COMPOSITION SERIES FOR DEGENERATE PRINCIPAL SERIES OF GL(n)

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Abstract. In this note we consider representations of the group $GL(n, F)$, where F is the field of real or complex numbers or, more generally, an arbitrary local field, in the space of equivariant line bundles over Grassmannians over the same field F. We study reducibility and composition series of such representations.

Similar results were obtained already in [HL99, Al12, Zel80], but we give a short uniform proof in the general case, using the tools from [AGS15a]. We also indicate some applications to cosine transforms in integral geometry.

1. Introduction

Let F be a local field, *i.e.* either $\mathbb{R}$ or $\mathbb{C}$ or a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers for some prime p, or the field $\mathbb{F}_q((t))$ of Laurent series over a finite field. Let $G = G_n := GL(n, F)$ be the group of invertible matrices of size n with entries in F.

Then $C^\infty(G)$ is a $G \times G$-module with left and right actions given by $L_g f(x) = f(g^{-1}x)$, $R_g f(x) = f(xg)$. Let $P \subset G$ be a Lie subgroup with modular function Δ_P and let χ be a character of P. The induced representation $I(P, \chi) := Ind_P^G(\chi)$ is the right G-action on the space

$$C^\infty(G, P, \chi) := \left\{ f \in C^\infty(G) \mid L_{p^{-1}} f = \chi(p) \Delta_P^{1/2}(p) f \text{ for all } p \in P \right\}, \tag{1}$$

whose elements may also be regarded as smooth sections of a line bundle on G/P.

We are interested in the so-called maximal parabolic subgroups $P = P_{p_1,p_2}$, with $p_1 + p_2 = n$, consisting of matrices $x \in G_n$ of the form

$$x = \begin{bmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{bmatrix}; x_{ij} \in Mat_{p_i \times p_j}. \tag{2}$$

In this case G/P is the Grassmannian of p_1-dimensional subspaces of F^n. The characters of P are of the form $\chi_1 \otimes \chi_2(x) = \chi_1(x_{11})\chi_2(x_{22})$, where χ_i is a character of G_{p_i}. Following [BZ77] we write $\chi_1 \times \chi_2$ instead of $I(P, \chi_1 \otimes \chi_2)$.

It is well known that the representation $\chi_1 \times \chi_2$ is irreducible for most characters χ_1, χ_2. We explicitly describe the exceptional characters for which $\chi_1 \times \chi_2$ is reducible, and show that the composition series are monotone in a certain sense. We also show that if F is non-archimedean, the length of $\chi_1 \times \chi_2$ is at most two, *i.e.* it has at most one non-trivial G-invariant subspace.

In 1.1 below we describe the main results precisely. In 1.2 below we give applications to integral geometry.

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Most of the results are known, see 1.3. However, we give short proofs that work uniformly for all fields. Our main tool are the Bernstein-Zelevinsky derivatives ([BZ77]) and their archimedean analogs developed in [AGS15a, AGS15b]. We describe these, as well as other preliminaries, in §2.

1.1. **Main results.** We fix some notation to describe our main results succinctly. For $z \in F$ let $\nu(z)$ denote the positive scalar by which the additive Haar measure on F transforms under multiplication by z. We will also regard ν as a character of G_k defined by $\nu(g) := \nu_k(g) := \nu(\det g)$, and we note that the modular function of $P = P_{p_1,p_2}$ is $\Delta_P = \nu^{p_2} \otimes \nu^{-p_1}$. We also let 1 denote the trivial character of G_k, ε denote the $sgn(\det)$ character of $\mathrm{GL}_k(\mathbb{R})$, and α denote the character $\det|\det|^{-1}$ of $\mathrm{GL}_k(\mathbb{C})$. Twisting by a character of G_n we can assume $\chi_2 = 1$ and consider $\chi \times 1$. Let $r := \min(p_1, p_2)$.

Theorem 1.1. *$\chi \times 1$ is reducible if and only if one of the following holds*

(i) $\chi = \nu^{\pm(k-n/2)}$, *where k is an integer, and $0 \le k \le r-1$.*

(ii) $F = \mathbb{R}$, *and* $\chi = \varepsilon^{k+1}\nu^{\pm(k-r+n/2)}$ *for some positive integer k.*

(iii) $F = \mathbb{R}$, $r > 1$ *and* $\chi = \nu^{\pm(k-r+n/2+1)}$ *or* $\chi = \varepsilon\nu^{\pm(k-r+n/2+1)}$ *for some positive integer k.*

(iv) $F = \mathbb{C}$, *and either χ or χ^{-1} equals $\alpha^k\nu^s$ where $s - n/2$ and k are integers and* $s+k, s-k > n/2 - r$.

