INVARIANT GENERALIZED FUNCTIONS SUPPORTED ON AN ORBIT

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Abstract. We study the space of invariant generalized functions supported on an orbit of the action of a real algebraic group on a real algebraic manifold. This space is equipped with the Bruhat filtration. We study the generating function of the dimensions of the filtras, and give some methods to compute it. To illustrate our methods we compute those generating functions for the adjoint action of $\mathrm{GL}_3(\mathbb{C})$. Our main tool is the notion of generalized functions on a real algebraic stack, introduced recently in [Sak16].

Contents

1. Introduction

The study of invariant distributions plays important role in representation theory and related topics (see e.g. [HC63, HC65, GK75, Sha74, Ber84, JR96, Bar03, AGRS10, AG09a, AG09b, SZ12]). In many cases this study can be reduced to the consideration of distributions supported on a single orbit (see e.g. [Ber84, §1.5], [KV96], [AG09a, Appendix D], [AG13, Appendix B]). While for non-Archimedean fields this case is very simple, for Archimedean fields it is much more involved. In this paper we establish some infrastructure in order to analyze the Archimedean case.

Let a Nash[1] group G act on a Nash manifold M. Let O be an orbit of G in X. The space $\mathcal{G}(X \smallsetminus (\bar{O} \smallsetminus O))^G$ of tempered G-invariant generalized functions defined in a neighborhood of O and supported in O is equipped with the Bruhat filtration (see e.g. [AG08]). Let $\bar{\delta}_{\mathcal{O}}^X(i)$ denote the dimension of the i-s filtra and

$$\bar{\mathfrak{G}}_{\mathcal{O}}^X(t) := (1-t)\sum_i t^i \bar{\delta}_{\mathcal{O}}^X(i)$$

denotes the corresponding generating function.

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[1]Nash manifolds are generalizations of real algebraic manifolds. In most places in this paper the reader can safely replace the word Nash by “smooth real algebraic”. For more details on Nash manifolds and Schwartz functions over them see [AG08].

In this paper we introduce several techniques for the computation of this function. We illustrate our methods on the case of the adjoint action of $\mathrm{GL}_3(\mathbb{C})$. Our main tool is the notion of generalized functions on a real algebraic stack, introduced recently in [Sak16].

1.1. **Results.**

(1) In the case of when O is (locally) a fiber of a G-invariant submersion we prove that $\bar{\mathfrak{G}}_O^X(t) = (1-t)^{\dim O - \dim X}$ (see Corollary 6.4).
(2) We prove that $\bar{\delta}_{\mathcal{O}}^X(i) - \bar{\delta}_{\mathcal{O}}^X(i-1)$ is bounded by $\dim(\mathrm{Sym}^i(N_{\mathcal{O},x}^X))^{G_x}$ (see Lemma 4.1), and in the case when the stabilizer of a point in O is reductive, this bound is achieved (see Theorem 4.2)
(3) We prove that $\bar{\mathfrak{G}}_O^X(t)$ is multiplicative in an appropriate sense (see Lemma 4.1).
(4) In the general case we reduce the computation of $\bar{\mathfrak{G}}_O^X(t)$ to the computation of certain subspace of distributions supported on a point in a manifold of dimension $\dim X - \dim O$ (see Theorem 6.1). Under certain connectivity assumptions this can be reduced to an infinite dimensional linear algebra problem (see Corollary 6.3).
(5) For the case of the adjoint action of $\mathrm{GL}_3(\mathbb{C})$ on its lie algebra (or equivalently on itself) we compute $\bar{\mathfrak{G}}_O^X(t)$ for all orbits (see §7).

1.2. **Ideas in the proof.** Results (2,3) follows easily from the existing knowledge on invariant distributions. Result (1) follows easily from (4). Result (5) is a computation based on (4). In order to formulate and prove Result (4) we use [Sak16]. Namely we find a different presentation of the quotient stack $G\backslash X$, and use the fact that the space of generalized functions on a stack does not depend on the presentation (See [Sak16, Theorem 3.3.1]). In order to compute generalized functions in the new presentation we replace our groupoid structure by an infinitesimal one. We do it in Theorem 3.1.

1.3. **Structure of the paper.** In §2 we fix notation for generalized functions on Nash manifolds, Nash groupoids and Nash stacks.

In §3 we analyze generalized functions on groupoids. We prove Theorem 3.1 which states that, under certain continuity assumptions, generalized functions on a groupoid are generalized functions on the objects manifold, satisfying a certain system of PDE.

In §4 we define the function $\bar{\mathfrak{G}}_O^X(t)$, which is the main object of study in this paper, and establish its basic properties.

In §5 we introduce the stack slice, which is our main geometric tool for the computation of $\bar{\mathfrak{G}}_O^X(t)$.

In §6 we present a method to compute $\bar{\mathfrak{G}}_O^X(t)$ using the stack slice. We implement this method for regular orbits.

In §7 we compute $\bar{\mathfrak{G}}_O^X(t)$ for the adjoint action of $\mathrm{GL}_3(\mathbb{C})$.

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2. Preliminaries on generalized functions

In this section we fix some notation concerning generalized functions on manifolds, and tempered generalized functions on Nash manifolds and Nash stacks. We refer the reader to [Hör90, AG08, Sak16] for more details.

