$.*

We would like to reformulate it in terms of defect. For this we will need the following notation.

Notation 4.2.10.
(i) Let π be a representation of sl_2. We denote by $\overline{\pi}$ the representation of sl_2 on the same space defined by $\overline{\pi}(e) := -\pi(e)$, $\overline{\pi}(f) := -\pi(f)$ and $\overline{\pi}(h) := \pi(h)$.
(ii) We define grading on $\pi \oplus \overline{\pi}$ by the involution $s(v \oplus w) := w \oplus v$.

Proposition 4.2.9 can be reformulated in the following way.

Proposition 4.2.11. *Let π be a representation of sl_2. Then $def(\pi \oplus \overline{\pi}) < 0$.*

In [AG2] we also deduced from this proposition the following theorem (see [AG2], 7.6.2).

Theorem 4.2.12. *For any reductive group G, the pairs $(G \times G, \Delta G)$ and $(G_{E/F}, G)$ are of negative normalized defect and hence weakly linearly tame. Here ΔG is the diagonal in $G \times G$.*

In [AG2, §§2.7] we proved the following theorem.

Theorem 4.2.13. *The pair $(GL(V \oplus V), GL(V) \times GL(V))$ is of negative normalized defect and hence regular.*

Note that in the case $dim V \neq dim W$ the pair $(GL(V \oplus W), GL(V) \times GL(W))$ is obviously regular by Proposition 2.2.22.

5. Proof of regularity and tameness

5.1. **The pair $(GL(V), O(V))$.**

In this subsection we prove that the pair $(GL(V), O(V))$ is weakly linearly tame. For $\dim V \leq 1$ it is obvious. Hence it is enough to prove the following theorem.

Theorem 5.1.1. *Let V be a quadratic space of dimension at least 2. Then the pair $(GL(V), O(V))$ has negative normalized defect.*

We will need the following notation.

Notation 5.1.2. *Let π be a representation of sl_2. We define grading on $\pi \otimes \overline{\pi}$ by the involution $s(v \otimes w) := -w \otimes v$.*

Theorem 5.1.1 immediately follows from the following one.

Theorem 5.1.3. *Let π be a representation of sl_2 of dimension at least 2. Then $def(\pi \otimes \overline{\pi}) < -1$.*

This theorem in turn follows from the following lemma.

Lemma 5.1.4. *Let V_λ and V_μ be irreducible representations of sl_2. Then*
(i) $def(V_\lambda \otimes \overline{V_\lambda}) = -(\lambda + 1)(\frac{\lambda}{2} + 1)$.
(ii) $def(V_\lambda \otimes \overline{V_\mu} \oplus V_\mu \otimes \overline{V_\lambda}) < 0$.

Proof.
(i) Follows from the fact that $V_\lambda \otimes \overline{V_\lambda} = \bigoplus_{i=0}^{\lambda} V_{2\lambda-2i}^{-1}$ and from Lemma 4.2.2.
(ii) Follows from Proposition 4.2.11. $\qquad\square$

5.2. **The pair $(O(V_1 \oplus V_2), O(V_1) \times O(V_2))$.**

In this subsection prove that the pair $(O(V_1 \oplus V_2), O(V_1) \times O(V_2))$ is regular. For that it is enough to prove the following theorem.

Theorem 5.2.1. *Let V_1 and V_2 be quadratic spaces. Assume $\dim V_1 = \dim V_2$. Then the pair $(O(V_1 \oplus V_2), O(V_1) \times O(V_2))$ has negative normalized defect.*

This theorem immediately follows from the following one.

Theorem 5.2.2. *Let π be a (non-zero) graded representation of sl_2 such that $\dim \pi_0 = \dim \pi_1$ and $\pi \simeq \pi^*$. Then $\Lambda^2(\pi)$ has negative defect.*

For this theorem we will need the following lemma.

Lemma 5.2.3. *Let $V_{\lambda_1}^{w_1}$ and $V_{\lambda_2}^{w_2}$ be irreducible graded representations of sl_2. Then*
(i) $def(V_{\lambda_1}^{w_1} \otimes V_{\lambda_2}^{w_2}) =$

$$= -\frac{1}{2} \begin{cases} \min(\lambda_1, \lambda_2) + 1 - \frac{w_1 w_2}{2}(\lambda_1 + \lambda_2 + 1 + (-1)^{\min(\lambda_1, \lambda_2)}(|\lambda_1 - \lambda_2| - 1)), & \lambda_1 \neq \lambda_2 & \mod 2; \\ \min(\lambda_1, \lambda_2) + 1 - w_1 w_2(\max(\lambda_1, \lambda_2) + 1), & \lambda_1 \equiv \lambda_2 \equiv 0 & \mod 2; \\ \min(\lambda_1, \lambda_2) + 1 - w_1 w_2(\min(\lambda_1, \lambda_2) + 1), & \lambda_1 \equiv \lambda_2 \equiv 1 & \mod 2; \end{cases}$$

(ii) $def(\Lambda^2(V_{\lambda_1}^{w_1})) = -\frac{\lambda^2}{4} - \frac{\lambda}{2} - \frac{1+(-1)^{\lambda+1}}{8}$

Proof. This lemma follows by straightforward computations from Lemmas 4.1.4 and 4.2.2. $\square$

Proof of Theorem 5.2.2. Since $\pi \simeq \pi^*$, π can be decomposed to a direct sum of irreducible graded representations in the following way

$$\pi = (\bigoplus_{i=1}^{l} V_{\lambda_i}^1) \oplus (\bigoplus_{j=1}^{m} V_{\mu_j}^{-1}) \oplus (\bigoplus_{k=1}^{n} V_{\nu_k}^1 \oplus V_{\nu_k}^{-1}).$$

Here, all λ_i and μ_j are even and ν_k are odd. Since $\dim \pi_0 = \dim \pi_1$, $l = m$.

