NASH MANIFOLDS AND SCHWARTZ FUNCTIONS ON THEM

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Abstract. These are the lecture notes for my talk on December 1, 2014 at the BIRS workshop "Motivic Integration, Orbital Integrals, and Zeta-Functions".

1. Semi-algebraic sets and the Seidenberg-Tarski theorem

In this section we follow [BCR].

Definition 1.1. *A subset $A \subset \mathbb{R}^n$ is called a* ***semi-algebraic set*** *if it can be presented as a finite union of sets defined by a finite number of polynomial equalities and inequalities. In other words, if there exist finitely many polynomials $f_{ij}, g_{ik} \in R[x_1, ..., x_n]$ such that*

$$A = \bigcup_{i=1}^{r} \{x \in \mathbb{R}^n | f_{i1}(x) > 0, ..., f_{is_i}(x) > 0, g_{i1}(x) = 0, ..., g_{it_i}(x) = 0\}.$$

Lemma 1.2. *The collection of semi-algebraic sets is closed with respect to finite unions, finite intersections and complements.*

Example 1.3. *The semi-algebraic subsets of $\mathbb{R}$ are unions of finite number of (finite or infinite) intervals.*

In fact, a semi-algebraic subset is the same as a union of connected components of an affine real algebraic variety.

Definition 1.4. *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be semi-algebraic sets. A mapping $\nu : A \to B$ is called* ***semi-algebraic*** *iff its graph is a semi-algebraic subset of $\mathbb{R}^{m+n}$.*

Proposition 1.5. *Let ν be a bijective semi-algebraic mapping. Then the inverse mapping ν^{-1} is also semi-algebraic.*

Proof. The graph of ν is obtained from the graph of ν^{-1} by switching the coordinates. □

One of the main tools in the theory of semi-algebraic spaces is the Tarski-Seidenberg principle of quantifier elimination. Here we will formulate and use a special case of it. We start from the geometric formulation.

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Theorem 1.6. *Let $A \subset \mathbb{R}^n$ be a semi-algebraic subset and $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the standard projection. Then the image $p(A)$ is a semi-algebraic subset of $\mathbb{R}^{n-1}$.*

By induction and a standard graph argument we get the following corollary.

Corollary 1.7. *An image of a semi-algebraic subset of $\mathbb{R}^n$ under a semi-algebraic map is semi-algebraic.*

Sometimes it is more convenient to use the logical formulation of the Tarski-Seidenberg principle. Informally it says that any set that can be described in semi-algebraic language is semi-algebraic. We will now give the logical formulation and immediately after that define the logical notion used in it.

Theorem 1.8 (Tarski-Seidenberg principle, see e.g.[BCR, Proposition 2.2.4])**.** *Let Φ be a formula of the language $L(\mathbb{R})$ of ordered fields with parameters in $\mathbb{R}$. Then there exists a quantifier - free formula Ψ of $L(\mathbb{R})$ with the same free variables $x_1, \ldots, x_n$ as Φ such that $\forall x \in \mathbb{R}^n, \Phi(x) \Leftrightarrow \Psi(x)$.*

Definition 1.9. *A **formula of the language of ordered fields with parameters in** $\mathbb{R}$ is a formula written with a finite number of conjunctions, disjunctions, negations and universal and existential quantifiers ($\forall$ and $\exists$) on variables, starting from atomic formulas which are formulas of the kind $f(x_1, \ldots, x_n) = 0$ or $g(x_1, \ldots, x_n) > 0$, where f and g are polynomials with coefficients in $\mathbb{R}$. The free variables of a formula are those variables of the polynomials which are not quantified. We denote the language of such formulas by $L(\mathbb{R})$.*

Notation 1.10. *Let Φ be a formula of $L(\mathbb{R})$ with free variables $x_1, \ldots, x_n$. It defines the set of all points $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ that satisfy Φ. We denote this set by S_Φ. In short,*

$$S_\Phi := \{x \in \mathbb{R}^n | \Phi(x)\}.$$

Corollary 1.11. *Let Φ be a formula of $L(\mathbb{R})$. Then S_Φ is a semi-algebraic set.*

Proof. Let Ψ be a quantifier-free formula equivalent to Φ. The set S_Ψ is semi-algebraic since it is a finite union of sets defined by polynomial equalities and inequalities. Hence S_Φ is also semi-algebraic since $S_\Phi = S_\Psi$. □

Proposition 1.12. *The logical formulation of the Seidenberg-Tarski principle implies the geometric one.*

Proof. Let $A \subset \mathbb{R}^n$ be a semi-algebraic subset, and $pr : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the standard projection. Then there exists a formula $\Phi \in L(\mathbb{R})$ such that $A = S_\Phi$. Then $pr(A) = S_\Psi$ where

$$\Psi(y) = \text{“}\exists x \in \mathbb{R}^n \, (\mathrm{pr}(x) = y \wedge \Phi(x))\text{”}.$$

Since $\Psi \in L(\mathbb{R})$, the proposition follows now from the previous corollary. □

In fact, it is not difficult to deduce the logical formulation from the geometric one. Let us now demonstrate how to use the logical formulation of the Seidenberg-Tarski theorem.