For a smooth manifold M we denote by $C^{-\infty}(M)$ the space of generalized functions, i.e. continuous functionals on the space of compactly supported smooth measures. If M has a fixed smooth invertible measure then this space can be identified with the space of distributions on M.

For a smooth real algebraic manifold (or, more generally, a Nash manifold) M we denote by $\mathcal{S}(M)$ the space of Schwartz functions on M (see e.g. [AG08]), and by $\mathcal{S}^*(M)$ the dual space. We call the elements of $\mathcal{S}^*(M)$ *tempered distributions* (Schwartz distributions in [AG08]). We also denote by $\mathcal{G}(M)$ the space of tempered generalized functions, i.e. functionals on the space of Schwartz measures $\mathcal{S}(M, D_M)$ (see [AG08]).

For a distribution or a generalized function ξ on a manifold M we denote by $\mathrm{WF}(\xi)$ its wave-front set (see [Hör90, §8.1]).

Definition 2.1. *A Lie (resp. Nash) groupoid is a diagram $\{Mor \overset{s}{\underset{t}{\rightrightarrows}} Ob\}$ of smooth (resp. Nash) manifolds such that s and t are submersions, a smooth (resp. Nash) composition map* $\mathrm{comp} : Mor \times_{Ob} Mor \to Mor$*, a smooth (resp. Nash) identity section* $\mathrm{I} : Ob \to Mor$ *and a smooth (resp. Nash) inversion map $inv : Mor \to Mor$ satisfying the usual groupoid axioms.*

Definition 2.2. *A generalized function $\xi \in C^{-\infty}(S)$ on a Lie groupoid $S = \{Mor \overset{s}{\underset{t}{\rightrightarrows}} Ob\}$ is a generalized function on Ob such that $t^*\xi = s^*\xi$. If S is a Nash groupoid, we also define the space $\mathcal{G}(S)$ of tempered generalized functions in a similar way.*

In [Sak16, Theorem 3.3.1] it is shown that $\mathcal{G}(S)$ depends only on the Nash stack corresponding to S (see [Sak16, §2.2] for the definition of the Nash stack corresponding to a Nash groupoid). Note that [Sak16] uses the notation $\mathcal{S}$ for Schwartz measures and $\mathcal{S}^*$ for generalized functions.

3. Generalized functions on smooth groupoids

Theorem 3.1. *Let $S = \{Mor \overset{s}{\underset{t}{\rightrightarrows}} Ob\}$ be a Lie groupoid. Let $\xi \in C^{-\infty}(Ob)$. Consider the following properties of ξ:*

(1) $\xi \in C^{-\infty}(S)$.
(2) For any open subset $U \subset Ob$ and any section $\varphi : U \to Mor$ of s such that $\psi := t \circ \varphi : U \to Ob$ is an open embedding we have $\psi^*\xi = \xi|_U$.
(3) For any $m \in Mor$, there exist smooth manifolds U, V and a submersion $\varphi : V \times U \to Mor$ with $m \in \operatorname{Im}\varphi$ such that for any $x \in V$, the maps $\varphi_x^s := s \circ \varphi|_{\{x\}\times U}$ and $\varphi_x^t := t \circ \varphi|_{\{x\}\times U}$ are open embeddings and we have $(\varphi_x^t)^*\xi = (\varphi_x^s)^*\xi$.
(4) For any section α of I^*TMor, where $I : Ob \to Mor$ is the identity section, with $ds(\alpha) = 0$ we have $dt(\alpha)\xi = 0$. Here, $dt(\alpha)$ and $ds(\alpha)$ are the vector fields given by $ds(\alpha)_x := d_{Id_x}s(\alpha_x)$, $dt(\alpha)_x := d_{Id_x}t(\alpha_x)$.

Then (1)⇔(2)⇔(3)⇒(4) *and if for all $x \in Ob$, $s^{-1}(x)$ is connected then* (3)⇔(4)*.*

For the proof we will need the following lemmas.

Lemma 3.2. *Let X, Y be smooth manifolds. Let $\xi \in C^{-\infty}(X \times Y)$ such that for any $x \in X$, $\mathrm{WF}(\xi) \cap CN_{\{x\}\times Y}^{X\times Y} \subset \{x\} \times Y$ and $\xi|_{\{x\}\times Y} = 0$. Then $\xi = 0$.*

Here, the restriction $\xi|_{\{x\}\times Y}$ is in the sense of [Hör90, Corollary 8.2.7].

This lemma follows from the next one in view of [Hör90, Theorem 8.2.4 and the proof of Theorem 8.2.3].