By the last lemma, $def(V_{\lambda_i}^1 \otimes (V_{\nu_k}^1 \oplus V_{\nu_k}^{-1})) = -(\min(\lambda_1, \lambda_2) + 1) < 0$. Similarly, $def(V_{\mu_j}^{-1} \otimes (V_{\nu_k}^1 \oplus V_{\nu_k}^{-1})) < 0$. Also, $def((V_{\nu_{k_1}}^1 \oplus V_{\nu_{k_1}}^{-1}) \otimes (V_{\nu_{k_2}}^1 \oplus V_{\nu_{k_2}}^{-1})) < 0$ and $def(\Lambda^2(V_\lambda^w)) \leq 0$ for all λ and w.

Hence if $l = 0$ we are done. Otherwise we can assume $n = 0$. Now,

$$def(\Lambda^2(\pi)) \leq \sum_{1 \leq i < j \leq l} |\lambda_i - \lambda_j| + \sum_{1 \leq i < j \leq l} |\mu_i - \mu_j| - \sum_{1 \leq i,j \leq l} (\lambda_i + \mu_j + 2) <$$

$$< \sum_{1 \leq i < j \leq l} (\lambda_i + \lambda_j) + \sum_{1 \leq i < j \leq l} (\mu_i + \mu_j) - \sum_{1 \leq i,j \leq l} (\lambda_i + \mu_j) = -\sum_{i=1}^{l}(\lambda_i + \mu_i) \leq 0.$$

$\square$

5.3. **The pairs** $(GL(V), U(V))$, $(U(V_1 \oplus V_2), U(V_1) \times U(V_2))$ **and** $(U(V_D), O(V))$**.**

In this subsection prove that the pairs $(GL(V), U(V))$ and $(U(V_D), O(V))$ are weakly linearly tame and the pair $(U(V_1 \oplus V_2), U(V_1) \times U(V_2))$ is regular.

Let V be a hermitian space. Note that $(GL(V), U(V))$ is a form of $(GL(W) \times GL(W), \Delta GL(W))$ for some W and $(U(V \oplus V), U(V) \times U(V))$ is a form of $(GL(W \oplus W), GL(W) \times GL(W))$ for some W.

Also, for any quadratic space V of dimension at least 2 and any quadratic extension D/F, the pair $(U(V_D), O(V))$ is a form of $(GL(W), O(W))$ for some quadratic space W.

Hence by Lemma 4.2.8 and Theorems 4.2.12, 4.2.13 and 5.1.1 those 3 pairs are of negative normalized defect and hence are weakly linearly tame. If $\dim V \leq 1$ then the pair $(U(V_D), O(V))$ is obviously linearly tame.

If V_1 and V_2 are non-isomorphic hermitian spaces then $(U(V_1 \oplus V_2), U(V_1) \times U(V_2))$ is regular by Proposition 2.2.22.

6. Computation of descendants

In this section we compute the descendants of the pairs we discussed before. For this we use a technique of computing centralizers of semisimple elements of orthogonal and unitary groups, which is described in [SpSt]. The proofs in this section are rather straightforward but technically involved. The most important things in this section are the formulations of the main theorems: Theorems 6.2.1, 6.3.1, 6.4.1, 6.5.1, 6.6.1. Those theorems are summarized graphically in subsection 6.7.

6.1. **Preliminaries and notation for orthogonal and unitary groups.**

6.1.1. *Orthogonal group.*

Notation 6.1.1. *Let V be a linear space over F. Let $x \in GL(V)$ be a semi-simple element and let $Q = \sum_{i=0}^{n} a_i \xi^i \in F[\xi]$ (where $a_n \neq 0$) be an irreducible polynomial.*
- *Denote $F_Q := F[\xi]/Q$*
- *Denote $inv(Q) := \sum_{i=0}^{n} a_{n-i} \xi^i$*
- *Denote $V_{Q,x}^0 := Ker(Q(x))$ and $V_{Q,x}^1 := Ker(inv(Q)(x))$*
  We define an F_Q-linear space structure on $V_{Q,x}^i$ by letting ξ act on $V_{Q,x}^0$ by x and on $V_{Q,x}^1$ by x^{-1}. We will consider $V_{Q,x}^i$ as linear spaces over F_Q.
- *In case Q is proportional to $inv(Q)$ we define an involution μ on F_Q by $\mu(P(\xi)) := P(\xi^{-1})$.*
- *For a linear space W over F_Q we can consider its dual space W^* over F_Q and the dual space of W over F which we denote by W_F^*. The space W_F^* has a canonical structure of a linear space over F_Q. The spaces W_F^* and W^* can be identified as linear spaces over F_Q. For this identification one has to choose an F-linear functional $\lambda : F_Q \to F$. We will fix such functional λ such that $\lambda(\mu(d)) = \lambda(d)$ if μ is defined.*
  From now on we will identify W_F^* and W^*.

The following two lemmas are straightforward.

Lemma 6.1.2. *Let V be a quadratic space over F. Let $x \in GL(V)$ and let $P, Q \in F[\xi]$ be irreducible polynomials. Suppose that either*
(i) $x = x^t$ and P is not proportional to Q or
(ii) $x \in O(V)$ and P is not proportional to $inv(Q)$
Then $Ker(Q(x))$ is orthogonal to $Ker(P(x))$.