Corollary 1.13. *The closure of a semi-algebraic set is semi-algebraic.*

Proof. Let $A \subset \mathbb{R}^n$ be a semi-algebraic subset, and let $\overline{A}$ be its closure. Then $\overline{A} = S_\Psi$ where

$$\Psi(x) = \text{“}\forall(\varepsilon > 0\, \exists y \in A\, |x-y|^2 < (\varepsilon\text{”}.$$

Clearly, $\Psi \in L(\mathbb{R})$ and hence $\overline{A}$ is semi-algebraic. □

Corollary 1.14. *The derivative f' of any differentiable semi-algebraic function $f : \mathbb{R} \to \mathbb{R}$ is semi-algebraic.*

Proof. The graph of f' equals S_Ψ, where

$$\Psi(x,y) = \text{“}\forall(\varepsilon > 0\, \exists\delta > 0, \text{ s.t. } \forall 0 \neq \delta' \in (-\delta,\delta) \text{ we have } (f(x+\delta') - f(x) - y\delta')^2 < (\varepsilon\delta')^2\text{”}.$$

Clearly, $\Psi \in L(\mathbb{R})$ and hence f' is semi-algebraic. □

Corollary 1.15.
(i) The composition of semi-algebraic mappings is semi-algebraic.
(ii) The $\mathbb{R}$-valued semi-algebraic functions on a semi-algebraic set A form a ring, and any nowhere vanishing semi-algebraic function is invertible in this ring.
(iii) Images and preimages of semi-algebraic sets under semi-algebraic mappings are semi-algebraic.

Proposition 1.16. [BCR, Proposition 2.4.5] *Any semi-algebraic set in $\mathbb{R}^n$ has a finite number of connected components.*

Remark 1.17 (Loeser)**.** *Over a non-archimedean local field F (e.g. $F = \mathbb{Q}_p$) one considers sets that are finite unions of finite intersections of sets of the form*

$$\{x \in F^n \text{ s.t. } p(x) \text{ is a } k\text{-th power}\},$$

An analog of the Seidenberg - Tarski theorem holds for such sets.

2. Nash manifolds

Let us now define the category of Nash manifolds, i.e. smooth semi-algebraic manifolds. I like this category since the Nash manifolds behave as tamely as algebraic varieties (e.g. posses some finiteness properties, and admit an analog of Hironaka's desingularization theorem), and in addition their local structure is almost as easy as that of differentiable manifolds. In particular, they are locally trivial, and analogs of the implicit function theorem and the tubular neighborhood hold for them. The only thing we "loose" is the partition of unity.

Nash has shown that any compact smooth manifold has a unique structure of an affine Nash manifold. It was later shown that it also has uncountably many structures of non-affine Nash manifold. It is Artin-Mazur who first used the term of Nash manifold. They gave a fundamental theorem which states that an affine Nash manifold can be imbedded in a Euclidean space so that the image contains no singular points of its Zariski closure.

In this section we follow [BCR, Shi].

Definition 2.1. *A* ***Nash map*** *from an open semi-algebraic subset U of $\mathbb{R}^n$ to an open semi-algebraic subset $V \subset \mathbb{R}^m$ is a smooth (i.e. infinitely differentiable) semi-algebraic function. The ring of $\mathbb{R}$-valued Nash functions on U is denoted by $\mathcal{N}(U)$. A* ***Nash diffeomorphism*** *is a Nash bijection whose inverse map is also Nash.*

As we are going to do semi-algebraic differential geometry, we will need a semi-algebraic version of implicit function theorem.