Lemma 3.3. *Let $V = \mathbb{R}^n, W = \mathbb{R}^k$ be real vector spaces. Let $\xi \in C^{-\infty}(V \times W)$ such that for any $x \in V$, $\mathrm{WF}(\xi) \cap CN_{\{x\}\times W}^{V\times W} \subset \{x\} \times W$. Fix Lebesgue measures V and W. Let $f \in C_c^\infty(V \times W)$. Let $e_i \in C_c^\infty(V \times W)$ be a sequence satisfying $\int_{V\times W} e_i(z)dz = 1$ and $e_i(z) = 0$ for any z with $||z|| > 1/i$. For any $x \in V$ denote $g(x) := \langle \xi|_{\{x\}\times W}, f|_{\{x\}\times W}\rangle$ and $g_n(x) := \langle (\xi * e_n)|_{\{x\}\times W}, f|_{\{x\}\times W}\rangle$. Then $g_n \to g$ uniformly as $n \to \infty$.*

Proof. Let $U \supset \operatorname{Supp} f$ be an open centrally symmetric set with compact closure. Let

$$\Gamma := \left(V \times pr_{W\times(V\times W)^*}\left(\mathrm{WF}(\xi) \cap (\bar{U} \times (V \times W)^*)\right)\right) \cup (((V \times W) \smallsetminus U) \times (V \times W)^*) \subset T^*(V \times W).$$

For any $x \in V$ denote $\xi_x := Sh_x(\xi)$, where Sh_x is the translation by x. Denote

$$C_\Gamma^{-\infty}(V \times W) := \{\eta \in C^{-\infty}(V \times W),\ \mathrm{WF}(\eta) \subset \Gamma\},$$

with the topology of [Hör90, Definition 8.2.2]. It is easy to see that $x \mapsto \xi_x$ defines a continuous map $V \to C_\Gamma^{-\infty}(V \times W)$. Let $\xi_{n,x} := \xi_x * e_n$. The proof of [Hör90, Theorem 8.2.3] implies that $\xi_{n,x} \to \xi_x$ as $n \to \infty$ in the topology of $C_\Gamma^{-\infty}(V \times W)$ uniformly in x. Thus, by [Hör90, Theorem 8.2.4], $\xi_{n,x}|_{\{0\}\times W} \to \xi_x|_{\{0\}\times W}$ as $n \to \infty$ in the weak topology of $C^{-\infty}(W)$ uniformly in x. This implies the assertion. $\Box$

The following lemma is standard.

Lemma 3.4. *Let $\varphi : M \to N$ be a submersion of smooth manifolds with connected fibers. Let $s_0, s_1 : N \to M$ be its (smooth) sections. Then, for any $y \in N$, there exists an open neighborhood U of y and a smooth homotopy $h : [0,1] \times U \to M$ such that $h|_{\{0\}\times U} = s_0$, $h|_{\{1\}\times U} = s_1$, and $h|_{\{t\}\times U}$ is a section of φ for any t.*

Corollary 3.5. *Let $\varphi_1 : M_1 \to N$ and $\varphi_2 : M_2 \to N$ be submersions of smooth manifolds. Assume that all the fibers of φ_2 are connected. Let $\psi_1, \psi_2 : M_1 \to M_2$ be smooth maps of N-manifolds (that is, smooth maps such that $\varphi_2 \circ \psi_i = \varphi_1$). Then, for any $y \in M_1$, there exists an open neighborhood U of y and a smooth homotopy $h : [0,1] \times U \to M_2$ such that $h|_{\{0\}\times U} = \psi_0$, $h|_{\{1\}\times U} = \psi_1$, and $h|_{\{t\}\times U}$ is a map of N-manifolds.*

Proof of Theorem 3.1.

(1)⇒(2): by functoriality of the pullback.

(2)⇒(3): It is enough to show that for any $m \in Mor$, there exist smooth manifolds U, V and a submersion $\varphi : V \times U \to Mor$ with $m \in \operatorname{Im}\varphi$ such that for any $x \in V$, the maps φ_x^s and φ_x^t are open embeddings. Since s and t are submersions, we can decompose $T_m(Mor) = V' \oplus U'$ such that $d_m s|_{U'}$ and $d_m t|_{U'}$ are isomorphisms. Let $\varphi' : T_m(Mor) \to Mor$ be such that $\varphi'(0) = m$ and $d\varphi' = Id$. By the implicit function theorem one can choose open subsets $U \subset U'$ and $V \subset V'$ such that $\varphi := \varphi'|_{U\times V}$ is a submersion and the maps φ_x^s and φ_x^t are open embeddings.

(3) ⇒ (1): by Lemma 3.2.

(2) ⇒ (4): Let α be a section of I^*TMor with $ds(\alpha) = 0$. Define a vector field β on Mor by

$$\beta_m := (d_{I(t(m)),m} comp)(\alpha_{t(m)}, 0),$$

where $comp : Mor \times_{Ob} Mor \to Mor$ is the composition map. By the existence and uniqueness theorem for ODE, we have an open neighborhood $\mathcal{O}$ of $Mor \times \{0\}$ in $Mor \times \mathbb{R}$ and a map $B : \mathcal{O} \to Mor$ that solves the ODE defined by β. Fix $x \in Ob$. There exists a neighborhood U of x and $\varepsilon > 0$ such that $U \times (-\varepsilon, \varepsilon) \subset \mathcal{O}$. For any $r \in (-\varepsilon, \varepsilon)$ define $\varphi_r(x) := B(x, r)$. Define $\psi_r := t \circ \varphi_r$. By (2) we have $\psi_r^* \xi = \xi|_U$. On the other hand, it is easy to see that

$$\frac{d}{dr}|_{r=0} \psi_r^* \xi = dt(\alpha)\xi|_U.$$

(4) ⇒ (3), for connected $s^{-1}(x)$: Let $m \in Mor$. By Corollary 3.5, there exist an open neighborhood V of m and a smooth homotopy $h : [0,1] \times V \to Mor$ such that

$$h|_{\{0\}\times V} = I \circ s,\ h|_{\{1\}\times V} = I \circ s \text{ and } s(h(r,x)) = s(x).$$

