UNIQUENESS OF SHALIKA FUNCTIONALS (THE ARCHIMEDEAN CASE)

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Abstract. Let F be either $\mathbb{R}$ or $\mathbb{C}$. Let (π, V) be an irreducible admissible smooth Fréchet representation of $GL_{2n}(F)$. A Shalika functional $\phi : V \to \mathbb{C}$ is a continuous linear functional such that for any $g \in GL_n(F)$, $A \in \mathrm{Mat}_{n\times n}(F)$ and $v \in V$ we have

$$\phi\left[\pi\begin{pmatrix} g & A \\ 0 & g \end{pmatrix})v\right] = \exp(2\pi i \operatorname{Re}(\operatorname{Tr}(g^{-1}A)))\phi(v).$$

In this paper we prove that the space of Shalika functionals on V is at most one dimensional.

For non-Archimedean F (of characteristic zero) this theorem was proven in [JR96].

Contents

1. Introduction

Let F be either $\mathbb{R}$ or $\mathbb{C}$. Let (π, V) be an admissible smooth Fréchet representation of $GL_{2n}(F)$. We assume that V is the canonical completion of an irreducible Harish-Chandra $(\mathfrak{g}, K)$- module in the sense of Casselman-Wallach (see e.g. [Wal92], chapter 11). A **Shalika functional** $\phi : V \to \mathbb{C}$ is a continuous linear functional such that for any $g \in GL_n(F)$, $A \in Mat_{n\times n}(F)$ and $v \in V$ we have

$$\phi\left[\pi\begin{pmatrix} g & A \\ 0 & g \end{pmatrix} v\right] = \exp\left(2\pi i \operatorname{Re}(\operatorname{Tr}(g^{-1}A))\right)\phi(v).$$

In this paper we prove the following theorem.

Theorem 1.1. *Let (π, V) be an irreducible admissible smooth Fréchet representation of $GL_{2n}(F)$. Then the space of Shalika functionals on V is at most one dimensional.*

For non-Archimedean F (of characteristic zero) this theorem was proven in [JR96]. The proof in [JR96] is based on the fact that $(\mathrm{GL}_{2n}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_n(F))$ is a Gelfand pair, which was also proven in [JR96], and the method of [FJ93, Section 3] of integration of Shalika functionals.

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In the Archimedean case those two ingredients also exist. Namely, [FJ93, Section 3] is valid also in the Archimedean case, and the fact that $(\mathrm{GL}_{2n}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_n(F))$ is a Gelfand pair is proven in [AG08b].

The proof that we present here is similar to the proof in [JR96]. The main difierence is that we have to prove the continuity of a certain linear form.

1.1. Structure of the proof.

We construct a linear map from the space of Shalika functionals to the space of linear periods (linear functionals on V that are invariant by $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$) and prove that the map is injective. Hence the uniqueness of the linear periods implies uniqueness of the Shalika functionals. The uniqueness of linear periods, i.e. the fact that $(\mathrm{GL}_{2n}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_n(F))$ is a Gelfand pair, is proven in [AG08b].

1.2. Structure of the paper.

In §2 we fix notation and terminology. In §3 we describe a way of obtaining a linear period from a Shalika functional by integration, as in [FJ93, Section 3]. In §4 we investigate the properties of the obtained period. In §5 we explain how this implies the uniqueness of Shalika functionals.

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2. Preliminaries and notation