We also show (Lemma 4.2) that if F is non-archimedean then $length(\chi \times 1) \le 2$.

Now we want to show that $\chi \times 1$ has a unique composition series, monotone in some sense. For Archimedean F we could use just the Gelfand - Kirillov dimension. For the general case we will use the notion of rank, see §2.4. Define $s(\chi) \in \mathbb{R}$ by $|\chi| = \nu^{s(\chi)}$.

Theorem 1.2. *If $s(\chi) > 0$ then the irreducible constituents of $\chi \times 1$ appear in strictly descending order of rank. Otherwise they appear in strictly ascending order of rank.*

Consider the standard intertwining operator $I_\chi : \chi \times 1 \to 1 \times \chi$. From Theorem 1.2 we obtain

Corollary 1.3. *The image of I_χ is the only irreducible quotient of $\chi \times 1$ and the only irreducible submodule of $1 \times \chi$. In particular, I_χ is invertible if and only if $\chi \times 1$ is irreducible.*

1.2. **Connections to integral geometry.** The α-cosine transform on real Grassmannian in a Euclidean space $\mathbb{R}^n$, including the case of even functions on the sphere, was studied by both analysts and geometers for a long period of time: [Sem61, Sch70, Kol97, OR05, Rub98, Rub02, Rub13]. The case $\alpha = 1$ plays a special role in convex and stochastic geometry, see [GH90a, GH90b, GH96], and in particular in valuation theory [AB04, Al03, Al04, Al11]. One of the reasons why the case of at least some values $\alpha \neq 1$ is relevant for geometry is the recent result [AlGS15] saying that the Radon transform between the Grassmannians of i- and $(n-i)$-dimensional linear subspaces in $\mathbb{R}^n$ coincides with the $\alpha = -\min\{i, n-i\}$-cosine transform [1], and moreover sometimes composition of two Radon transforms coincides with the α-cosine transform for appropriate $\alpha \neq 1$.

[1]For $i = 1$ this result is known for a long time and seems to be a folklore.

Let us recall the definition of the α-cosine transform. Let $Gr_i(\mathbb{R}^n)$ denote the Grassmannian of i-dimensional linear subspaces in the Euclidean space $\mathbb{R}^n$. For a pair of subspaces $E, F \in Gr_i(\mathbb{R}^n)$ one defines (the absolute value of) the cosine of the angle $|\cos(E,F)|$ between E and F as the coefficient of the distortion of measure under the orthogonal projection from E to F. More precisely, let $q\colon E \to F$ denote the restriction to E of the orthogonal projection $\mathbb{R}^n \to F$. Let $A \subset E$ be an arbitrary subset of finite positive Lebesgue measure. Then define

$$|\cos(E,F)| := \frac{vol_F(q(A))}{vol_E(A)},$$

where vol_F, vol_E are Lebesgue measures on F, E respectively induced by the Euclidean metric on $\mathbb{R}^n$ normalized so that the Lebesgue measures of unit cubes are equal to 1. It is easy to see that $|\cos(E,F)|$ is independent of the set A and is symmetric with respect to E and F.

Example 1.4. *If $i = 1$ then $|\cos(E,F)|$ is the usual cosine of the angle between two lines.*

For $\alpha \in \mathbb{C}$ with $Re(\alpha) \geq 0$ let us define the α-cosine transform

$$T_\alpha := T^i_\alpha\colon C^\infty(Gr_i(\mathbb{R}^n)) \to C^\infty(Gr_i(\mathbb{R}^n))$$

by

$$(T^i_\alpha f)(E) = \int_{F \in Gr_i(\mathbb{R}^n)} |\cos(E,F)|^\alpha f(F) dF, \tag{3}$$

where dF is the $O(n)$-invariant Haar probability measure on $Gr_i(\mathbb{R}^n)$.