For any $r \in [0,1]$ and $u \in V$ consider

$$\alpha(r,v) := d_{h(r,v),inv(h(r,v))} comp(\frac{d}{dr} h(r,v), 0) \in T_{I(t(h(r,v)))} Mor.$$

Extend $\alpha(r,v)$ to a smooth section of I^*TMor in a way that depends smoothly on (r,v) such that $ds(\alpha(r,v)) = 0$ for any $(r,v) \in [0,1] \times V$. By (4) we have $dt(\alpha(r,v))\xi = 0$. Define a vector field $\beta(r,v)$ on Mor by

$$\beta(r,v)_n := d_{I(t(n)),n} comp(\alpha(r,v)_{t(n)}, 0). \quad (1)$$

For any $v \in V$ we consider $\beta(\cdot, v)$ as a time-dependent vector field on Mor. By the existence and uniqueness theorem for ODE, we have an open neighborhood $\mathcal{O}$ of $Mor \times \{0\} \times V \cup \{h(r,v), r, v) \,|\, r \in [0,1],\, v \in V\}$ in $Mor \times [0,1] \times V$ and a map $B : \mathcal{O} \to Mor$ that solves the ODE defined by β. Let

$$\Xi := I \times Id^{-1}_{[0,1]\times V}(\mathcal{O}) \subset Ob \times [0,1] \times V, A := B \circ (I \times Id_{[0,1]\times V}|_{\Xi}) \text{ and } C := t \circ A.$$

Let U be a neighborhood of $s(m)$ in Ob such that $U \times [0,1] \times V \subset \Xi$. Define $\varphi : U \times V \to Mor$ by $\varphi := A|_{U\times\{1\}\times V}$. It is enough to prove that for any $x \in V$:

(i) The map $\varphi^s_x := s \circ \varphi|_{\{x\}\times U}$ is an open embedding.
(ii) The map $\varphi^t_x := t \circ \varphi|_{\{x\}\times U}$ is an open embedding.
(iii) We have $(\varphi^t_x)^*\xi = (\varphi^s_x)^*\xi$.

Note that $\varphi^s_x = Id$ and thus (i) holds.

For $(v, r) \in V \times [0,1]$ let $\gamma(v,r) := dt(\alpha(v,r))$ be a vector field on Ob. By (4), $\gamma(v,r)\xi = 0$. It is easy to see that

$$\frac{\partial}{\partial r} C(x,v,r) = \gamma(v,r) C(x,v,r).$$

Thus, for any $(x,v) \in Ob \times V$, the (partially defined) curve $C(x,v,\cdot)$ is a solution of the ODE defined by the time-dependent vector field $\gamma(v,\cdot)$. Note that $\varphi^t_x = C|_{U\times\{x\}\times\{1\}}$ and thus (ii) holds.

Finally, (iii) follows from the equality $\gamma(v,r)\xi = 0$.

$\square$

4. The dimension growth function of an orbit

Let a Nash group G act on a Nash manifold X. Let $\mathcal{O} \subset X$ be an orbit. Let F_i be the Bruhat filtration on $\mathcal{G}_{\mathcal{O}}(X \smallsetminus (\bar{\mathcal{O}} \smallsetminus \mathcal{O}))$ (see [AG08, Corollary 5.5.4]). Let

$$V_i := \{\xi \in F_i \,|\, \exists \eta \in \mathcal{G}(X) \text{ s.t. } \eta|_{(X\smallsetminus(\bar{\mathcal{O}}\smallsetminus\mathcal{O}))} = \xi\}.$$

Define the distributional dimension growth function of $\mathcal{O}$ in X by

$$\delta^X_{\mathcal{O}}(i) := \dim V_i.$$

Define also the distributional normal dimension of $\mathcal{O}$ in X by

$$Ddim(\mathcal{O}, X) := \limsup_i \frac{\ln \delta^X_{\mathcal{O}}(i)}{\ln i},$$

and the distributional normal degree of $\mathcal{O}$ in X by

$$Ddeg(\mathcal{O}, X) := \limsup_i \left(Ddim(\mathcal{O}, X)! \delta^X_{\mathcal{O}}(i) i^{-Ddim(\mathcal{O},X)} \right).$$

Define the distributional dimension generating function by

$$\mathfrak{G}^X_{\mathcal{O}}(t) := (1-t) \sum_i t^i \delta^X_{\mathcal{O}}(i).$$

Finally, define the reduced versions of the above notions by

$$\bar{\delta}^X_{\mathcal{O}} := \delta^{X\smallsetminus(\bar{\mathcal{O}}\smallsetminus\mathcal{O})}_{\mathcal{O}},\ \overline{Ddim}(\mathcal{O}, X) := Ddim(\mathcal{O}, X \smallsetminus (\bar{\mathcal{O}} \smallsetminus \mathcal{O})),\ \overline{Ddeg}(\mathcal{O}, X) := Ddeg(\mathcal{O}, X \smallsetminus (\bar{\mathcal{O}} \smallsetminus \mathcal{O}))$$

For a point $x \in \mathcal{O}$ we will denote

$$\delta^X_x(i) := \delta^X_{\mathcal{O}}(i), \quad \bar{\delta}^X_x(i) := \bar{\delta}^X_{\mathcal{O}}(i), \quad Ddim(x, X) := Ddim(\mathcal{O}, X),$$
$$\overline{Ddim}(x, X) := \overline{Ddim}(\mathcal{O}, X), \quad Ddeg(x, X) := Ddeg(\mathcal{O}, X), \quad \overline{Ddeg}(x, X) := \overline{Ddeg}(\mathcal{O}, X).$$

The following lemma follows from [AG08, Corollary 5.5.4] and [AG10, Corollary 2.6.3].