Lemma 6.1.3. *Let (V, B) be a quadratic space over F. Let $x \in GL(V)$ be a semi-simple element and let $Q \in F[\xi]$ be an irreducible polynomial. Then*
(i) If $x = x^t$ then B defines an F_Q-linear isomorphism $V_{Q,x}^i \cong (V_{Q,x}^i)^*$.
(ii) If $x \in O(V)$ then B defines an F_Q-linear isomorphism $V_{Q,x}^i \cong (V_{Q,x}^{1-i})^*$.

6.1.2. *Unitary group.*
From now and till the end of the paper we fix a quadratic extension D of F and denote by τ the involution that fixes F.

Notation 6.1.4. *Let V be a hermitian space over (D, τ). Let $x \in GL(V)$ be a semi-simple element and let $Q = \sum_{i=0}^{n} a_i \xi^i \in D[\xi]$ (where $a_n \neq 0$) be an irreducible polynomial.*
- *Denote $D_Q := D[\xi]/Q$*

- *Denote*
$$inv(Q) := \sum_{i=0}^{n} a_{n-i} \xi^i, \quad Q^* := \tau(inv(Q))$$

- *Denote*
$$V_{Q,x}^{00} := Ker(Q(x)), \quad V_{Q,x}^{01} := Ker(Q^*(x)), \quad V_{Q,x}^{10} := Ker(inv(Q)(x)), \quad V_{Q,x}^{11} := Ker(\tau(Q)(x)).$$

  We twist the action of D on $V_{Q,x}^{i1}$ by τ. We define D_Q-linear space structure on $V_{Q,x}^{ij}$ by letting ξ act on $V_{Q,x}^{0j}$ by x and on $V_{Q,x}^{1j}$ by x^{-1}. We will consider $V_{Q,x}^{ij}$ as linear spaces over D_Q.
- *If Q is proportional to Q^* we define an involution μ^{01} on D_Q by $\mu^{01}(P(\xi)) := \tau(P)(\xi^{-1})$.*
  If Q is proportional to $inv(Q)$ we define an involution μ^{10} on D_Q by $\mu^{10}(P(\xi)) := P(\xi^{-1})$.
  If Q is proportional to $\tau(Q)$ we define an involution μ^{11} on D_Q by $\mu^{11}(P(\xi)) := \tau(P)(\xi)$.
- *For a linear space W over D_Q we can consider its dual space W^* over D_Q and the dual space of W over D which we denote by W_D^*. The space W_D^* has a canonical structure of a linear space over D_Q. The spaces W_D^* and W^* can be identified as linear spaces over D_Q. For this identification one has to choose a D-linear functional $\lambda : D_Q \to D$. We will fix such functional λ such that*
$$\lambda(\mu^{ij}(d)) = \tau^j(\lambda(d)) \quad \text{if } \mu^{ij} \text{ is defined.}$$

From now on we will identify W_D^* and W^*.

The following two lemmas are straightforward.

Lemma 6.1.5. *Let V be a hermitian space over (D, τ). Let $x \in GL(V)$ and let $P, Q \in D[\xi]$ be irreducible polynomials. Suppose that either*
(i) $x = x^*$ and P is not proportional to $\tau(Q)$ or
(ii) $x \in U(V)$ and P is not proportional to Q^*
Then $Ker(Q(x))$ is orthogonal to $Ker(P(x))$.

Lemma 6.1.6. *Let (V, B) be a hermitian space over (D, τ). Let $x \in GL(V)$ be a semi-simple element and let $Q \in D[\xi]$ be an irreducible polynomial. Then*
(i) If $x = x^*$ then B defines a D_Q-linear isomorphism $V_{Q,x}^{ij} \cong (V_{Q,x}^{1-i,1-j})^*$.
(ii) If $x \in U(V)$ then B defines a D_Q-linear isomorphism $V_{Q,x}^{ij} \cong (V_{Q,x}^{i,1-j})^*$.

6.2. The pair $(GL(V), O(V))$.

Theorem 6.2.1. *Let V be a quadratic space over F. Then all the descendants of the pair $(GL(V), O(V))$ are products of pairs of the type $(GL(W), O(W))$ for some quadratic space W over some field F' that extends F.*

Proof. Note that in this case the anti-involution σ is given by $\sigma(x) = x^t$. Let $x \in GL(V)^\sigma$ be a semi-simple element. Let P be the minimal polynomial of x. We will now discuss a special case and then deduce the general case from it.

Case 1. P is irreducible over F.
Clearly $GL(V)_x \cong GL(V_{P,x}^0)$. The isomorphism $V_{P,x}^0 \cong (V_{P,x}^0)^*$ gives a quadratic structure on $V_{P,x}^0$. Now $O(V)_x \cong O(V_{P,x}^0)$.

Case 2. General case
Let $P = \prod_{i \in I} P_i$ be the decomposition of P to irreducible polynomials. Clearly $V = \bigoplus V_{P_i,x}^0$ and $V_{P_i,x}^0$ are orthogonal to each other. Hence the pair $(GL(V)_x, O(V)_x)$ is a product of pairs from Case 1. $\square$

6.3. The pair $(GL(V), U(V))$.

Theorem 6.3.1. *Let (V, B) be a hermitian space over (D, τ). Then all the descendants of the pair $(GL(V), U(V))$ are products of pairs of the types*
(i) $(GL(W) \times GL(W), \Delta GL(W))$ for some linear space W over some field D' that extends D
(ii) $(GL(W), U(W))$ for some hermitian space W over some (D', τ') that extends (D, τ).

Proof. Note that in this case the anti-involution σ is given by $\sigma(x) = x^*$. Let $x \in GL(V)^\sigma$ be a semi-simple element. Let P be the minimal polynomial of x. Note that $\tau(P)$ is proportional to P. We will now discuss 2 special cases and then deduce the general case from them.