Remark 1.5. *One can show that the integral (3) absolutely converges for $Re(\alpha) > -1$ (see* [Al12, Lemma 2.1]*).*

It is well known that T_α has a meromorphic continuation in $\alpha \in \mathbb{C}$. For $\alpha_0 \in \mathbb{C}$ we will denote by S_{α_0} the first non-zero coefficient in the decomposition of the meromorphic function T_α near α_0, namely

$$T_\alpha = (\alpha - \alpha_0)^k \cdot (S_{\alpha_0} + O(\alpha - \alpha_0)) \text{ as } \alpha \to \alpha_0,\ S_{\alpha_0} \neq 0.$$

Thus

$$S_{\alpha_0}\colon C^\infty(Gr_i(V)) \to C^\infty(Gr_i(V))$$

is defined and non-zero for any $\alpha_0 \in \mathbb{C}$, and coincides with T_{α_0} for all but countably many α_0 (necessarily $Re(\alpha_0) \leq -1$ for exceptional values of α_0). S_α will also be called the α-cosine transform and sometimes denoted S^i_α. The geometric meaning of S_{α_0} for the exceptional values of α_0 is investigated in [AlGS15].

Obviously the α-cosine transform commutes with the natural action of the orthogonal group $O(n)$. It was observed in [AB04] for $\alpha = 1$ and in [Al12] for general α that the α-cosine transform can be rewritten in such a way to commute with an action of the much larger full linear group $GL(n, \mathbb{R})$. In that language it is a map

$$C^\infty(Gr_i(\mathbb{R}^n), L_\alpha) \to C^\infty(Gr_{n-i}(\mathbb{R}^n), M_\alpha), \tag{4}$$

where L_α, M_α are appropriately chosen $GL(n, \mathbb{R})$-equivariant line bundles over Grassmannians, and a choice of a Euclidean metric on $\mathbb{R}^n$ induces identification $Gr_{n-i}(\mathbb{R}^n) \simeq Gr_i(\mathbb{R}^n)$ by taking the orthogonal complement, and $O(n)$-equivariant identifications of L_α and M_α with trivial line bundles. In fact the α-cosine transform turned out to be a special case of the standard representation theoretical construction called intertwining integral. This opened the way to apply the infinite dimensional representation theory of the group $GL(n,\mathbb{R})$. In that language we have

$$C^\infty(Gr_i(\mathbb{R}^n), L_\alpha) \simeq \nu_i^{(n-i)/2} \times \nu_{n-i}^{-\alpha-i/2}, \quad C^\infty(Gr_{n-i}(\mathbb{R}^n), M_\alpha) \simeq \nu_{n-i}^{-\alpha-i/2} \times \nu_i^{(n-i)/2},$$

and S_α is the standard intertwining operator

$$S_\alpha^i : \nu_i^{(n-i)/2} \times \nu_{n-i}^{-\alpha-i/2} \to \nu_{n-i}^{-\alpha-i/2} \times \nu_i^{(n-i)/2}.$$

Thus Theorem 1.1 and Corollary 1.3 immediately imply the following corollary.

Corollary 1.6. *(1) S_α^i is not invertible if and only if there exists a non-negative integer k such that one of the following holds: $(a)\,\alpha = 2k$, $(b)\,\alpha = -n-2k$, $(c)\,r > 1$ and $\alpha = 1 - r + k$, $(d)\,r > 1$ and $\alpha = r - n - 1 - k$.*

(2) If S_α^i is not invertible then its range is the intersection of all closed G-invariant proper subspaces of $\nu_{n-i}^{-\alpha-i/2} \times \nu_i^{(n-i)/2}$.

(3) If S_α^i is invertible then its inverse is a scalar multiple of $S_{-n-\alpha}^{n-i}$.

1.3. **Related works.** In the archimedean cases Theorem 1.1 was proven earlier in [HL99], the non-archimedean case follows from [Zel80]. In fact [HL99] computes the length of $\chi_1 \times \chi_2$ in the archimedean cases. For $F = \mathbb{R}$ and $\chi = \nu^s$, where $|s| + n/2$ is not an integer, or is bigger than $n-1$ Theorem 1.2 was proven in [Al12]. For $F \in \{\mathbb{R}, \mathbb{C}\}$ and $p_1 \geq p_2$ a similar theorem was proven in [HL99], where the irreducible constituents are characterized by their K-types, as opposed to the rank (or Gelfand-Kirillov dimension). The assumption $p_1 \geq p_2$ can be removed using the Cartan involution $g \mapsto {}^t g^{-1}$. For $F = \mathbb{R}$, $n = 2r$ this theorem was proven in [BSS90].