Lemma 4.1. *Let $x \in \mathcal{O}$. Then* *(i)* $\delta^X_{\mathcal{O}}(i) - \delta^X_{\mathcal{O}}(i-1) \leq \dim(\mathrm{Sym}^i(N^X_{\mathcal{O},x}))^{G_x}$ *and* $Ddim(\mathcal{O}, X) \leq \dim(N^X_{\mathcal{O},x})$.

(ii) *Let U be an open G-invariant neighborhood of $\mathcal{O}$ in X. Then*

$$\delta^U_{\mathcal{O}}(i) \geq \delta^X_{\mathcal{O}}(i) \text{ and } Ddim(\mathcal{O}, U) \geq Ddim(\mathcal{O}, X).$$

(iii) *Let another Nash group G' act on a Nash manifold X', and $\mathcal{O}'$ be an orbit. Consider the action of $G \times G'$ on $X \times X'$. Then*

$$\mathfrak{G}^{X\times X'}_{\mathcal{O}\times\mathcal{O}'}(t) = \mathfrak{G}^X_{\mathcal{O}}(t)\mathfrak{G}^{X'}_{\mathcal{O}'}(t).$$

The following theorem follows from the proof of [AG09a, Theorem 3.1.1].

Theorem 4.2. *Let a reductive group G act on an affine algebraic manifold X. Let $\mathcal{O} \subset X$ be a closed orbit. Then $\delta^X_{\mathcal{O}}(i) - \delta^X_{\mathcal{O}}(i-1) = \dim(\mathrm{Sym}^i(N^X_{\mathcal{O},x}))^{G_x}$.*

Remark 4.3.

(i) *One can replace the assumption that G is reductive and X is affine by the weaker assumption that the stabilizer of a point $x \in \mathcal{O}$ is reductive. For that one needs to use the version of the Luna slice theorem appearing in* [AHR, Theorem 2.1].

(ii) *Since the Poincare series of a finitely generated graded algebra is a rational function whose poles are roots of unity (see e.g.* [AM69, Theorem 11.1]*), Theorem 4.2 implies that $\mathfrak{G}^X_{\mathcal{O}}(t)$ is also a rational function whose poles are roots of unity.*

(iii) *In a similar way one can show that under the conditions of Theorem 4.2, the dimension $Ddim(\mathcal{O}, X)$ equals the dimension of the categorical quotient $N^X_{\mathcal{O},x}//G_x$ of $N^X_{\mathcal{O},x}$ by G_x.*

(iv) *Similarly, if in Lemma 4.1 the categorical quotient $N^X_{\mathcal{O},x}//G_x$ exists then $Ddim(\mathcal{O}, X) \leq \dim(N^X_{\mathcal{O},x}//G_x)$.*

5. Restriction of a Nash stack to a slice

Let a Nash group G act on a Nash manifold X.

Definition 5.1. *Choose a point $x \in X$ and let $\mathcal{O} := Gx$ be its orbit.*

(1) *We call a locally closed Nash submanifold $S \subset X$ a slice to the action of G at x if $x \in S$, the action map $a : G \times S \to X$ is a submersion, and $\dim \mathcal{O} + \dim S = \dim X$.*

(2) *Let S be a slice to the action of G at x. Define $M_S := a^{-1}(S) \subset G \times S$. Consider the quotient Nash groupoid $G\times X \underset{a}{\overset{pr}{\rightrightarrows}} X$ and its subgroupoid $M_S \underset{a}{\overset{pr}{\rightrightarrows}} S$. We will call this subgroupoid a* groupoid slice *to the action of G at x, and call the corresponding Nash stack a* stack slice *to the action of G at x.*

Lemma 5.2. *For any $x \in X$ there exists a slice to the action of G at x.*

Proof. Choose a direct complement W to T_xX in T_xGx. It is a standard fact that there exists a Nash manifold $S' \subset X$ containing x such that $T_xS' = W$. Consider the action map $a : G \times S' \to X$. Let $S := \{x \in S' \mid a \text{ is a submersion at } (1, x)\}$. It is easy to see that S satisfies the conditions. □

The following proposition follows from the definition in [Sak16, §2.2].