2.1. Notation.

- Henceforth we fix an Archimedean field F (i.e. F is $\mathbb{R}$ or $\mathbb{C}$).
- For a group G acting on a vector space V we denote by V^G the space of G-invariant vectors in V. For a character χ of G we denote by $V^{G,\chi}$ the space of (G,χ)-equivariant vectors in V.
- For a smooth real algebraic variety M we denote by $\mathcal{S}(M)$ the space of Schwartz functions on M, i.e. the space of smooth functions that are rapidly decreasing as well as all their derivatives. For precise definition see e.g. [AG08a].
- We fix a natural number n and denote $G := GL_{2n}(F)$.
- We fix a norm on G by
$$||g|| := \sum_{1\leq i,j\leq 2n} |g_{ij}|^2 + \sum_{1\leq i,j\leq 2n} |(g^{-1})_{ij}|^2.$$
- We denote
$$G_1 := \left\{ \begin{pmatrix} g & 0 \\ 0 & Id \end{pmatrix} \mid g \in GL_n(F) \right\} \subset G$$
- We denote by $\nu : GL_n(F) \to G_1$ the isomorphism defined by
$$\nu(g) := \begin{pmatrix} g & 0 \\ 0 & Id \end{pmatrix}.$$
Note that for any $X \in \mathrm{Mat}(n \times n, F)$, $d\nu(X) = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$.
- We denote
$$H := \left\{ \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \mid g \in GL_n(F) \right\} \subset G$$
- We denote
$$U := \left\{ \begin{pmatrix} Id & A \\ 0 & Id \end{pmatrix} \mid A \in Mat_{n\times n}(F) \right\} \subset G$$

- We denote by $\mu : \mathrm{Mat}(n \times n, F) \to U$ the isomorphism defined by
$$\mu(A) := \begin{pmatrix} Id & A \\ 0 & Id \end{pmatrix}.$$
Note that for any $X \in \mathrm{Mat}(n \times n, F)$, $d\mu(X) = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$.
- We denote by $\tau : U \to F$ the homomorphism given by
$$\tau(\mu(A)) := \mathrm{Tr}(A).$$
- We let ψ be the additive character of F defined by $\psi(x) := e^{2\pi \mathrm{i} \,\mathrm{Re}\, x}$. We define an homomorphism $\Psi : U \to F^\times$ by
$$\Psi := \psi \circ \tau.$$
We extend Ψ to an homomorphism $\Psi : HU \to F^\times$ trivial on H.
- We denote by K the standard maximal compact subgroup of G. Thus $K = O(2n)$ if $F = \mathbb{R}$ and $K = U(2n)$ if $F = \mathbb{C}$.

2.2. **Admissible representations.**
In this paper we consider admissible smooth Fréchet representations of G, i.e. smooth admissible representations (π, V) of G such that V is a Fréchet space and, for any continuous semi-norm α on V, there exist another continuous semi-norm β on V and a natural number M such that for any $g \in G$,
$$\alpha(\pi(g)v) \leq \beta(v)||g||^M.$$

By Casselman - Wallach theorem (see e.g. [Wal92], chapter 11), V may be regarded as the canonical model of an irreducible Harish-Chandra $(\mathfrak{g}, K)-$module. By Casselman embedding theorem ([Cas80]), V can be realized as a closed subspace of a principal series representation. We denote by $\widetilde{V}$ the canonical model of the contragredient Harish-Chandra $(\mathfrak{g}, K)-$module. It is a subspace of the topological dual V^* of V.

3. Integration of Shalika functionals

In this section we fix:

- an irreducible admissible smooth Fréchet representation (π, V) of G
- a Shalika functional λ on V, i.e. $\lambda \in (V^*)^{HU,\Psi}$.

Theorem 3.1. *There exists $M \in \mathbb{R}$ such that for any $v \in v$ and over the region of $s \in \mathbb{C}$ with $\mathrm{Re}(s) > M$, the integral*
$$L_{\lambda,v}(s) := \int_{g \in G_1} \lambda(\pi(g)v) |\det(g)|^{s-\frac{1}{2}} dg$$
converges absolutely and is a holomorphic function of s.

Moreover, $L_{\lambda,v}(s)$ has meromorphic continuation to the complex plane and is a holomorphic multiple of the L-function L_π of the representation π. Finally, for any $\lambda \neq 0$ there exists $v \in V$ such that $L_{\lambda,v} = L_\pi$.

In [FJ93, Proposition 3.1] this theorem is proven under the following assumption:
(*) There exists a continuous semi-norm β on V such that $|\lambda(\pi(g)v)| \leq \beta(v)$ for any $g \in G$.
This may not be true in general. However, we have the following result.