1.4. **Acknowledgements.** The main ideas of this paper appeared in connection to the work [GS13b]. We thank Siddhartha Sahi for allowing us to use them in this work. We thank Semyon Alesker for explaining us the connections to integral geometry.

2. Preliminaries

In this section we recall some of the necessary background for the benefit of the reader.

2.1. **Degenerate principal series.** Let G be a reductive algebraic group over an arbitrary local field F. In this section we discuss some basic properties of the induced representation $I(P,\chi)$ on $C^\infty(G,P,\chi)$ as in (1). For detailed proofs we refer the reader to [BW80, Wall88] or to other standard texts on representation theory.

Let $\mathcal{E}'(G)$ denote the set of compactly supported distributions on G, regarded as a left and right G-module as usual via the pairing $\langle \cdot, \cdot \rangle : \mathcal{E}'(G) \times C^\infty(G) \to \mathbb{C}$.

Lemma 2.1. *Let $\varepsilon \in \mathcal{E}'(G)$ denote evaluation at $1 \in G$, then we have*

$$\left\langle R_{p^{-1}}\varepsilon, f \right\rangle = \chi(p)\, \Delta_P^{1/2}(p) \langle \varepsilon, f \rangle \text{ for all } p \in P, f \in C^\infty(G,P,\chi)$$

Proof. Indeed both sides are equal to $f(p)$. □

Lemma 2.2 ([Wall88, V.5.2.4]). *The representations $I(P,\chi)$ and $I\left(P,\chi^{-1}\right)$ are contragredient.*

Let $\bar{P}$ denote the parabolic subgroup opposite to P. Then the characters of P and $\bar{P}$ can be identified with those of the common Levi subgroup $L = P \cap \bar{P}$.

Proposition 2.3 ([KnSt71, Wald03]). *There is a nonzero intertwining operator* $I(P,\chi) \to I(\bar{P},\chi)$.

Lemma 2.4. *If K is a closed Lie subgroup (or an l-subgroup) of G such that $PK = G$ then the restriction $I(P,\pi)|_K$ is isomorphic to the induced representation $Ind_{P\cap K}^{K}(\pi|_{P\cap K})$. In particular, the space $I(P,\pi)^K$ of K-fixed vectors is isomorphic to $\pi^{P\cap K}$.*

The isomorphism is given just by restricting the functions to K. We will use this lemma for the maximal compact subgroup $K \subset G$ which in the non-Archimedean case is (up to conjugation) the subgroup $GL(n,\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of F. For $F = \mathbb{R}$, K is the orthogonal group $O(n)$ and for $F = \mathbb{C}$, $K = U(n)$.

In both Archimedean cases, $P \cap K$ is a symmetric subgroup of K and we will use the following well-known lemma.

Lemma 2.5 (see e.g. [Wol07, Theorem 12.2.4]). *If G is a connected and compact Lie group, M is a symmetric subgroup of G and χ is a character of M then $Ind_M^G(\chi)$ is a multiplicity-free representation.*

2.2. **Finite dimensional representations.** Let (ϕ, V) be an irreducible finite dimensional representation of a reductive group G. We are interested in the possibility of realizing ϕ as a submodule or quotient of some $I(P,\chi)$, which we denote by $\phi \hookrightarrow I(P,\chi)$ and $I(P,\chi) \twoheadrightarrow \phi$ respectively. Let (ϕ^*, V^*) be the contragredient representation of (ϕ, V).

From Lemmas 2.1 and 2.2 we obtain

Lemma 2.6. *We have $\phi \hookrightarrow I(P,\chi)$ (resp. $I(P,\chi) \twoheadrightarrow \phi$) if and only if $(V^*)^{P,\chi^{-1}\Delta_P^{-1/2}}$ (resp. $V^{P,\chi\Delta_P^{-1/2}}$) is nonzero.*

Corollary 2.7. *Let ψ be a character of G_n. Then* $\dim \mathrm{Hom}(\psi, \chi \times 1) = 1$ *if and only if $\psi = \nu^{-p_1/2}$ and $\chi = \nu^{-n/2}$. Otherwise* $\mathrm{Hom}(\psi, \chi \times 1) = 0$.