Proposition 5.3. *For any $x \in X$ and any stack slice $\mathfrak{S}$ to the action of G at x there exists an open Nash G-invariant neighborhood U of x and such that $G\backslash U \cong \mathfrak{S}$.*

6. Description of the space of invariant generalized functions supported on an orbit

Proposition 5.3 and [Sak16, Theorem 3.3.1] imply the following theorem

Theorem 6.1. *Let a Nash group G act on a Nash manifold X. Let $x \in X$ such that the orbit Gx is closed. Then for any groupoid slice $\mathfrak{S}$ to the action of G at x we have a canonical isomorphism $\mathcal{G}_{Gx}(X)^G \cong \mathcal{G}_{\{x\}}(\mathfrak{S})$. Here, we consider $\{x\}$ as a closed subset in $\mathfrak{S}$.*

Notation 6.2. *Let a Lie group G act on a smooth manifold X. Let $S \subset X$ be a (locally closed) smooth submanifold. Let $\varphi : S \to \mathfrak{g}$ be a smooth map. For any $s \in S$ define $\alpha_\varphi(s) \in T_sX$ by $\alpha_\varphi(s) := d_e(a_s(\varphi(s)))$, where $a_s : G \to X$ is the action map on s and $e \in G$ is the unit element. Suppose that α_φ defines a vector field on S, i.e. $\alpha_\varphi(s) \in T_sS$ for any $s \in S$. Then we call this field* strongly tangential *to the action of G.*

Theorems 3.1 and 6.1 give the following corollary.

Corollary 6.3. *Let a Nash group G act on a Nash manifold X. Let $x \in X$ such that the orbit Gx is closed. Let S be a slice to the action of G at x. Then we have a canonical embedding of $\mathcal{G}_{Gx}(X)$ into the space*

$$\{\xi \in \mathcal{G}_{\{x\}}(S) \,|\, \alpha\xi = 0 \text{ for any vector field } \alpha \text{ on } S \text{ strongly tangential to the action of } G\}.$$

Moreover, if for all $x \in S$, the set of all $g \in G$ with $gx \in S$ is connected then this embedding is an isomorphism.

Corollary 6.4. *Let $\varphi : X \to Y$ be a Nash submersion of Nash manifolds. Let a Nash group G act on X preserving φ. Let $y \in Y$ and assume that G acts transitively on the fiber $\varphi^{-1}(y)$. Then $\mathcal{G}_{\varphi^{-1}(y)}(X)^G$ is isomorphic as a filtered vector space to $\mathbb{C}[t_1, \dots, t_{\dim Y}]$. In particular,*

$$\mathfrak{G}^X_{\varphi^{-1}(y)}(t) = (1-t)^{-\dim Y},\ Ddim(\varphi^{-1}(y), X) = \dim Y,\ \text{and } Ddeg(\varphi^{-1}(y), X) = 1. \tag{2}$$

Proof. Let $x \in \varphi^{-1}(y)$. By Lemma 5.2 there exists a slice S to the action of G on X at x. Shrinking S, we can assume that $\varphi|_S$ is an open embedding. Let $M_S \overset{pr}{\underset{a}{\rightrightarrows}} S$ be as in Definition 5.1. By the assumption, $pr = a$. Thus the corollary follows from Theorem 6.1. □

7. Computation of $\bar{\delta}$ for the adjoint action of $\mathrm{GL}_3(\mathbb{C})$

Theorem 7.1. *Consider the adjoint action of $G := \mathrm{GL}_3(\mathbb{C})$ on its Lie algebra $\mathfrak{g}$. Let $x \in \mathfrak{g}$ and let m_x denote its minimal polynomial. Then*

$$\bar{\mathfrak{G}}^{\mathfrak{g}}_x(t) = \begin{cases} (1-t)^{-6} & \deg m_x = 3 \\ (1-t)^{-6}(1+t)^{-4}(t^2-t+2)^2 & m_x = (x-\lambda)^2 \\ (1-t)^{-6}(1+t)^{-2} & m_x = (x-\lambda)(x-\mu),\ \lambda \neq \mu \\ (1-t)^{-6}(1+t)^{-2}(1+t+t^2)^{-2} & \deg m_x = 1 \end{cases},$$

$\overline{Ddim}(\mathcal{O}, X) = 6$ *and* $\overline{Ddeg}(\mathcal{O}, X) = ((3 - \deg m_x)!)^{-2}$.

The case $\deg m_x = 3$ follows from Corollary 6.4. The case $\deg m_x = 1$ follows from Theorem 4.2. The case $m_x = (x-\lambda)(x-\mu)$, $\lambda \neq \mu$ follows from Theorem 4.2 and Lemma 4.1(iii). Thus it is enough to prove the following proposition.

Proposition 7.2. *Let $G := \mathrm{GL}_3(\mathbb{C})$ act on $X := \mathfrak{sl}_3(\mathbb{C}) \smallsetminus 0$ by conjugation. Let $x \in X$ be the subregular nilpotent matrix. Then $\bar{\mathfrak{G}}^X_x(t) = (1-t)^{-4}(1+t)^{-4}(t^2-t+2)^2$.*

Let $e := E_{12} \in \mathcal{O}$. Let $f := E_{21}$ and let $\mathfrak{s}_{\mathbb{C}} := e + \mathfrak{gl}_3(\mathbb{C})^f$ and $\mathfrak{s}_{\mathbb{R}} := e + \mathfrak{gl}_3(\mathbb{R})^f$ be the Slodowy slices. For the proof we will need the following lemma.