Theorem 2.2 (Implicit Function Theorem for Nash manifolds, see e.g. [BCR, Corollary 2.9.8])**.** *Let $(x^0, y^0) \in \mathbb{R}^{n+p}$, and let $f_1, ..., f_p$ be semi-algebraic smooth functions on an open neighborhood of (x^0, y^0), such that $f_j(x^0, y^0) = 0$ for $j = 1, .., p$ and the matrix $[\frac{\partial f_j}{\partial y_i}(x^0, y^0)]$ is invertible. Then there exist open semi-algebraic neighborhoods U (resp. V) of x^0 (resp. y^0) in $\mathbb{R}^n$ (resp. $\mathbb{R}^p$) and a Nash mapping ϕ, such that $\phi(x^0) = y^0$ and $f_1(x, y) = ... = f_p(x, y) = 0 \Leftrightarrow y = \phi(x)$ for every $(x, y) \in U \times V$.*

Definition 2.3. *A* ***Nash submanifold*** *of $\mathbb{R}^n$ is a semi-algebraic subset of $\mathbb{R}^n$ which is a smooth submanifold.*

By the implicit function theorem it is easy to see that this definition is equivalent to the following one, given in [BCR]:

Definition 2.4. *A semi-algebraic subset M of $\mathbb{R}^n$ is said to be a* ***Nash submanifold of*** $\mathbb{R}^n$ ***of dimension*** d *if, for every point x of M, there exists a Nash diffeomorphism ϕ from an open semi-algebraic neighborhood Ω of the origin in $\mathbb{R}^n$ onto an open semi-algebraic neighborhood Ω' of x in $\mathbb{R}^n$ such that $\phi(0) = x$ and $\phi(\mathbb{R}^d \times \{0\} \cap \Omega) = M \cap \Omega'$.*

Definition 2.5. *A* ***Nash map*** *from a Nash submanifold M of $\mathbb{R}^m$ to a Nash submanifold N of $\mathbb{R}^n$ is a semi-algebraic smooth map.*

Any open semi-algebraic subset of a Nash submanifold of $\mathbb{R}^n$ is also a Nash submanifold of $\mathbb{R}^n$.

Theorem 2.6 ([BCR, §2])**.** *Let $M \subset \mathbb{R}^n$ be a Nash submanifold. Then it has the same dimension as its Zarisky closure.*

Unfortunately, open semi-algebraic sets in $\mathbb{R}^n$ do not form a topology, but only a restricted topology. That is, the collection of open semi-algebraic sets is closed only under finite intersections and unions but not under infinite unions. For this reason we will consider only finite covers.

We will use this restricted topology to "glue" affine Nash manifolds and define Nash manifolds exactly in the same way as algebraic varieties are glued from affine algebraic varieties.

Definition 2.7. *A* $\mathbb{R}$***-space*** *is a pair* $(M, \mathcal{O}_M)$ *where* M *is a restricted topological space and* $\mathcal{O}_M$ *a sheaf of* $\mathbb{R}$*-algebras over* M *which is a subsheaf of the sheaf* $\mathbb{R}[M]$ *of real-valued functions on* M*.*

A ***morphism between*** $\mathbb{R}$***-spaces*** $(M, \mathcal{O}_M)$ *and* $(N, \mathcal{O}_N)$ *is a continuous map* $f : M \to N$*, such that the induced morphism of sheaves* $f^* : f^*(\mathbb{R}[N]) \to \mathbb{R}[M]$ *maps* $\mathcal{O}_N$ *to* $\mathcal{O}_M$*.*

Example 2.8. *Take for* M *a Nash submanifold of* $\mathbb{R}^n$*, and for* $\mathring{\mathfrak{S}}(M)$ *the family of all open subsets of* M *which are semi-algebraic in* $\mathbb{R}^n$*. For any open (semi-algebraic) subset* U *of* M *we take as* $\mathcal{O}_M(U)$ *the algebra* $\mathcal{N}(U)$ *of Nash functions* $U \to \mathbb{R}$*.*

Definition 2.9. *An* ***affine Nash manifold*** *is an* $\mathbb{R}$*-space which is isomorphic to an* $\mathbb{R}$*-space of a closed Nash submanifold of* $\mathbb{R}^n$*. A morphism between two affine Nash manifolds is a morphism of* $\mathbb{R}$*-spaces between them.*

Example 2.10. *Any real nonsingular affine algebraic variety has a natural structure of an affine Nash manifold.*

Remark 2.11. *Let* $M \subset \mathbb{R}^m$ *and* $N \subset \mathbb{R}^n$ *be Nash submanifolds. Then a Nash map between them is the same as a morphism of affine Nash manifolds between them.*

Let $f : M \to N$ *be a Nash map. Since an inverse of a semi-algebraic map is semi-algebraic,* f *is a diffeomorphism if and only if it is an isomorphism of affine Nash manifolds. Therefore we will call such* f *a Nash diffeomorphism.*

In [Shi] there is another but equivalent definition of affine Nash manifold.

