Derivatives for representations of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$

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Definition

$$P_n = \left\{ \begin{pmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\} \subset G_n := GL_n(F)$$

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Proof.

$$\mathcal{M}(P_n) = \mathcal{M}(\mathcal{H}(P_n)) = \mathcal{M}(\mathcal{H}(G_{n-1} \ltimes V_n)) =$$
$$= \mathcal{M}(\mathcal{H}(G_{n-1}) \otimes \mathcal{H}(V_n)) \cong \mathcal{M}(\mathcal{H}(G_{n-1}) \otimes \mathcal{S}(V_n))$$

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We have a short exact sequence

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- $D^k = \Psi \circ \Phi^{k-1}$

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- *π is finitely generated over $\mathfrak{n}$.*

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Theorem (Casselman-Wallach)

The functor $HC : \mathcal{M}_{\infty}(G) \to \mathcal{M}_{HC}(G)$ is an equivalence of categories.

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- $D^k(\pi) :== (E^k(\pi))_{gen,\mathfrak{v}_{n-k+1}}$. Here $\mathfrak{v}_{n-k+1}$ is the nil-radical of $\mathfrak{p}_{n-k+1}$ and $\cdot_{gen,\mathfrak{v}_{n-k+1}}$ denotes the generalized co-invariants.

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- $B^k(\pi) := (E^k(\pi))_{\mathfrak{v}_{n-k+1}}$.
- $depth(\pi)$ – the largest part in the associated partition of π

Associated partition

$\mathcal{U}(\mathfrak{g}_n)$ has a filtration by the order of the tensor.
$\mathrm{Gr}(\mathcal{U}(\mathfrak{g}_n)) = \mathrm{Sym}(\mathfrak{g}_n) = \mathrm{Pol}(\mathfrak{g}_n^*)$.

$$\mathcal{V}(\pi) := \mathrm{Zeroes}(\mathrm{Gr}(\mathrm{Ann}(\pi)))$$

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Theorem (Joseph)

If π is irreducible then $\mathcal{V}(\pi)$ is the closure of a single orbit.

By Jordan's theorem this orbit is described by a partition of n, that we call associated partition of π.

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- $E^n = D^n = B^n = (\Phi)^{n-1}$ is the Whittaker functor.

$$depth(\pi) = n \iff D^n(\pi) \neq 0$$

Whittaker spaces

Let $N_n < G_n$ denote the subgroup of unipotent upper-triangular matrices, and define a character ψ of N_n to be the sum of superdiagonal elements. The Whittaker space is the space of co-equivariants

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Then

$$Wh_\lambda(\pi) = B^{n_k}(B^{n_{k-1}}(...(B^{n_1}(\pi))))$$

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- *For a unitarizable representation* π

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6. Using the exactness we prove the product formula in the smooth category
7. We deduce from the product formula that for a unitarizable representation π

$$E^d(\pi) = D^d(\pi) = B^d(\pi) = A(\pi)$$

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Theorem (Baruch, Bernstein, Sahi)

$\forall \pi \in \widehat{G_n}$, $\pi|_{P_n} \in \widehat{P_n}$

We define $A\pi := A(\pi|_{P_n})$.

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$$\Delta_{4m-4} \twoheadleftarrow \chi_1|_{G_{m-1}} \times \chi_2|_{G_{m-1}} \times \chi_3|_{G_{m-1}} \times \chi_4|_{G_{m-1}} =$$
$$= E^4(\chi_1 \times \chi_2 \times \chi_3 \times \chi_4) \twoheadrightarrow E^4(\Delta_{4m}) \twoheadrightarrow A(\Delta_{4m})$$

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$$depth(\pi) = d \Rightarrow \text{constrains on } \mathcal{V}_{\mathfrak{g}}(\pi) \Rightarrow$$
$$\Rightarrow AV_{\mathfrak{n}_{n-d+1}}(E^d(\pi)) \subset \mathfrak{n}^*_{n-d} \Rightarrow E^d(\pi) \text{ is f.g. over } \mathfrak{n}_{n-d}$$

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Solution – to introduce a class of “good" $\mathfrak{p}_n$ representations

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Method – exactness, key lemma, induction