Lemma 7.3. *$\mathfrak{s}_{\mathbb{C}}$ is a slice for the action of G at the point e, and for any $x \in \mathfrak{s}_{\mathbb{C}}$, the Nash manifold $\{g \in G \,|\, gx \in \mathfrak{s}_{\mathbb{C}}\}$ is connected.*

Proof. The fact that $\mathfrak{s}_{\mathbb{C}}$ is a slice for the action of G is standard. Since all the stabilizers of the action of G are connected, in order to prove that $\{g \in G \,|\, gx \in \mathfrak{s}_{\mathbb{C}}\}$ is connected it is enough to prove that the intersection of any G-orbit $\mathcal{O}$ with $\mathfrak{s}_{\mathbb{C}}$ is connected. For this it is enough to show that $\bar{\mathcal{O}} \cap \mathfrak{s}_{\mathbb{C}}$ is an irreducible algebraic variety. We divide the proof into two cases.

Case 1 $\bar{\mathcal{O}} = \{x \in X \mid \det(x - \lambda \operatorname{Id}) = -\lambda^3 + \gamma_1 \lambda + \gamma_0\}$ for some fixed γ_0 and γ_1.
Choose the following coordinates on $\mathfrak{s}_{\mathbb{R}}$:

$$\mathfrak{s}_{\mathbb{R}} = \left\{ \begin{pmatrix} a & 1 & 0 \\ b & a & c \\ d & 0 & -2a \end{pmatrix} \right\}. \tag{3}$$

In these coordinates, $\bar{\mathcal{O}} \cong \{(a,b,c,d) \mid 3a^2 + b = \gamma_1 \text{ and } cd - 2a^3 + 2ab = \gamma_0\}$. This variety is isomorphic to $\{(a,c,d) \mid -8a^3 + 2\gamma_1 a + cd = \gamma_0\}$. Thus this case follows from the irreducibility of the polynomial $-8a^3 + 2\gamma_1 a + cd - \gamma_0$ for any γ_1, γ_2.

Case 2 $\bar{\mathcal{O}} = \{x \in X \mid (x - \gamma \operatorname{Id})(x + 2\gamma \operatorname{Id}) = 0\}$ for some fixed γ.
In the coordinates above $\bar{\mathcal{O}}$ is given by the irreducible polynomial

$$cd - (2a + \gamma)^2 (2a - 2\gamma).$$

□

Lemma 7.4. *The collection of vector fields on $\mathfrak{s}_{\mathbb{R}}$ strongly tangential to the action of G is generated over $C^\infty(\mathfrak{s}_{\mathbb{R}})$ by the fields $v_1, \dots, v_4$, where*

$$v_1(A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ -A_{31} & 0 & 0 \end{pmatrix}, \; v_2(A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -A_{11}A_{23} \\ A_{11}A_{31} & 0 & 0 \end{pmatrix}$$

$$v_3(A) = \begin{pmatrix} A_{31}/2 & 0 & 0 \\ -3A_{11}A_{31} & A_{31}/2 & 9A_{11}^2 - A_{21} \\ 0 & 0 & -A_{31} \end{pmatrix}, \; v_4(A) = \begin{pmatrix} -A_{23}/2 & 0 & 0 \\ 3A_{11}A_{23} & -A_{23}/2 & 0 \\ -9A_{11}^2 + A_{21} & 0 & A_{23} \end{pmatrix}$$

This lemma is proven by a direct computation.

Proof of Proposition 7.2. Let

$$V_{\mathbb{C}} := \{\xi \in \mathcal{G}_{\{e\}}(\mathfrak{s}_{\mathbb{C}}) \mid \alpha\xi = 0 \text{ for any vector field } \alpha \text{ on } \mathfrak{s}_{\mathbb{C}} \text{ strongly tangential to the action of } G\}.$$

and

$$V_{\mathbb{R}} := \{\xi \in \mathcal{G}_{\{e\}}(\mathfrak{s}_{\mathbb{R}}) \mid \alpha\xi = 0 \text{ for any vector field } \alpha \text{ on } \mathfrak{s}_{\mathbb{R}} \text{ strongly tangential to the action of } \operatorname{GL}_n(\mathbb{R})\}.$$

By Lemma 7.3 and Corollary 6.3 $\mathcal{G}_{\mathcal{O}}(X) \cong V_{\mathbb{C}}$. It is easy to see that $V_{\mathbb{C}} \cong V_{\mathbb{R}} \otimes V_{\mathbb{R}}$ as a filtered vector space. By Lemma 7.4,

$$V_{\mathbb{R}} = \{\xi \in \mathcal{G}_{\{x\}}(\mathfrak{s}_{\mathbb{R}}) \mid v_i \xi = 0 \; \forall 1 \le i \le 4\}.$$

Choose the following coordinates on $\mathfrak{s}_{\mathbb{R}}$:

$$\mathfrak{s}_{\mathbb{R}} = \left\{ \begin{pmatrix} a & 1 & 0 \\ b & a & c \\ d & 0 & -2a \end{pmatrix} \right\}.$$

In these coordinates we have

$$v_1 = c\frac{\partial}{\partial c} - d\frac{\partial}{\partial d}, \; v_2 = -av_1, \; v_3 = \frac{d}{2}\frac{\partial}{\partial a} - 3ad\frac{\partial}{\partial b} + (9a^2 - b)\frac{\partial}{\partial c}, \; v_4 = -\frac{c}{2}\frac{\partial}{\partial a} + 3ac\frac{\partial}{\partial b} + (b - 9a^2)\frac{\partial}{\partial d}$$