Case 1. $P = Q\tau(Q)$ where Q is irreducible over D.
Clearly $GL(V)_x \cong GL(V_{Q,x}^{00}) \times GL(V_{Q,x}^{11})$. Recall that B gives a non-degenerate pairing between $V_{Q,x}^{00}$ and $V_{Q,x}^{11}$, and the spaces $V_{Q,x}^{ii}$ are isotropic. Therefore

$$GL(V_{Q,x}^{00}) \cong GL(V_{Q,x}^{11}), \quad GL(V)_x \cong GL(V_{Q,x}^{00})^2 \text{ and } U(V)_x \cong \Delta GL(V_{Q,x}^{00}) < GL(V_{Q,x}^{00})^2.$$

Case 2. P is irreducible over D.
Clearly $GL(V)_x \cong GL(V_{P,x}^{00})$ and $V_{P,x}^{00}$ is identical to $V_{P,x}^{11}$ as F-linear spaces but the actions of D_P differ by a twist by μ^{11}. Hence the isomorphism $V_{P,x}^{00} \cong (V_{P,x}^{11})^*$ gives a hermitian structure on $V_{P,x}^{00}$ over (D_P, μ^{11}). Now $U(V)_x \cong U(V_{P,x}^{00}) < GL(V_{P,x}^{00})$.

Case 3. General case
Let $P = \prod_{i \in I} P_i$ be the decomposition of P to irreducible polynomials. Then $\tau(P_i)$ is proportional to

$P_{s(i)}$ where s is some permutation of I of order ≤ 2. Let $I = \bigsqcup I_\alpha$ be the decomposition of I to orbits of s. Denote $V_\alpha := Ker(\prod_{i \in \alpha} P_i(x))$. Clearly $V = \bigoplus V_\alpha$ and V_α are orthogonal to each other. Hence the pair $(GL(V)_x, U(V)_x)$ is a product of pairs from the first 2 cases. $\square$

6.4. **The pair** $(U(V_D), O(V))$.

Theorem 6.4.1. *Let (V, B) be a quadratic space over F. Let $V_D := V \otimes_F D$ be its extension of scalars with the corresponding hermitian structure.*

Then all the descendants of the pair $(U(V_D), O(V))$ are products of pairs of the types
(i) $(GL(W), O(W))$ for some quadratic space W over some field F' that extends F.
(ii) $(U(W_{D'}), O(W))$ for some extension (D', τ') of (D, τ) and some quadratic space W over $D'^{\tau'}$.

For the proof of this theorem we will need the following notation and lemma.

Notation 6.4.2. *Let (V, B) be a quadratic space over F. The involution τ defines an involution $\widetilde{\tau}$ on V_D. The form B defines a quadratic form B_D on V_D and a hermitian form B_D^τ on V_D.*

Lemma 6.4.3. *Let (V, B) be a quadratic space over F. Let P be an irreducible polynomial. Let $x \in U(V_D)$ be a semi-simple element such that $x = x^t$ (where x^t is defined by B_D). Then the involution $\widetilde{\tau}$ gives a D_P-linear isomorphism $V_{P,x}^{ij} \cong V_{P,x}^{i,1-j}$.*

Proof. We will show that $\widetilde{\tau}$ maps $V_{P,x}^{00}$ to $V_{P,x}^{11}$, and the other cases are done similarly. Let $v \in V_{P,x}^{ij}$. We have
$$P^*(x)(\widetilde{\tau}(v)) = \widetilde{\tau}(inv(P)(x^*)(v)) = \widetilde{\tau}(inv(P)(x^{-1})(v)) = \widetilde{\tau}(x^{-degP} P(x)(v)) = 0.$$
$\square$

Proof of Theorem 6.4.1. Note that in this case the anti-involution σ is given by $\sigma(x) = x^t$. Let $x \in U(V_D)^\sigma$ be a semi-simple element. Let P be the minimal polynomial of x. Then P is proportional to P^*. We will now discuss 2 special cases and then deduce the general case from them.

Case 1. $P = QQ^*$ where Q is irreducible over D.
Clearly $GL(V)_x \cong GL(V_{Q,x}^{00}) \times GL(V_{Q,x}^{01})$. Recall that B_D^τ gives a non-degenerate pairing between $V_{Q,x}^{00}$ and $V_{Q,x}^{01}$, and the spaces $V_{Q,x}^{0i}$ are isotropic. Therefore
$$GL(V_{Q,x}^{00}) \cong GL(V_{Q,x}^{01}), \quad GL(V)_x \cong GL(V_{Q,x}^{00})^2, \quad U(V)_x \cong \Delta GL(V_{Q,x}^{00}) < GL(V_{Q,x}^{00})^2$$
Compose the isomorphism $V_{Q,x}^{00} \cong V_{Q,x}^{01}$ given by $\widetilde{\tau}$ with the isomorphism $V_{Q,x}^{01} \cong (V_{Q,x}^{00})^*$ given by B_D^τ. This gives a quadratic structure on $V_{Q,x}^{00}$. Now
$$O(V)_x \cong \Delta O(V_{Q,x}^{00}) < \Delta GL(V_{Q,x}^{00}).$$

Case 2. P is irreducible over D.
Clearly $GL(V)_x \cong GL(V_{P,x}^{00})$ and $V_{P,x}^{00}$ is identical to $V_{P,x}^{01}$ as F-linear spaces but the actions of D_P on them differ by a twist by μ^{01}. Hence the isomorphism $V_{P,x}^{00} \cong (V_{P,x}^{01})^*$ given by B_D^τ gives a hermitian structure on $V_{P,x}^{00}$ over (D_P, μ^{01}) and the isomorphism $V_{P,x}^{00} \cong V_{P,x}^{01}$ given by $\widetilde{\tau}$ gives an antilinear involution of $V_{P,x}^{00}$. Now
$$U(V)_x \cong U(V_{P,x}^{00}) < GL(V_{P,x}^{00}) \text{ and } O(V)_x \cong O(V_{P,x}^{00}) < U(V_{P,x}^{00}).$$