Lemma 3.2. *There exist $M > 0$ and a continuous semi-norm β on V such that $|\lambda(\pi(g)v)| \leq |\det g|^{-M} \beta(v)$ for any $g \in G_1$.*

Before proving the Lemma, we check that, with the help of this Lemma, the proof of Theorem 3.1 is still valid. Indeed, the functions $g \mapsto \lambda(\pi(g)v)$ are bounded in [FJ93] and satisfy a sharper estimate ([FJ93, Lemma 3.1]). Here they satisfy the following estimates.

Lemma 3.3. *There is a continuous semi-norm γ on V such that, for any $v \in V$,*

$$|\lambda(\pi(g)v)| \leq |\det b^{-1}a|^{-M}\gamma(v)$$

for

$$g = u \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} k$$

with $a, b \in GL(n, F)$, $u \in U$, $k \in K$. Furthermore, for any $v \in V$, there is $\Phi_v \in \mathcal{S}(\mathrm{Mat}(n \times n, F))$ such that

$$|\lambda(\pi(g)v)| \leq \Phi(b^{-1}a)|\det b^{-1}a|^{-M},$$

for g of the above form.

Proof. For the first assertion, we have

$$\lambda(\pi(g)v) = \Psi(u)\lambda(\pi(\nu(b^{-1}a))\pi(k)v).$$

Hence

$$|\lambda(\pi(g)v)| \leq |\det b^{-1}a|^{-M}\beta(\pi(k)v).$$

There is another continuous semi-norm γ such that, for all $k \in K$,

$$\beta(\pi(k)v) \leq \gamma(v).$$

The first assertion follows.

For the second assertion, we go through the proof of [FJ93, Lemma 3.1] (which is the above estimate with $M = 0$) and arrive at once at the present estimate. □

The proof of Theorem 3.1 is still valid. The only modification is that we need to check that, under our weaker assumption, two integrals in [FJ93] which depend on $s \in \mathbb{C}$, are still absolutely convergent for $\mathrm{Re}\, s >> 0$.

The first integral is integral [FJ93, 45]:

$$\int \lambda(\pi(g)v)\Phi(g)|\det g|^{s+n-\frac{1}{2}} d^\times g$$

where $\Phi \in \mathcal{S}(\mathrm{Mat}(2n \times 2n, F))$. We write

$$g = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k.$$

Then

$$d^\times g = |\det a|^{-n} d^\times a d^\times b dx dk.$$

By Lemma 3.3, the integral of the absolute value is bounded by

$$\int |\det a|^{\mathrm{Re}\, s - M - \frac{1}{2}} |\det b|^{\mathrm{Re}\, s + M + n - \frac{1}{2}} \left| \Phi \left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k \right] \right| d^\times a d^\times b dx dk.$$

This does converge absolutely for $\mathrm{Re}\, s >> 0$.

The second integral is integral [FJ93, 48]. It has the form

$$\int \lambda \left[\pi \begin{pmatrix} a & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \pi(x)v \right] |\det a|^{s-\frac{1}{2}} d^\times a d\mu(x)$$

where μ is the measure on $SL(2n, F)$ defined by

$$\int f(x) d\mu(x) =$$

$$\int f \left[\begin{pmatrix} b^{-1} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & u \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & b \end{pmatrix} k \right] \Upsilon(u, b^{-1}, b; k)\ |\det b|^n d^\times b du dk$$

In this formula k is integrated over $K' = K \cap SL(2n, F)$ and the function Υ is in $\mathcal{S}(\mathrm{Mat}(n \times n, F)^3 \times K')$. The integral of the absolute value of the integrand is bounded by

$$\int \Phi_v(ab^{-2})|\det ab^{-2}|^{-M}\ |\Upsilon|(u, b^{-1}, b; k)|\det b|^n d^\times b du dk\ |\det a|^{\mathrm{Re}\, s - \frac{1}{2}} d^\times a.$$

After changing a to ab^2, the integral decomposes into a product:

$$\int \Phi_v(a)|\det a|^{\operatorname{Re} s - M - \frac{1}{2}} d^\times a \times \int |\Upsilon|(u, b^{-1}, b; k)|\det b|^{n+2\operatorname{Re} s-1} d^\times b du dk.$$

The first integral converges for $\operatorname{Re} s >> 0$. The second integral converges for all s.