Fix a Lebesgue measure on $\mathfrak{s}_{\mathbb{R}}$. It defines the generalized function $\delta_e \in \mathcal{G}_{\{e\}}(\mathfrak{s}(\mathbb{R}))$. Let

$$\delta_{ijkl} := \left(\frac{\partial}{\partial c}\right)^i \left(\frac{\partial}{\partial c}\right)^j \left(\frac{\partial}{\partial c}\right)^k \left(\frac{\partial}{\partial c}\right)^l \delta_e.$$

If one of the indices i, j, k, l is negative we set $\delta_{ijkl} := 0$. We have

$$\begin{aligned} v_1\delta_{ijkl} &= -(k+1)\delta_{ijkl} + (l+1)\delta_{ijkl}, \\ v_3\delta_{ijkl} &= -\frac{l}{2}\delta_{i+1,j,k,l-1} - 3il\delta_{i-1,j+1,k,l-1} + 9i(i-1)\delta_{i-2,j,k+1,l} + j\delta_{i,j-1,k+1,l} \\ v_4\delta_{ijkl} &= \frac{k}{2}\delta_{i+1,j,k-1,l} + 3ik\delta_{i-1,j+1,k-1,l} - 9i(i-1)\delta_{i-2,j,k,l+1} - j\delta_{i,j-1,k,l+1} \end{aligned}$$

Let $\xi = \sum c_{ijkl}\delta_{ijkl}$ and note that $v_1\xi = 0$ if and only if $c_{ijkl} = 0 \, \forall k \neq l$. Set $\delta_{ijk} := \delta_{ijkk}$. Let $\xi = \sum c_{ijk}\delta_{ijk}$ we get

$$v_3\xi = \sum_{i,j\geq 0, k\geq 1} \left(-\frac{k}{2}c_{i-1,j,k} - 3(i+1)kc_{i+1,j-1,k} + 9(i+2)(i+1)c_{i+2,j,k-1} + (j+1)c_{i,j+1,k-1} \right) \delta_{i,j,k,k-1}$$

$$v_4\xi = \sum_{i,j,k\geq 0} \left(\frac{k+1}{2}c_{i-1,j,k+1} + 3(i+1)(k+1)c_{i+1,j-1,k+1} - 9(i+2)(i+1)c_{i+2,j,k} - (j+1)c_{i,j+1,k} \right) \delta_{i,j,k,k+1}$$

Here, if one of the indices i, j, k is negative we set $c_{i,j,k} = 0$.

We obtain that $V_{\mathbb{R}}$ is the collection of all finite combinations $\sum c_{ijk}\delta_{ijk}$ that satisfy

$$c_{i-1,j,k+1}\frac{k+1}{2} + 3c_{i+1,j-1,k+1}(i+1)(k+1) - 9c_{i+2,j,k}(i+2)(i+1) - c_{i,j+1,k}(j+1) = 0$$

for all $i, j, k \geq 0$.

Let F^n be the Bruhat filtration on $V_{\mathbb{R}}$ and G^l be the filtration on $F^n(V_{\mathbb{R}})$ given by

$$G^l(F^n(V_{\mathbb{R}})) = \left\{ \sum c_{ijk}\delta_{ijk} \in F^n(V_{\mathbb{R}}) \,|\, \forall k > l \text{ we have } c_{ijk} = 0 \right\}.$$

It is easy to compute that

$$\dim G^l(F^n(V_{\mathbb{R}})) - \dim G^{l-1}(F^n(V_{\mathbb{R}})) = n - 2l.$$

Thus

$$\dim F^{2m}(V_{\mathbb{R}}) = m(m+1) \text{ and } \dim F^{2m+1}(V_{\mathbb{R}}) = (m+1)^2.$$

Define the power series

$$f(s) := \sum_n s^{n+1} = s/(1-s) \text{ and } g(t) := \sum_n \dim F^n(V_{\mathbb{R}})t^n.$$

Then

$$\sum_m m(m+1)s^m = sf''(s) = 2(1-s)^{-3} \text{ and } \sum s^m(m+1)^2 = (sf'(s))' = (1+s)(1-s)^{-3}.$$

We get

$$\begin{aligned} g(t) = \sum_m (t^2)^m(m+1) + t\sum_m (t^2)^m(m+1)^2 &= 2(1-t^2)^{-3} + t(1+t^2)(1-t^2)^{-3} \\ &= (t^3+t+2)(1-t^2)^{-3} = (t^2-t+2)(1-t)^{-3}(1+t)^{-2} \end{aligned}$$

Thus

$$\sum_n (\dim F^n(V_{\mathbb{R}}) - \dim F^{n-1}(V_{\mathbb{R}}))t^n = (t^2-t+2)(1-t)^{-2}(1+t)^{-2},$$

and hence

$$\bar{\mathfrak{G}}_x^X(t) = \sum_n (\dim F^n(V_{\mathbb{C}}) - \dim F^{n-1}(V_{\mathbb{C}}))t^n = (t^2-t+2)^2(1-t)^{-4}(1+t)^{-4}.$$

$\square$

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