From highest weight theory we also obtain

Corollary 2.8. *$\chi \times 1$ has a finite-dimensional submodule if and only if either*

(i) $\chi = \nu^{-n/2}$.

(ii) $F = \mathbb{R}$, *and* $\chi = \varepsilon^k \nu^{-k-n/2}$ *for some non-negative integer k.*

(iii) $F = \mathbb{C}$ *and* $\chi = \alpha^{(l-k)/2}\nu^{-(k+l+n)/2}$ *for some non-negative integers k, l of the same parity.*

2.3. **Derivatives.** If χ is a character of G_p with $p > 0$, we write χ' for its restriction to G_{p-1}.

We will prove the main results by induction on n, using results from [BZ77, AGS15a]. We refer the reader to those papers for the notion of *depth* for an admissible representation of G_n, and for the definition of the functor Φ which maps admissible representations of G_n of depth ≤ 2 to admissible representations of G_{n-2}. In [BZ77] this functor is denoted Φ^-.

Proposition 2.9. *(*[BZ77, AGS15a]*)*

(1) Φ *is an exact functor and* $\Phi(\chi_1 \times \chi_2) = \chi_1' \times \chi_2'$.

(2) Every subquotient of $\chi_i \times \chi_j$ has depth ≤ 2.

(3) If π has depth 1 then π is finite dimensional and $\Phi(\pi) = 0$.

(4) If π has depth 2 then $\Phi(\pi) \neq 0$.

If $p_1 = 1$ (resp. $p_2 = 1$) then $\chi_1' \times \chi_2'$ means just χ_2' (resp. χ_1').

2.4. **Rank.** The rank of representations of $GL_n(F)$ for F of zero characteristic was defined in [GS13a] using the notion of wave-front set, which is a certain union of coadjoint nilpotent orbits. The rank is defined to be the maximal among the ranks of all the matrices in the wave-front set. For archimedean F the dimension of the wave-front set equals the Gelfand-Kirillov dimension.

By Jordan's theorem the nilpotent orbits in $\mathfrak{gl}_n$ are given by partitions of n, i.e. the sizes of Jordan blocks. By e.g. [BB89, Theorem 2] the wave-front set of $\chi_1 \times \chi_2$ is the closure of the nilpotent orbit given by the nilpotent orbit with r blocks of size 2 and $n-r$ blocks of size 1. We will shortly denote such a partition by $2^r 1^{n-r}$. Thus, $rank(\chi_1 \times \chi_2) = r$. The closure ordering on the orbits in the closure of $2^r 1^{n-r}$ is linear, and thus the wave-front set of any subquotient π of $\chi_1 \times \chi_2$ is the closure of the nilpotent orbit given by the partition $2^{rank(\pi)} 1^{n-rank(\pi)}$. The dimension of this orbit is $2rank(\pi)(n - rank(\pi))$.

Since in this paper we want to consider fields of arbitrary characteristic we give the following definition of rank

Definition 2.10. *For a subquotient π of $\chi_1 \times \chi_2$ we define the rank of π to be the number k such that $\Phi^k(\pi)$ is finite-dimensional and non-zero. In particular, if π is finite-dimensional then $rank(\pi) = 0$ and if $\pi = \chi_1 \times \chi_2$ then $rank(\pi) = \min(p_1, p_2)$.*

For F of zero characteristic this definition coincides with the one given in [GS13a]. This follows from [GS, Theorem 5.0.4] for Archimedean F and from [GS13a, Theorem B] for non-Archimedean F.

2.5. **Infinitesimal characters.** In this subsection we assume that F is Archimedean. We denote the Lie algebra of G by $\mathfrak{g}_0$ and its complexification by $\mathfrak{g}$, and we adopt similar terminology for subgroups of G and their Lie algebras. Let $\mathcal{Z} = \mathcal{Z}(\mathfrak{g})$ be the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ and let $\mathcal{Z}'$ be the set of its multiplicative characters. We say that $\pi \in Rep(G)$ has infinitesimal character $\xi \in Z'$ if for any $z \in \mathcal{Z}$, $\pi(z)$ acts by the scalar $\xi(z)$. If π is irreducible then it has an infinitesimal character.