Case 3. General case
Let $P = \prod_{i \in I} P_i$ be the decomposition of P to irreducible polynomials. Then P_i^* is proportional to $P_{s(i)}$ where s is some permutation of I of order ≤ 2. Let $I = \bigsqcup I_\alpha$ be the decomposition of I to orbits of s. Denote $V_\alpha := Ker(\prod_{i \in \alpha} P_i(x))$. Clearly $V_D = \bigoplus V_\alpha$, V_α are orthogonal to each other and each V_α is invariant with respect to $\widetilde{\tau}$. Hence the pair $(GL(V)_x, U(V)_x)$ is a product of pairs from the first 2 cases. $\square$

6.5. **The pair** $(O(V_1 \oplus V_2), O(V_1) \times O(V_2))$**.**

Theorem 6.5.1. *Let (V, B) be a quadratic space over F.*
Let $\varepsilon \in O(V)$ be an element of order 2. Then all the descendants of the pair $(O(V), O(V)_\varepsilon)$ are products of pairs of the types
(i) $(GL(W), O(W))$ for some quadratic space W over some field F' that extends F
(ii) $(U(W_E), O(W))$ for some quadratic space W over some field F' that extends F, and some quadratic extension E of F'.
(iii) $(O(W), O(W)_\varepsilon)$ for some quadratic space W over F.

For the proof of this theorem we will need the following straightforward lemma.

Lemma 6.5.2. *Let (V, B) be a quadratic space over F.*
Let $\varepsilon \in O(V)$ be an element of order 2. Let $x \in O(V)$ such that $\varepsilon x \varepsilon = x^{-1}$. Let Q be an irreducible polynomial. Then ε gives an F_Q- linear isomorphism $V_{Q,x}^i \cong V_{Q,x}^{1-i}$.

Proof of Theorem 6.5.1. Note that the involution σ on $O(V)$ is given by $x \mapsto \varepsilon x^{-1} \varepsilon$. Let $x \in O(V)^\sigma$ be a semi-simple element and let P be its minimal polynomial.

Note that the minimal polynomial of x^{-1} is $inv(P)$ and hence P is proportional to $inv(P)$. We will now discuss 3 special cases and then deduce the general case from them.

Case 1. $P = Q inv(Q)$, where Q is an irreducible polynomial.
Note that $GL(V)_x \cong \prod_i GL(V_{Q,x}^i)$.

Since B defines a non-degenerate pairing $V_{Q,x}^0 \cong (V_{Q,x}^1)^*$, and $V_{Q,x}^i$ are isotropic, we have
$$O(V)_x \cong \Delta GL(V_{Q,x}^0) < GL(V_{Q,x}^0)^2.$$
Now, compose the isomorphism $V_{Q,x}^i \cong V_{Q,x}^{1-i}$ given by ε with the isomorphism $V_{Q,x}^{1-i} \cong (V_{Q,x}^i)^*$. This gives a quadratic structure on $V_{Q,x}^0$. Clearly, ε gives an isomorphism $V_{Q,x}^0 \cong V_{Q,x}^1$ as quadratic spaces and hence
$$(O(V)_\varepsilon)_x \cong \Delta O(V_{Q,x}^0) < \Delta GL(V_{Q,x}^0).$$

Case 2. P is irreducible and $x \neq x^{-1}$
In this case $GL(V)_x \cong GL(V_{P,x}^0)$. Also, $V_{P,x}^0$ and $V_{P,x}^1$ are identical as F-vector spaces but the action of F_P on them differs by a twist by μ. Therefore the isomorphism $V_{P,x}^0 \cong (V_{P,x}^1)^*$ gives a hermitian structure on $V_{P,x}^0$ over (F_P, μ) and ε gives an (F_P, μ)-antilinear automorphism of $V_{P,x}^0$. Now
$$O(V)_x \cong U(V_{P,x}^0).$$
Denote $W := (V_{P,x}^0)^\varepsilon$. It is a linear space over $(F_P)^\mu$. It has a quadratic structure. Now
$$(O(V)_\varepsilon)_x \cong O(W) < U(V_{P,x}^0).$$

Case 3. P is irreducible and $x = x^{-1}$.
Again, $GL(V)_x \cong GL(V_{P,x}^0)$. However, in this case $F_P = F$ and $V_{P,x}^0 = V$. Also $O(V)_x \cong O(V_{P,x}^0)$. Now, ε commutes with x and hence $\varepsilon \in O(V)_x \cong O(V_{P,x}^0)$. Hence
$$(O(V)_\varepsilon)_x \cong (O(V_{P,x}^0))_\varepsilon < O(V_{P,x}^0).$$

Case 4. General case
Let $P = \prod_{i \in I} P_i$ be the decomposition of P to irreducible multiples. Since P is proportional to $inv(P)$, every P_i is proportional to $P_{s(i)}$ where s is some permutation of I of order ≤ 2.

Let $I = \bigsqcup I_\alpha$ be the decomposition of I to orbits of s. Denote $V_\alpha := Ker(\prod_{i \in \alpha} P_i(x))$. Clearly $V = \bigoplus V_\alpha$ and V_α are orthogonal to each other and ε-invariant. Hence the pair $(O(V)_x, (O(V)_\varepsilon)_x)$ is a product of pairs from the first 3 cases.