It remains to prove Lemma 3.2. We will prove the following more general lemma.

Lemma 3.4. *There exists $M_0 > 0$ such that, for any polynomial P on the real vector space $\operatorname{Mat}(n\times n, F)$, there exists a continuous semi-norm β_P on V such that for any $g \in \operatorname{GL}_n(F)$ we have*

$$|\lambda\left(\pi(\nu(g))v\right)| \le \beta_P(v)\frac{1}{|P(g)|}|\det g|^{-M_0}.$$

Proof. We have

$$\lambda(\pi(\mu(X))v) = \psi(\operatorname{Tr} X)\lambda(v) \quad \forall X \in \operatorname{Mat}(n\times n, F).$$

We have then

$$\lambda(d\pi(d\mu(X))v) = 2\pi \mathrm{i} \operatorname{Re}\operatorname{Tr}(X)\lambda(v) \quad \forall X \in \operatorname{Mat}(n\times n, F)$$

and hence

$$\lambda(\pi(\nu(g))d\pi(d\mu(X))v) = 2\pi \mathrm{i} \operatorname{Re}\operatorname{Tr}(gX)\lambda(\pi(\nu(g))v) \quad \forall X \in \operatorname{Mat}(n\times n, F) \text{ and } g \in \operatorname{GL}_n(F).$$

Similarly, if Q is a polynomial on the real vector space $\operatorname{Mat}(n \times n, F)$, there is an element X_Q of the enveloping algebra of $\mathfrak{gl}_{2n}(F)$ such that

$$\lambda(\pi(\nu(g))d\pi(X_Q)v) = Q(g)\lambda(\pi(\nu(g))v) \quad \forall g \in \operatorname{GL}_n(F).$$

We know that there exist a continuous semi-norm β on V and a natural number M such that $|\lambda(\pi(g)v)| \le \beta(v)||g||^M$ for any $g \in G$. Therefore for any $g \in \operatorname{GL}_n(F)$ we have

$$|Q(g)\lambda(\pi(\nu(g))v)| = |\lambda(\pi(\nu(g))d\pi(\mu(X_Q))v)| \le \beta(d\pi(X_Q)v)||\nu(g)||^M.$$

Note that $||\nu(g)||^M = P_0(g)|\det g|^{-2M}$ for a suitable polynomial P_0 on the real vector space $\operatorname{Mat}(n\times n, F)$. Therefore, we have, with $M_0 = 2M$,

$$|\lambda(\pi(\nu(g))v)| \le \beta(d\pi(X_Q)v)\frac{P_0(g)}{|Q(g)|}|\det g|^{-M_0}.$$

We may take Q of the form $Q = P_0 P$ where P is another polynomial. Since $v \mapsto \beta(d\pi(X_Q)v)$ is a continuous semi-norm the Lemma follows. ☐

4. Properties of $L_{\lambda,v}$

Theorem 4.1. *Let (π, V) be an irreducible admissible smooth Fréchet representation of G. Fix a Shalika functional $\lambda \in (V^*)^{HU,\Psi}$ and a vector $v \in V$. Then, for any polynomial p, the product $p(s)L_{\lambda,v}(s)$ is bounded at infinity on every vertical strip of finite width.*

In [FJ93, §§3.3] the following statement is proven.

Lemma 4.2. *For $\operatorname{Re}(s)$ large enough, $L_{\lambda,v}(s)$ is a finite sum of functions of the type*

$$\mathcal{L}_{u,\xi,\Phi}(s) := \int_{g\in G} \Phi(g)\xi(\pi(g)u)|\det g|^{s+n-\frac{1}{2}} dg,$$

where $\Phi \in \mathcal{S}(\operatorname{Mat}(2n \times 2n, F))$, $u \in V$, $\xi \in V^*$.

Now Theorem 4.1 follows from the following one.