Lemma 2.11. *If $\pi, \pi' \in Rep(G)$ have infinitesimal characters ξ, ξ', then $Hom_G(\pi, \pi') = 0$ unless $\xi = \xi'$.*

Fix a Cartan subalgebra and a Borel sublagebra $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. Let $W = W(\mathfrak{g}, \mathfrak{h})$ be the corresponding Weyl group and let ρ be the half-sum of roots of $\mathfrak{h}$ in $\mathfrak{b}$.

Theorem 2.12. *(Harish-Chandra) There is an isomorphism $\mu \mapsto \xi_\mu : \mathfrak{h}^*/W \to \mathcal{Z}'$.*

We often say that a representation has infinitesimal character μ instead of ξ_μ.

Lemma 2.13. *If ϕ is a finite dimensional representation with highest weight $\mu \in \mathfrak{h}^*$ then ϕ has infinitesimal character $\mu + \rho$.*

If $P = LN$ is a parabolic subgroup, then we may choose $\mathfrak{h}$ so that $\mathfrak{h} \subset \mathfrak{l}$; then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{l}$ as well.

Lemma 2.14. *If $\pi \in Rep(L)$ has infinitesimal character $\mu \in \mathfrak{h}^*$, then $I(P, \pi)$ has infinitesimal character μ also.*

3. Finite-dimensional subquotients of $\chi \times 1$

3.1. **Archimedean case.** Consider first the case $F = \mathbb{R}$. By the Harish-Chandra isomorphism, the characters of $\mathcal{Z}(G_n)$ can be identified with $\mathbb{C}^n/S_n$ where S_n is the symmetric group. Thus infinitesimal characters can be thought of as *multisets* of complex numbers.

Definition 3.1. *Let $\xi = \{z_1, \dots, z_n\}$ be a multiset of complex numbers with* $\operatorname{Re} z_i \geq \operatorname{Re} z_{i+1}$*. We say that ξ is a* segment *if $z_i - z_{i+1} = 1$ for all i, and a* generalized segment *if $z_i - z_{i+1}$ is an integer for all i. For $s \in \mathbb{C}$, $p \in \mathbb{N}$ we denote the segment of length p with center at s by ξ_p^s.*

It is well-known that the infinitesimal character of a character $\chi = \varepsilon^k \nu^s$ of $GL_p(\mathbb{R})$ is ξ_p^s, and that the infinitesimal character of an irreducible finite-dimensional representation is a generalized segment. From Lemma 2.14 we obtain that the infinitesimal character of $\chi \times 1$ is the disjoint (multiset) union $\xi_{p_1}^0 \sqcup \xi_{p_2}^s$.

For a representation of $GL_n(\mathbb{C})$ the infinitesimal character is a pair of multisets (η, ζ), the infinitesimal character of $\chi = \alpha^k \nu^s$ is $(\xi_p^{s+k}, \xi_p^{s-k})$, and the infinitesimal character of $\chi \times 1$ is $(\xi_{p_1}^{s+k} \sqcup \xi_{p_2}^0, \xi_{p_2}^{s-k} \sqcup \xi_{p_2}^0)$.

Let us now consider the K-types of $\chi \times 1$, for $F \in \{\mathbb{R}, \mathbb{C}\}$. Here, K is the maximal compact subgroup of G_n given by $K = O(n)$ for $F = \mathbb{R}$ and $K = U(n)$ for $F = \mathbb{C}$. We denote by K^0 the connected component of K, thus $K^0 = SO(n)$ for $F = \mathbb{R}$ and $K^0 = K$ for $F = \mathbb{C}$.

Lemma 3.2. *(i) If $\chi \times 1$ has the trivial K-type then χ is a power of ν.*
(ii) The restriction of $\chi \times 1$ to K^0 is multiplicity free
(iii) Suppose $F = \mathbb{R}$, $\chi = \varepsilon^k \nu^s$ and let $\psi = \varepsilon^l \nu^t$ be a character of G_{p_1}, such that $\chi \times 1$ and $\psi \times 1$ have a common K-type. Then $k = l$ (mod 2).

Proof. Let $P := P_{p_1,p_2}, M := K^0 \cap P$ and M^0 be the connected component of M. Consider the involution on K^0 given by conjugation by the block-diagonal matrix $\operatorname{diag}(Id_{p_1}, -Id_{p_2})$. The fixed points group of this involution is M. Thus both M and M^0 are symmetric subgroups of K^0.