$\square$

6.6. **The pair** $(U(V_1 \oplus V_2), U(V_1) \times U(V_2))$**.**

Theorem 6.6.1. *Let (V, B) be a hermitian space over (D, τ).*
Let $\varepsilon \in U(V)$ be an element of order 2. Then all the descendants of the pair $(U(V), U(V)_\varepsilon)$ are products of pairs of the types
(i) $(GL(W) \times GL(W), \Delta GL(W))$ for some linear space W over some field D' that extends D
(ii) $(U(W) \times U(W), \Delta U(W))$ for some hermitian space W over some extension (D', τ') of (D, τ)
(iii) $(GL(W), U(W))$ for some hermitian space W over some extension (D', τ') of (D, τ)
(iv) $(GL(W_{D'}), GL(W))$ where F' is a field extension of D, D'/F' is a quadratic extension, W is a linear space over F' and $W_{D'} := W \otimes_{F'} D'$ is its extension of scalars to D'
(v) $(GL(W), GL(W)_\varepsilon)$ where W is a linear space over D and $\varepsilon \in GL(W)$ is an element of order ≤ 2.
(vi) $(U(W_E), U(W))$ where W is a hermitian space over some extension (D', τ') of (D, τ), (E, τ'') is some quadratic extension of (D', τ') and $W_E = W \otimes_{D'} E$ is an extension of scalars with the corresponding hermitian structure.
(vii) $(U(W), U(W)_\varepsilon)$ where W is a hermitian space over (D, τ).

For the proof of this theorem we will need the following straightforward lemma.

Lemma 6.6.2. *Let (V, B) be a hermitian space over (D, τ).*
Let $\varepsilon \in U(V)$ be an element of order 2. Let $x \in U(V)$ such that $\varepsilon x \varepsilon = x^{-1}$. Let Q be an irreducible polynomial. Then ε gives an D_Q- linear isomorphism $V_{Q,x}^{ij} \cong V_{Q,x}^{1-i,j}$.

Proof of Theorem 6.6.1. Let $x \in U(V)^\sigma$ be a semi-simple element and let P be its minimal polynomial.

Note that the minimal polynomial of x^* is P^* and hence P^* is proportional to P. Since $x \in U(V)^\sigma$, we have $x^{-1} = \varepsilon x \varepsilon$ and hence its minimal polynomial is P. Hence P is proportional to $inv(P)$. We will now discuss 7 special cases and then deduce the general case from them.

Case 1. $P = QQ^*inv(Q)\tau(Q)$, where Q is an irreducible polynomial.
Note that $GL(V)_x \cong \prod_{ij} GL(V_{Q,x}^{ij}) \cong GL(V_{Q,x}^{00})^4$. This identifies $U(V)_x$ with a diagonal $\Delta GL(V_{Q,x}^{00})^2 < GL(V_{Q,x}^{00})^4$ and $(U(V)_\varepsilon)_x$ with a diagonal $\Delta GL(V_{Q,x}^{00}) < GL(V_{Q,x}^{00})^4$.

Case 2. $P = Qinv(Q)$, where Q is an irreducible polynomial and $Q^* = Q$.
Note that $GL(V)_x \cong \prod_i GL(V_{Q,x}^{i0}) \cong GL(V_{Q,x}^{00})^2$. Note also that in this case $V_{Q,x}^{i0}$ and $V_{Q,x}^{i1}$ are identical as sets and F-vector spaces but the actions of D_Q on them differ by a twist by μ^{01}. Now the isomorphism $V_{Q,x}^{i0} \cong (V_{Q,x}^{i1})^*$ gives a (D_Q, μ^{01})-hermitian structure on $V_{Q,x}^{i0}$. Therefore, $U(V)_x \cong U(V_{Q,x}^{00}) \times U(V_{Q,x}^{10})$. Note that ε gives an isomorphism of (D_Q, μ^{01})-hermitian spaces between $V_{Q,x}^{00}$ and $V_{Q,x}^{01}$. Hence

$$U(V)_x \cong U(V_{Q,x}^{00})^2 \text{ and } (U(V)_\varepsilon)_x \cong \Delta U(V_{Q,x}^{00}) < U(V_{Q,x}^{00})^2.$$

Case 3. $P = Qinv(Q)$, where Q is an irreducible polynomial and $Q^* = inv(Q)$.
Note that $GL(V)_x \cong \prod_i GL(V_{Q,x}^{i0}) \cong \prod_j GL(V_{Q,x}^{0j})$.

Since B defines a non-degenerate pairing $V_{Q,x}^{00} \cong (V_{Q,x}^{01})^*$ and $V_{Q,x}^{0i}$ are isotropic, we have

$$U(V)_x \cong \Delta GL(V_{Q,x}^{00}) < (GL(V_{Q,x}^{00}))^2.$$

Note that in this case $V_{Q,x}^{ij}$ and $V_{Q,x}^{1-i,1-j}$ are identical as sets and as F-vector spaces but the action of D_Q on them differs by a twist by μ^{11}.

Now, compose the isomorphism $V_{Q,x}^{00} \cong V_{Q,x}^{10}$ given by ε with the isomorphism $V_{Q,x}^{10} \cong (V_{Q,x}^{11})^*$. This gives a (D_Q, μ^{11}) unitary structure on $V_{Q,x}^{00}$. Similarly we get a unitary structure on $V_{Q,x}^{10}$. Finally, ε gives an isomorphism $V_{Q,x}^{00} \cong V_{Q,x}^{10}$ as unitary spaces and hence

$$(U(V)_\varepsilon)_x \cong \Delta U(V_{Q,x}^{00}) < \Delta GL(V_{Q,x}^{00}).$$

Case 4. $P = QQ^*$, where Q is an irreducible polynomial, $Q = inv(Q)$ and $x \neq x^{-1}$.