Theorem 4.3. *Let (π, V) be an irreducible admissible smooth Fréchet representation of G. Let $\Phi \in \mathcal{S}(\operatorname{Mat}(2n \times 2n, F))$, $u \in V$ and $\xi \in \widetilde{V}$. Then $\mathcal{L}_{u,\xi,\Phi}(s)$ has a meromorphic continuation to $\mathbb{C}$ whose product by any polynomial is bounded at infinity on any vertical strip. The continuation is a holomorphic multiple of $L_\pi(s) = L(s, \pi)$. It satisfies the functional equation*

$$\int \widehat{\Phi}(g)\xi(\pi({}^t g^{-1})u)|\det g|^{1-s+n-\frac{1}{2}} dg = \gamma(s, \pi, \psi)\mathcal{L}_{u,\xi,\Phi}(s)$$

where
$$\gamma(s,\pi,\psi):=\varepsilon(s,\pi,\psi)\frac{L(1-s,\widetilde{\pi})}{L(s,\pi)}.$$
and
$$\widehat{\Phi}(X):=\int_{\mathrm{Mat}(2n\times 2n,F)}\Phi(Y)\psi(\mathrm{tr}(XY^t))dY.$$
Finally, these assertions remain true if ξ is in V^* (topological dual of V).

This theorem is proven in [GJ72] in slightly narrower generality: the vectors u and ξ are $K-$finite and the function Φ is the product of a Gaussian function and a polynomial. For the convenience of the reader we indicate how to extend the results of [GJ72].

We will need the following lemma.

Lemma 4.4. *Let $T\subset G$ be the torus of diagonal matrices. We will also regard T as the subset $(F^\times)^{2n}$ of F^{2n}. Let $\chi:T\to\mathbb{C}^\times$ be a multiplicative character. Let (π,V) be the corresponding representation of principal series of G. Let $v\in V$ and $\xi\in\widetilde{V}$. Let Φ be a Schwartz function on* $\mathrm{Mat}(2n\times 2n,F)$.

Then there exists a Schwartz function $\phi\in\mathcal{S}(F^{2n})$ such that
$$\int_{g\in G}\Phi(g)\xi(\pi(g)v)|\det g|^{s+n-\frac{1}{2}}dg=\int_{t\in T}\phi(t)\chi(t)|\det t|^s dt$$
for any $s\in\mathbb{C}$ such that the integral on the right converges absolutely.

Proof. Let N denote the group of upper triangular matrices with unit diagonal. Let $B=TN$ and δ_B be the module of the group B. Realize V as the space of smooth functions on G that satisfy
$$f(tg)=\chi(t)f(g)\delta_B^{1/2}(t)\text{ and }f(ug)=f(g)\text{ for any }t\in T\text{ and }u\in N.$$
Realize also $\widetilde{V}$ in the corresponding way. Then
$$\xi(\pi(g)v)=\int_{k\in K}v(kg)\xi(k)dk\,,$$
where K is the standard maximal compact subgroup. Now
$$\int_G\Phi(g)\xi(\pi(g)v)|\det g|^{s+\frac{n-1}{2}}dg=\int_G\int_K\Phi(g)v(kg)\xi(k)|\det g|^{s\frac{n-1}{2}}dgdk=\int_G\int_K\Phi(k^{-1}g)v(g)\xi(k)|\det g|^{s+\frac{n-1}{2}}dgdk$$
To compute this integral we set
$$g=\begin{pmatrix}a_1 & u_{1,2} & \cdots & u_{1,2n}\\ 0 & a_2 & \cdots & u_{2,2n}\\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & a_{2n}\end{pmatrix}k'\,.$$
Then
$$dg=|a_1|^{1-2n}|a_2|^{2-2n}\cdots|a_{2n-1}|^{-1}\otimes d^\times a_i\otimes du_{i,j}dk'$$
We set
$$\phi(a_1,...,a_{2n}):=\int\Phi\left[k^{-1}\begin{pmatrix}a_1 & u_{1,2} & \cdots & u_{1,2n}\\ 0 & a_2 & \cdots & u_{2,2n}\\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & a_{2n}\end{pmatrix}k'\right]v(k')\xi(k)dkdk'\otimes du_{i,j}\,.$$
Clearly ϕ is a Schwartz function on F^{2n} and
$$\int_{g\in G}\Phi(g)\xi(\pi(g)v)|\det g|^{s+\frac{n-1}{2}}dg=\int_{t\in T}\phi(t)\chi(t)|\det t|^s dt$$
for any $s\in\mathbb{C}$ such that the integral on the right converges. □

Now we can prove Theorem 4.3.