Let $\chi_K := (\chi \otimes 1)|_{P \cap K}$. By Lemma 2.4 we have $(\chi \times 1)|_K = Ind_{P \cap K}^K(\chi_K)$ and $(\chi \times 1)|_{K^0} = Ind_M^{K^0}(\chi_K|_M)$.

(i) follows now from Lemma 2.6 and (ii) from Lemma 2.5.

For (iii) note that if k and l have different parities then one of the representations $(\chi \times 1)|_{K^0}$ and $(\psi \times 1)|_{K^0}$ is $Ind_M^{K^0}(1)$ and the other is $Ind_M^{K^0}(\varepsilon \otimes 1)$. Thus, $(\chi \times 1)|_{K^0} \oplus (\psi \times 1)|_{K^0} = Ind_{M^0}^{K^0}(1)$ is multiplicity free by Lemma 2.5, and thus $\chi \times 1$ and $\psi \times 1$ cannot have a common K-type. □

Corollary 3.3. *$\chi \times 1$ has at most one irreducible finite-dimensional subquotient.*

Proof. Different irreducible finite-dimensional representations of the connected component G_n^0 of G_n have different infinitesimal characters. Thus only one irreducible finite-dimensional representation of G_n^0 can appear as a subquotient. It appears only once by Lemma 3.2(ii). □

3.2. **Non-archimedean case.** In this subsection we assume that F is non-archimedean. Thus every irreducible finite-dimensional representation of G_n is a character.

We need the next lemma, which follows from [BZ77, Theorem 2.9].

Lemma 3.4. *Let $B_n \subset G_n$ be the usual Borel subgroup, ϕ_1 and ϕ_2 be characters of B_n given by n-tuples X_1 and X_2 of characters of $F^\times$. Suppose that the principal series $I(B_n, \phi_i)$ have a common constituent. Then X_1 is a permutation of X_2.*

Corollary 3.5. *If $\chi \times 1$ has a one-dimensional subquotient ψ then ψ is unique and either $\chi = \nu^{-n/2}$, $\psi = \nu^{-p_2/2}$ and ψ is a submodule or $\chi = \nu^{n/2}$, $\psi = \nu^{p_1/2}$ and ψ is a quotient.*

Proof. Define a_1, a_2, a_3 to be characters of $F^\times$ such that $\chi = a_1 \circ \det$, $\psi = a_3 \circ \det$ and $a_2 = 1$. Let $p_3 := n$ and

$$Y_i := (a_i\nu^{-(p_i-1)/2}, a_i\nu^{-(p_1-3)/2}, \dots, a_i\nu^{(p_i-1)/2}), X_1 := Y_3 \text{ and } X_2 := (Y_1, Y_2).$$

Then $\chi \hookrightarrow I(B_{p_1}, Y_1)$, $1 \hookrightarrow I(B_{p_2}, Y_2)$ and $\psi \hookrightarrow I(B_{p_3}, Y_3)$. Thus $\chi \times 1 \hookrightarrow I(B_n, X_2)$ and thus ψ is a joint subquotient of $I(B_n, X_1)$ and $I(B_n, X_2)$. Thus X_1 is a permutation of X_2, which implies that either $a_3 = \nu^{-p_2/2}$ and $a_1 = \nu^{-n/2}$ or $a_3 = \nu^{p_2/2}$ and $a_1 = \nu^{n/2}$. In the first case we get $\chi = \nu^{-n/2}$, $\psi = \nu^{-p_2/2}$ and, by Corollary 2.7 ψ is a submodule, and in the second case $\chi = \nu^{n/2}$, $\psi = \nu^{p_1/2}$ and ψ is a quotient.

The uniqueness of ψ follows from the uniqueness of K-fixed vector (Lemma 2.4). □

4. Proof of the main results

Denote $\pi := \chi \times 1$, $r := \min(p_1, p_2)$.

Proof of Theorem 1.1. We prove the statement by induction on r. For the base of the induction, note that if $r = 1$ then $\Phi(\pi)$ is 1-dimensional and thus, by Proposition 2.9, π is reducible if and only if it has either a finite-dimensional submodule or a finite-dimensional quotient. By Corollary 2.8 and Lemma 2.2, this is equivalent to the conditions of the proposition.