Definition 2.12. *An* ***affine*** C^∞ ***Nash manifold*** *is an* $\mathbb{R}$*-space over* $\mathbb{R}$ *which is isomorphic to an* $\mathbb{R}$*-space of a Nash submanifold of* $\mathbb{R}^n$*.*

The equivalence of the definitions follows from the following theorem, which immediately follows from [BCR, Theorem 8.4.6] and Proposition 1.16.

Theorem 2.13. *Any affine* C^∞ *Nash manifold is Nash diffeomorphic to a union of finite number of connected components of a real nonsingular affine algebraic variety.*

The book [Shi] usually uses the notion of affine C^ω Nash manifold instead of affine C^∞ Nash manifold, that we called here just Nash manifold. The two notions are equivalent by the theorem of Malgrange (see [Mal] or [Shi, Corollary I.5.7]) and hence equivalent to what we call just affine Nash manifold. In other words, any Nash manifold has a natural structure of a real analytic manifold and any Nash map between Nash manifolds is analytic.

One also considers C^r-Nash manifolds for any $0 \leq r < \infty$. These satisfy all the properties listed here, and in addition partition of unity.

Definition 2.14. *A **Nash manifold** is an $\mathbb{R}$-space $(M, \mathcal{N}_M)$ which has a finite cover (M_i) by open sets M_i such that the $\mathbb{R}$-spaces $(M_i, \mathcal{N}_M|_{M_i})$ are isomorphic to $\mathbb{R}$-spaces of affine Nash manifolds.*

*A **morphism between Nash manifolds** is a morphism of $\mathbb{R}$-spaces between them. Such morphisms are called Nash maps, and isomorphisms are called Nash diffeomorphisms.*

By Proposition 1.16, any Nash manifold is a union of a finite number of connected components. Any semi-algebraic set can be stratified by Nash manifolds.

Any Nash manifold has a natural structure of a smooth manifold. Any real nonsingular algebraic variety has a natural structure of a Nash manifold.

It is well-known that the real projective space $\mathbb{RP}^n$ is affine, see e.g. [BCR, Theorem 3.4.4]. Since any number of polynomial equations over $\mathbb{R}$ have the same set of solutions as a single equation (which is the sum of squares of the left hand sides), we get that any quasiprojective Nash manifold is affine.

Remark 2.15. *Note that the additive group of real numbers and and the multiplicative group of positive real numbers are isomorphic as Lie groups and as Nash manifolds, but are not isomorphic as Nash groups. Recently, the structure theory of (almost) linear Nash groups was developed in [Sun].*

The following theorem is a version of Hironaka's theorem for Nash manifolds.

Theorem 2.16 ([Shi, Corollary I.5.11])**.** *Let M be an affine Nash manifold. Then there exists a compact affine nonsingular algebraic variety N and a closed algebraic subvariety Z of N, which is empty if M is compact, such that Z has only normal crossings in N and M is Nash diffeomorphic to a union of connected components of $N - Z$.*

It implies that Nash manifolds are locally trivial.

Theorem 2.17 ([Shi, Theorem I.5.12])**.** *Any Nash manifold has a finite cover by open submanifolds Nash diffeomorphic to $\mathbb{R}^n$.*

Theorem 2.18 ([AG10, Theorem 2.4.3]). *Let M and N be Nash manifolds and $\nu : M \to N$ be a surjective submersive Nash map. Then locally (in the restricted topology) it has a Nash section, i.e. there exists a finite open cover $N = \bigcup_{i=1}^{k} U_i$ such that ν has a Nash section on each U_i.*

This implies that any etale Nash map is a local diffeomorphism.

In our work on Schwartz functions we frequently use the following

Theorem 2.19. *(Nash Tubular Neighborhood). Let $Z \subset M \subset \mathbb{R}^n$ be closed affine Nash submanifolds. Equip M with the Riemannian metric induced from $\mathbb{R}^n$. Then Z has a Nash tubular neighborhood.*

3. Schwartz functions on Nash manifolds

The Fréchet space $\mathcal{S}(\mathbb{R}^n)$ of Schwartz functions on $\mathbb{R}^n$ was defined by Laurant Schwartz to be the space of all smooth functions such that they and all their derivatives decay faster than $1/|x|^n$ for all n. In other words, $\mathcal{S}(\mathbb{R}^n)$ is the space of all $f \in C^\infty$ such that $|df|$ is bounded for every differential operator d with polynomial coefficients. This definition makes sense verbatim on any smooth affine algebraic variety and was extended in [dCl] to affine Nash manifolds, and in [AG08] (using some ideas from unpublished notes of Casselman) to arbitrary Nash manifolds. One can also define Schwartz sections of Nash bundles.

As Schwartz functions cannot be restricted to open subsets, but can be continued by 0 from open subsets, they form a cosheaf rather than a sheaf.