Proof of Theorem 4.3. The representation (π, V) is a sub-representation of a principal series representation determined by a character χ of T and the representation $(\widetilde{\pi}, \widetilde{V})$ is then a quotient of the representation determined by χ^{-1}. For $u \in V$ and $\xi \in \widetilde{V}$ (or ξ in the principal series determined by χ^{-1}) we have

$$\mathcal{L}_{u,\xi,\Phi}(s) = \int_{(a_1,..,a_{2n}) \in F^{\times 2n}} \phi(a_1, a_2, ..., a_{2n}) \chi_1(a_1)|a_1|^s ... \chi_{2n}(a_{2n})|a_{2n}|^s d^\times a_1 ... d^\times a_{2n} \,.$$

The right hand side extends to a meromorphic function of s and the product of this function by any polynomial is bounded at infinity in any vertical strip. Moreover, the function ϕ depends continuously on $\Phi, u \in V, \xi \in \widetilde{V}$. Therefore the analytic continuation depends continuously on $\Phi, v \in V, \xi \in \widetilde{V}$. By continuity, it has the properties stated in the Theorem. To extend further to the case where ξ is in the topological dual V^* we appeal to the Dixmier Malliavin Lemma ([DM78]) applied to the representation of $SL_{2n}(F)$ on $\mathcal{S}(\mathrm{Mat}_{2n\times 2n}(F))$ defined by

$$g_1\Phi(X) := \Phi(g_1^{-1}X)$$

Thus we may assume Φ is of the form

$$\Phi(X) = \int_{SL_{2n}(F)} \Phi_1(g_1^{-1}X) f(g_1) dg_1$$

where f_1 is a C^∞ function of compact support on $SL_{2n}(F)$. Then

$$\mathcal{L}_{u,\xi,\Phi} = \mathcal{L}_{u,\xi_1,\Phi_1}$$

where

$$\xi_1(v) := \xi(\pi(f_1)v) \,.$$

Now ξ_1 is in $\widetilde{V}$ and our assertion follows.

□

Remark 4.5. *The previous result with $\xi \in V^*$ is used without comment in* [FJ93]*, formula (57). This is why we included a sketch of the proof.*

Theorem 4.6. *There exists $M > 0$ such that for any even integer $M' \geq 2$ and any polynomial p on $\mathbb{C}$, there exists a semi-norm β on V such that $|pL_{\lambda,v}|_{M'+M+\mathrm{i}\mathbb{R}}| \leq \beta(v)$.*

First, we will prove the following lemma.

Lemma 4.7. *There exists $M > 0$ such that, for any even integer $M' \geq 2$, there exists a continuous semi-norm β on V such that $|L_{\lambda,v}(s)| \leq \beta(v)$ for $\mathrm{Re}\, s = M' + M$.*

Proof. By Lemma 3.4 there exists $M_0 > 0$, and for any polynomial P on $\mathrm{Mat}(n \times n, F)$, a continuous semi-norm β_P such that

$$\lambda\left(\pi(\nu(g))v\right) \leq \beta_P(v) \frac{1}{|P(g)|} |\det g|^{-M_0}.$$

Let $M := 1/2 + n^2 + M_0$. Let $M' \geq 2$ be an even integer and let $s \in \mathbb{C}$ with $\mathrm{Re}(s) = M + M'$. Let

$$P(X) = |\det X|^{M'} \prod_{i,j} (1 + X_{ij}\overline{X}_{ij}).$$

Let

$$\beta(v) := \beta_P(v) \int_{X \in \mathrm{Mat}(n\times n,F)} \frac{dX}{\prod_{i,j}(1 + X_{ij}\overline{X}_{ij})}.$$