Note that $GL(V)_x \cong \prod_j GL(V_{Q,x}^{0j})$ and as before

$$U(V)_x \cong \Delta GL(V_{Q,x}^{00}) < (GL(V_{Q,x}^{00}))^2.$$

In this case $V_{Q,x}^{0j}$ and $V_{Q,x}^{1j}$ are identical as sets and as F-vector spaces but the action of D_Q on them differs by a twist by μ^{10}. Hence ε gives a (D_Q, μ^{10}) anti-linear automorphism of $V_{Q,x}^{0j}$. Let $W_j := (V_{Q,x}^{0j})^\varepsilon$. This is a linear space over $(D_Q)^{\mu^{10}}$. Therefore,

$$(U(V)_\varepsilon)_x \cong \Delta GL(W_0) < \Delta GL(V_{Q,x}^{00}).$$

Case 5. $P = QQ^*$, where Q is an irreducible polynomial, $Q = inv(Q)$ and $x = x^{-1}$.
As in the previous case,

$$GL(V)_x \cong \prod_j GL(V_{Q,x}^{0j}) \text{ and } U(V)_x \cong \Delta GL(V_{Q,x}^{00}) < (GL(V_{Q,x}^{00}))^2.$$

In this case $D_Q = D$ and μ^{10} is trivial. Hence $V_{Q,x}^{0j}$ and $V_{Q,x}^{1j}$ are identical as D_Q-linear spaces.

Also, ε gives a D_Q-linear automorphism of $V_{Q,x}^{0j}$. So we can interpret ε as an element in $GL(V_{Q,x}^{00})$. Therefore,

$$(U(V)_\varepsilon)_x \cong \Delta(GL(V_{Q,x}^{00}))_\varepsilon < \Delta GL(V_{Q,x}^{00}).$$

Case 6. P is irreducible and $x \neq x^{-1}$
In this case $GL(V)_x \cong GL(V_{P,x}^{00})$. Also, $V_{P,x}^{00}$ and $V_{P,x}^{01}$ are identical as F-vector spaces but the action of D_P on them differs by a twist by μ^{01}. Again, the isomorphism $V_{P,x}^{00} \cong (V_{P,x}^{01})^*$ gives a (D_P, μ^{01}) hermitian structure on $V_{P,x}^{00}$ and

$$U(V)_x \cong U(V_{P,x}^{00}).$$

Note that $V_{P,x}^{00}$ and $V_{P,x}^{10}$ are identical as F-vector spaces but the action of D_P on them differs by a twist by μ^{10}. Hence, ε gives a (D_P, μ^{10}) anti-linear automorphism of $V_{P,x}^{00}$. Denote $W := (V_{P,x}^{00})^\varepsilon$. It is a linear space over $(D_P)^{\mu^{10}}$. It has a $((D_P)^{\mu^{10}}, \mu^{01}|_{(D_P)^{\mu^{10}}})$ hermitian structure. Now

$$(U(V)_\varepsilon)_x \cong U(W) < U(V_{P,x}^{00}).$$

Case 7. P is irreducible and $x = x^{-1}$.
Again,

$$GL(V)_x \cong GL(V_{P,x}^{00}) \text{ and } U(V)_x \cong U(V_{P,x}^{00}).$$

In this case $D_P = D$ and $\mu^{01} = \tau$. Also, ε commutes with x and hence $\varepsilon \in U(V)_x \cong U(V_{P,x}^{00})$. Hence

$$(U(V)_\varepsilon)_x \cong U(V_{P,x}^{00})_\varepsilon < U(V_{P,x}^{00}).$$
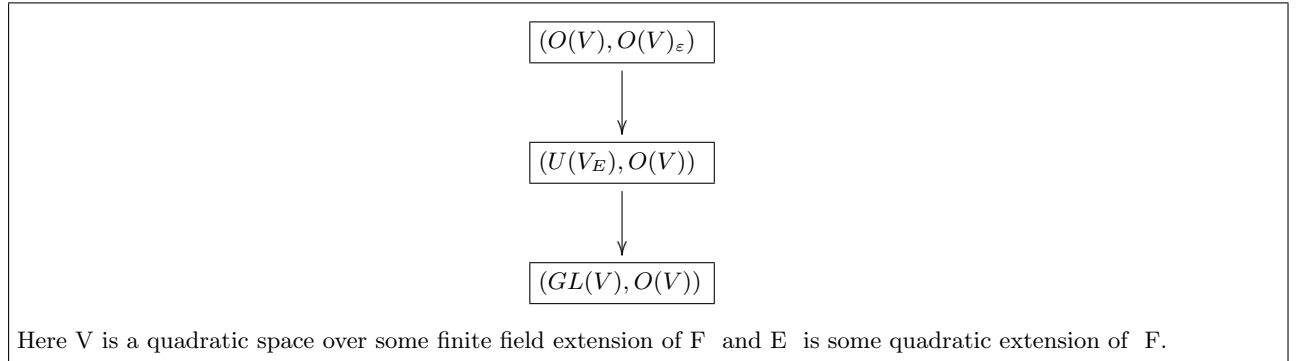
Case 8. General case
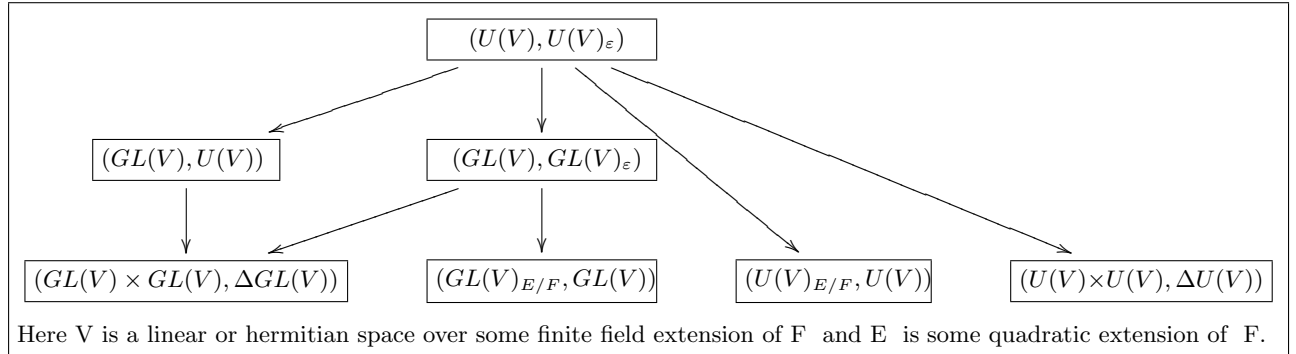Let $P = \prod_{i \in I} P_i$ be the decomposition of P to irreducible multiples. Since P is proportional to $inv(P)$, every P_i is proportional to $P_{s_1(i)}$ for some permutation s_1 of I of order ≤ 2. Since P is proportional to P^*, every P_i is proportional to some $P_{s_2(i)}$. This gives rise to an action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on I.