Note that if one of the conditions holds for χ' and $n-2$ then it holds for χ and n. Conversely, if one of the conditions holds for χ (and n) then either it holds for χ' (and $n-2$) or $\pi = \chi \times 1$ has a finite-dimensional submodule or quotient.

Now suppose π is irreducible but one of the conditions holds. Since we have standard intertwining operators between π and $1 \times \chi$, this implies that π is a direct summand of $1 \times \chi$. Applying Φ we get that $\Phi(\pi) = \chi' \times 1$ is a direct summand of $1 \times \chi'$. By the induction hypothesis, $\chi' \times 1$ is reducible. Applying Φ again and again if needed, we can assume (Corollary 2.8 and Lemma 2.2) that $\chi' \times 1$ has a finite-dimensional submodule or a finite-dimensional quotient. Then Corollary 2.8 and Lemma 2.2 imply that $1 \times \chi'$ cannot have a finite-dimensional constituent from the same side, contradicting having $\chi' \times 1$ as a direct summand.

Now, suppose π is reducible. If it has more than one infinite-dimensional constituent then $\chi' \times 1$ is reducible and thus, by the induction hypothesis, one of the conditions holds. Otherwise, π has either a finite-dimensional submodule or a finite-dimensional quotient. Dualizing if needed we can assume that π has a submodule and then Corollary 2.8 implies that one of the conditions holds. □

For the proof of Theorem 1.2 we will need the following lemma.

Lemma 4.1. *If $\chi \times 1$ has a finite-dimensional irreducible subquotient ϕ, then either ϕ is a quotient and $s(\chi) > 0$ or ϕ is a submodule and $s(\chi) < 0$.*

Proof. For non-archimedean F this follows from Corollary 3.5, and thus we assume that F is archimedean and the infinitesimal character ξ is a generalized segment (if $F = \mathbb{R}$) or a pair of generalized segments (if $F = \mathbb{C}$). Assume (dualizing if needed) that $s(\chi) < 0$. If $F = \mathbb{C}$ then Theorem 1.1 and Corollary 2.8 imply that $\chi \times 1$ has a finite-dimensional submodule ϕ'. If $F = \mathbb{R}$ we let $\chi = \varepsilon^k \nu^s$ and note that there exists a character $\psi = \varepsilon^l \nu^t$ of G_{p_1} such that ϕ can be realized as a submodule of $\psi \times 1$. This implies that $s + n/2, t + n/2,$ and l have the same parity. By Lemma 3.2(iii) k also has the same parity, and thus, by Corollary 2.8, $\chi \times 1$ has a finite-dimensional submodule ϕ'. By Corollary 3.3 ϕ' must coincide with ϕ. □

Proof of Theorem 1.2. We can assume that $s(\chi) > 0$. Then it follows from the previous lemma that π has at most one irreducible finite-dimensional subquotient ϕ, which is necessary a quotient. Now, let $\tau_1, \ldots, \tau_k$ be the other irreducible constituents. Then $\Phi(\tau_1), \ldots, \Phi(\tau_k)$ are non-zero (not necessary irreducible) constituents of $\chi_1' \times \chi_2'$. Arguing by induction, we can assume that $rank(\Phi(\tau_1)) > \cdots > rank(\Phi(\tau_k))$ and thus $rank(\tau_1) > \cdots > rank(\tau_k) > 0 = rank(\phi)$. □

Lemma 4.2. *If F is non-Archimedean then $\chi \times 1$ has length 1 or 2.*

Proof. We prove by induction on r. The base case $r = 0$ is trivial.

If $\chi \neq \nu^{\pm n/2}$ then, by Corollary 3.5, $\chi \times 1$ has no finite-dimensional constituents. Applying the functor Φ and using Proposition 2.9 we obtain length($\chi \times 1$) $\leq$ length($\chi' \times 1$). The induction hypotheses implies that length($\chi' \times 1$) ≤ 2.

Now consider the case $\chi = \nu^{\pm n/2}$. Theorem 1.1 implies that $\chi' \times 1$ is irreducible and thus $\chi \times 1$ has a unique infinite-dimensional irreducible constituent. By Corollary 3.5 it also has a unique one-dimensional constituent and thus has length two. □

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