Let M be a Nash manifold, E be a Nash bundle over M and let $\mathcal{S}(M, E)$ denote the space of Schwartz sections of E.

The following two theorems summarize some results from [dCl, AG08, AG10, AG13].

Theorem 3.1. *Let $U \subset M$ be an open (Nash) submanifold and $N \subset M$ be a closed (Nash) submanifold. Then*

1. *$\mathcal{S}(\mathbb{R}^n)$ = Classical Schwartz functions on $\mathbb{R}^n$.*
2. *For compact M, $\mathcal{S}(M, E)$ = smooth global sections of E.*
3. *The restriction maps $\mathcal{S}(M, E)$ onto $\mathcal{S}(Z, E|_Z)$.*
4. *$\mathcal{S}(U, E) := \mathcal{S}(U, E|_U) =$*

   $\{\xi \in \mathcal{S}(M, E) | \xi$ vanishes with all its derivatives on $M - U\}$.
5. *Partition of unity: Let $(U_i)_{i=1}^n$ be a finite cover by open Nash submanifolds. Then there exist smooth functions $\alpha_1, ..., \alpha_n$ such that supp$(\alpha_i) \subset U_i$, $\sum_{i=1}^{n} \alpha_i = 1$ and for any $g \in \mathcal{S}(M, E)$, $\alpha_i g \in \mathcal{S}(U_i, E)$.*
6. *$\mathcal{S}(M, E) = \mathcal{S}(M)\mathcal{S}(M, E)$.*
7. *$\mathcal{S}(M, E)$ is a nuclear Fréchet space.*

(8) *For any Nash manifold M' we have* $\mathcal{S}(M \times M') = \mathcal{S}(M)\hat{\otimes}\mathcal{S}(M')$.

Theorem 3.2. *Let $N \subset M$ be a closed submanifold. Denote*

$$\mathcal{S}_N(M)^i := \{\phi \in \mathcal{S}(M) \text{ s.t. } \phi \text{ is } 0 \text{ on } N \text{ with first} i-1 \text{ derivatives}\}.$$

Let CN_M^N denote the conormal bundle to N in M. Then

$$\mathcal{S}(M)^i/\mathcal{S}(M)^{i+1} \cong \mathcal{S}(N, Sym^{i+1}(CN_M^N)) \tag{1}$$

$$\mathcal{S}(M \setminus N) \cong \lim_{\leftarrow} \mathcal{S}(M)^i/\mathcal{S}(M)^{i+1}. \tag{2}$$

This theorem is helpful when analyzing the space of tempered distributions on M supported on N, i.e. the space of continuous linear functionals on $\mathcal{S}(M)$ that vanish on the space $\mathcal{S}(M \setminus N)$ consisting of Schwartz functions on the complement to N. This question appears in the analysis of equivariant distributions, that appear frequently in representation theory - as characters, relative characters, invariant functionals or orbital integrals. We frequently use this theorem when we have a natural stratification of M since it allows to reduce the analysis of equivariant distributions from M to single strata.

Let us finish with a theorem related to Igusa and orbital integrals.

Theorem 3.3 ([AG09], Theorem B.2.4)**.** *Let $\phi : M \to N$ be a Nash submersion of Nash manifolds. Let E be a Nash bundle over N. Fix Nash measures μ on M and ν on N.*

Then
(i) there exists a unique continuous linear map $\phi_* : \mathcal{S}(M) \to \mathcal{S}(N)$ such that for any $f \in \mathcal{S}(N)$ and $g \in \mathcal{S}(M)$ we have

$$\int_{x \in N} f(x)\phi_* g(x) d\nu = \int_{x \in M} (f(\phi(x)))g(x) d\mu.$$

In particular, we mean that both integrals converge.
(ii) If ϕ is surjective then ϕ_* is surjective.

In fact

$$\phi_* g(x) = \int_{z \in \phi^{-1}(x)} g(z) d\rho$$

for an appropriate measure ρ.

Finally, I would like to remark on a different, extrinsic, approach to Schwartz functions, applied in [CHM] and [KS]. We can compactify our manifold and define Schwartz functions on it as smooth functions on the (smooth) compactification that vanish to infinite order on the complement to M. If both M and the compactification are Nash manifolds then this definition will be equivalent, by Theorem 3.1(2,4). This allows to define Schwartz functions on non-Nash (say, subanalytic) manifolds, but this space will depend on the compactification.

Possibly, Schwartz functions on non-smooth semi-algebraic varieties (more precisely, C^0 Nash manifolds) can be defined by restriction from a smooth embient space. A student of mine is now working on that.

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