Let $I = \bigsqcup I_\alpha$ be the decomposition of I to orbits of this action. Denote $V_\alpha := Ker(\prod_{i \in \alpha} P_i(x))$. Clearly $V = \bigoplus V_\alpha$ and V_α are orthogonal to each other and ε-invariant. Hence the pair $(U(V)_x, (U(V)_\varepsilon)_x)$ is a product of pairs from the first 7 cases. $\qquad \square$

6.7. Genealogical trees of the symmetric pairs considered in this paper.

The following diagrams sum up the results of this section.

An arrow "$(G_1, H_1) \rightarrow (G_2, H_2)$" means that pairs of type (G_1, H_1) may have descendants with factor of the type (G_2, H_2). We will not draw the obvious arrows "$(G, H) \rightarrow (G, H)$" and when we draw "$(G_1, H_1) \rightarrow (G_2, H_2) \rightarrow (G_3, H_3)$" we mean also "$(G_1, H_1) \rightarrow (G_3, H_3)$".

Here V is a linear or hermitian space over some finite field extension of F and E is some quadratic extension of F.

Here V is a quadratic space over some finite field extension of F and E is some quadratic extension of F.

REFERENCES

[AG1] A. Aizenbud, D. Gourevitch, *Schwartz functions on Nash Manifolds,* International Mathematics Research Notices, Vol. 2008, 2008: rnm155-37 DOI: 10.1093/imrn/rnm155. See also arXiv:0704.2891 [math.AG].

[AG2] Aizenbud, A.; Gourevitch, D.:*Generalized Harish-Chandra descent, Gelfand pairs and an Archimedean analog of Jacquet-Rallis' Theorem,* postprint: arXiv:0812.5063v4[math.RT]. Originally published in: Duke Mathematical Journal, Volume 149, Number 3, pp 509-567 (2009).

[AGS] A. Aizenbud, D. Gourevitch, E. Sayag : $(GL_{n+1}(F), GL_n(F))$ *is a Gelfand pair for any local field F*, postprint: arXiv:0709.1273v4[math.RT]. Originally published in: Compositio Mathematica, **144** , pp 1504-1524 (2008), doi:10.1112/S0010437X08003746.

[BvD] E. E H. Bosman and G. Van Dijk, *A New Class of Gelfand Pairs,* Geometriae Dedicata **50**, 261-282, 261 @ 1994 KluwerAcademic Publishers. Printed in the Netherlands (1994).

[BZ] J. Bernstein, A.V. Zelevinsky, *Representations of the group* GL(n, F)*, where F is a local non-Archimedean field,* Uspekhi Mat. Nauk **10**, No.3, 5-70 (1976).

[GK] I.M. Gelfand, D. Kazhdan, *Representations of the group* GL(n, K) *where K is a local field*, Lie groups and their representations (Proc. Summer School, Bolyai Janos Math. Soc., Budapest, 1971), pp. 95–118. Halsted, New York (1975).

[Gro] B. Gross, *Some applications of Gelfand pairs to number theory,* Bull. Amer. Math. Soc. (N.S.) **24**, no. 2, 277–301 (1991).

[MVW] Mœglin, Colette and Vigneras, Marie-France and Waldspurger, Jean-Loup: *Correspondances de Howe sur un corps p-adique. (French) [Howe correspondences over a p-adic field]* Lecture Notes in Mathematics, **1291**, Springer-Verlag, Berlin, 1987. viii+163 pp. ISBN: 3-540-18699-9

[SpSt] T. A. Springer, R. Steinberg, *Conjugacy classes.* 1970, Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69) pp. 167-266 Lecture Notes in Mathematics, Vol. **131**, Springer, Berlin.

[vD] van Dijk, *On a class of generalized Gelfand pairs*, Math. Z. **193**, 581-593 (1986).

AVRAHAM AIZENBUD AND DMITRY GOUREVITCH, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, THE WEIZMANN INSTITUTE OF SCIENCE POB 26, REHOVOT 76100, ISRAEL.

E-mail address: `aizenr@yahoo.com`

E-mail address: `guredim@yahoo.com`