Now

$$|L_{\lambda,v}| = \left| \int_{GL_n(F)} \lambda(\pi(\nu(g))v) |\det g|^{s-1/2} dg \right| \leq \int_{GL_n(F)} \beta_P(v) \frac{1}{|P(g)|} |\det g|^{-M_0} |\det g|^{n^2+M'+M_0} dg =$$

$$\int_{GL_n(F)} \beta_P(v) \frac{1}{|P(g)|} |\det g|^{n^2+M'} dg = \int_{X \in \mathrm{Mat}(n\times n,F)} \beta_P(v) \frac{1}{|P(X)|} |\det X|^{M'} dX = \beta(v).$$

□

Proof of Theorem 4.6. For any $g \in GL_n(F)$ we have

$$L_{\lambda,\pi(\nu(g))v}(s) = |\det(g)|^{1/2-s} L_{\lambda,v}(s).$$

We can apply this to $g = \exp(tX)$, with $t \in \mathbb{R}$ and $X \in \mathrm{Mat}(n \times n, F)$. We get

$$L_{\lambda,\pi(\nu(g))v}(s) = |\det(\exp(tX))|^{\frac{1}{2}-s} L_{\lambda,v}(s).$$

Differentiating this identity with respect to t at $t = 0$, we get

$$L_{\lambda,d\pi(d\nu(X))v}(s) = (\frac{1}{2} - s)c(X)L_{\lambda,v}(s),$$

where $c(X) = \mathrm{Tr}\, X$ if $F = \mathbb{R}$ and $c(X) = 2\,\mathrm{Re}\,\mathrm{Tr}\, X$ if $F = \mathbb{C}$. Similarly, for any polynomial p on $\mathbb{C}$ there exists X_p in the universal enveloping algebra of $\mathfrak{gl}_{2n}(F)$ such that

$$L_{\lambda,d\pi(X_p)v}(s) = p(s)L_{\lambda,v}(s).$$

The theorem follows now from Lemma 4.7. □

Notation 4.8. *Define another representation π^θ on the same space V by $\pi^\theta(g) := \pi((g^t)^{-1})$. Recall that $\pi^\theta \cong \widetilde{\pi}$.*

For any Shalika functional $\lambda : \pi \to \mathbb{C}$ we define $\lambda^\theta : \pi^\theta \to \mathbb{C}$ by

$$\lambda^\theta(v) := \lambda\left(\pi\begin{pmatrix} 0_{nn} & Id_{nn} \\ -Id_{nn} & 0_{nn}\end{pmatrix} v\right).$$

It is easy to see that λ^θ is a Shalika functional for the rerpresentation π^θ.

Theorem 4.9 ([FJ93], Proposition 3.3)**.**

$$\gamma(s,\pi,\psi)L^{\pi}_{\lambda,v}(s) = L^{\pi^\theta}_{\lambda^\theta,v}(1-s).$$

Using Theorem 4.6 we obtain the following corollary.

Corollary 4.10. *There exists $N < 0$ such that for any odd integer $N' \leq -1$ and any polynomial p on $\mathbb{C}$ there exists a semi-norm β on V such that $|pL_{\lambda,v}|_{N'+N+i\mathbb{R}}| \leq \beta(v)$.*

5. Uniqueness of Shalika functionals

Theorem 5.1. *Let (π, V) an irreducible admissible representation of G. Let λ be a Shalika functional. Then the functional $L(\lambda) : V \to \mathbb{C}$ defined by*

$$L(\lambda)(v) := \frac{L_{\lambda,v}}{L_\pi}(\frac{1}{2})$$

is continuous.

Proof. By Theorem 4.6 we choose $M > 1$ such that for any polynomial p there exists a semi-norm β on V such that $|pL_{\lambda,v}|_{M+i\mathbb{R}}| \leq \beta(v)$. By Corollary 4.10 we choose $N < 0$ such for any polynomial p there exists a semi-norm β' on V such that $|pL_{\lambda,v}|_{N+i\mathbb{R}}| \leq \beta'(v)$. Let q be a polynomial such that the multiset of poles of $1/q$ (with multiplicities) coincides with the multiset of poles of $L_\pi|_{[N,M]+i\mathbb{R}}$. Here, $[N, M] + i\mathbb{R}$ denotes the strip $N \leq \mathrm{Re}(s) \leq M$. It is enough to show that the map $L'(\lambda)$ defined by $L'(\lambda)(v) := L_{\lambda,v}q(\frac{1}{2})$ is continuous. Now there exists a semi-norm α on V such that, for any $v \in V$,

$$|qL_{\lambda,v}|_{M+i\mathbb{R}}| \leq \alpha(v) \text{ and } |qL_{\lambda,v}|_{N+i\mathbb{R}}| \leq \alpha(v).$$

By Theorem 4.1, for any $v \in V$, there exists Δ such that $|qL_{\lambda,v}(s)| \leq \alpha(v)$ if $s \in [N, M] + i\mathbb{R}$ and $|\mathrm{Im}\, s| \geq \Delta$. Now by maximal modulus principle $L'(\lambda)(v) \leq \alpha(v)$ for any $v \in V$. □

Definition 5.2. *Let (π, V) an irreducible admissible representation of G. We define a map*

$$L : (V^*)^{HU,\Psi} \to (V^*)^{HG_1}$$

by

$$L(\lambda)(v) = \frac{L_{\lambda,v}}{L_\pi}(\frac{1}{2})$$

Theorem 3.1 implies

Proposition 5.3. *L is a monomorphism.*

Now we use the following theorem from [AG08b].

Theorem 5.4 (see [AG08b], Theorem I)**.** *The pair $(GL_{2n}, GL_n \times GL_n)$ is a Gelfand pair. Namely,*

$$\dim(V^*)^{\mathrm{GL}_n(F)\times\mathrm{GL}_n(F)} \leq 1.$$

Corollary 5.5. *Theorem 1.1 holds. Namely,*

$$\dim(V^*)^{HU,\Psi} \leq 1.$$

References

[AG08a] A. Aizenbud, D. Gourevitch, *Schwartz functions on Nash Manifolds,* International Mathematics Research Notices, Vol. 2008, n.5, Article ID rnm155, 37 pages. DOI: 10.1093/imrn/rnm155. See also arXiv:0704.2891 [math.AG].

[AG08b] A. Aizenbud, D. Gourevitch, *An Archimedean analog of Jacquet - Rallis theorem*, arXiv:0803.3397v3 [math.RT].

[Cas80] W. Casselman, *Jacquet modules for real reductive groups*, in Proceedings of the International Congress of Mathematicians (Helsinki, 1978), 557563, Acad. Sci. Fennica, Helsinki.

[CHM00] W. Casselman; H. Hecht; D. Miličić, *Bruhat filtrations and Whittaker vectors for real groups*, The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998), 151-190, Proc. Sympos. Pure Math., **68**, Amer. Math. Soc., Providence, RI, (2000).

[DM78] Jacques Dixmier and Paul Malliavin, *Factorisations de fonctions et de vecteurs indefiniment differentiables.* Bull. Sci. Math. (2) **102** (1978), no. 4, 307330. MR MR517765

[GJ72] R. Godement and H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Mathematics **260**, Berlin-Heidelberg-New York (1972).

[FJ93] Solomon Friedberg and Hervé Jacquet, *Linear periods.* J. Reine Angew. Math. **443** (1993), 91–139. MR MR1241129 (94k:11058)

[JR96] Hervé Jacquet and Stephen Rallis, *Uniqueness of linear periods*, Compositio Math. **102** (1996), no. 1, 65–123. MR MR1394521 (97k:22025)

[Wal88] N. Wallach, *Real Reductive groups I*, Pure and Applied Math. **132**, Academic Press, Boston, MA (1988).

[Wal92] N. Wallach, *Real Reductive groups II* , Pure and Applied Math. **132-II**, Academic Press, Boston, MA (1992).

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