

GENERALIZED WHITTAKER QUOTIENTS OF SCHWARTZ FUNCTIONS ON G -SPACES

DMITRY GOUREVITCH AND EITAN SAYAG

ABSTRACT. Let G be a reductive group over a local field F of characteristic zero, Archimedean or not. Let X be a G -space. In this paper we study the existence of generalized Whittaker quotients for the space of Schwartz functions on X , considered as a representation of G . We show that the set of nilpotent elements of the dual space to the Lie algebra such that the corresponding generalized Whittaker quotient does not vanish contains the nilpotent part of the image of the moment map, and lies in the closure of this image. This generalizes recent results of Prasad and Sakellaridis.

Applying our theorems to symmetric pairs (G, H) we show that there exists an infinite-dimensional H -distinguished representation of G if and only if the real reductive group corresponding to the pair (G, H) is non-compact. For quasi-split G we also extend to the Archimedean case the theorem of Prasad stating that there exists a generic H -distinguished representation of G if and only if the real reductive group corresponding to the pair (G, H) is quasi-split.

In the non-Archimedean case our result also gives bounds on the wave-front sets of distinguished representations.

1. INTRODUCTION

Let F be a local field of characteristic zero, Archimedean or not. Let \mathbf{G} be a reductive group defined over F , let $G := \mathbf{G}(F)$ be its F -points and \mathfrak{g} be the Lie algebra of G .

If F is non-Archimedean, we denote by $\mathcal{M}(G)$ the category of admissible smooth finitely-generated representations ([BZ76]). If F is Archimedean, we denote by $\mathcal{M}(G)$ the category of Casselman-Wallach representations, *i.e.* admissible smooth finitely-generated Fréchet representations of moderate growth (see [Wall92, §11], or [Cas89]). We denote by $\text{Irr}(G)$ the collection of irreducible representations in $\mathcal{M}(G)$.

A classical theme in representation theory of reductive groups over local fields is the study of representations *distinguished* with respect to a subgroup $H \subset G$. A representation $\pi \in \text{Irr}(G)$ is called H -distinguished if it has a non-zero H -equivariant linear functional (continuous if F is Archimedean). Denote by $\text{Irr}(G)_H \subset \text{Irr}(G)$ the subcollection of H -distinguished representations. Such are the local components of *distinguished* automorphic representations and such are the discrete series representations that contribute to the Plancherel decomposition of the Hilbert space $L^2(X)$, the space of square integrable functions (or sections of density bundle) on the G -space $X = G/H$.

Date: April 7, 2020.

2010 Mathematics Subject Classification. 20G05, 20G25, 22E35, 22E45, 14L30, 46F99.

Key words and phrases. Whittaker support, wave-front set, nilpotent orbit, moment map, reductive group actions, distinguished representations.

In this paper we offer unified approach to two seemingly unrelated questions concerning distinguished representations. The first is to clarify and generalize the relationship between distinction and genericity studied in [PrSa19] to arbitrary G -spaces over local fields, Archimedean or not. The second is a search for a non-Archimedean analogue of the qualitative study of the Plancherel decomposition provided by [HW17].

Regarding the first question, it was recently studied in the special case of linear periods in [SV20] both for Archimedean and non-Archimedean fields, and for symmetric pairs over non-Archimedean fields in [PrSa19]. It should be mentioned that the motivating example here is the disjointness result of [HR90] showing that irreducible generic representations of $GL(2n)$ never have non-zero symplectic invariant functionals (see [OS08] for a complete account).

As for the second, while a description of the wavefront set of the unitary representation $L^2(X)$ is of independent interest, it also yields a-priori bound on the wavefront of individual irreducible unitary distinguished representation that occurs in the Plancherel decomposition.

The unification of the topics mentioned above is obtained by studying the *Whittaker support* ([GGS]) of the non-admissible representation of G on spaces of Schwartz functions on X . Understanding Whittaker support allows, in the non-Archimedean case, to deduce exact information on wave front of individual distinguished representations leading to our answer to the first question. The study of $\text{WO}(\mathcal{S}(X))$, which is our smooth replacement to the problem studied in [HW17], is carried out using the theory of invariant distributions and in particular the *orbitwise* technique introduced by Gelfand-Kazhdan [GK75] in the non-Archimedean case (and its various extensions and ramifications), that in some cases reduces the study of invariant distributions on a space to the study of invariant distributions on each of the orbits separately.

In this language, the orbits that can support equivariant distributions are those for which a certain character is trivial on a stabilizer of one (hence any) point in the orbit. This can be reformulated as a condition tangling the point and the character as living in the image of a partial moment map attached to the G -space X and the orbit. Thus, as will also be clear from the formulation of our results, a key tool in our approach is the moment map attached to the G -space X . An unexpected aspect of this geometric approach is that it allows us to study the Whittaker supports of modules of the form $\mathcal{S}(X)$ even in cases where X is not G -homogenous.

We now turn to the results of the present paper. We need some notations. Let \mathbf{X} be a smooth \mathbf{G} -variety defined over F , and let $X := \mathbf{X}(F)$. Let $\mu : T^*X \rightarrow \mathfrak{g}^*$ denote the moment map, and let \mathcal{M} denote the closure of its image. Note that \mathcal{M} is a closed conical set. Let $\mathcal{N} \subset \mathfrak{g}^*$ denote the nilpotent cone. Let $\mathcal{S}(X)$ denote the space of Schwartz functions on X and let $\text{WO}(\mathcal{S}(X)) \subset \mathcal{N}$ denote the set of nilpotent elements φ such that $\mathcal{S}(X)$ has a generalized Whittaker quotient corresponding to φ (see §2.1 below). In §3 we prove the following theorem.

Theorem A (§3). (i) We have $\text{WO}(\mathcal{S}(X)) \subset \mathcal{M} \cap \mathcal{N}$.

(ii) If either F is non-Archimedean or \mathbf{X} is quasi-projective then $\text{Im } \mu \cap \mathcal{N} \subset \text{WO}(\mathcal{S}(X))$.

Let us now consider the homogeneous case, *i.e.* $X = G/H$. In this case we have $\text{Im } \mu = G \cdot \mathfrak{h}^\perp$. Here, \mathfrak{h}^\perp denotes the orthogonal complement to \mathfrak{h} in \mathfrak{g}^* , and $G \cdot \mathfrak{h}^\perp$ denotes the image of \mathfrak{h}^\perp under the coadjoint action of G on \mathfrak{g}^* . For homogeneous X we can formulate a twisted version of Theorem A.

Let $\zeta : H \rightarrow F$ and $\psi : H \rightarrow F^\times$ be algebraic characters. Let $d\zeta \in \mathfrak{h}^*$ denote the differential of ζ , and let $Fp_{\mathfrak{h}}^{-1}(d\zeta) \subset \mathfrak{g}^*$ denote the linear space spanned by the preimage of $d\zeta$ under the restriction map $p_{\mathfrak{h}} : \mathfrak{g}^* \rightarrow \mathfrak{h}^*$. Fix a character ξ_m of F^\times and a non-trivial unitary character ξ_a

of F , and let χ be the character of H given by the product

$$(1) \quad \chi = (\xi_a \circ \zeta) \cdot (\xi_m \circ \psi).$$

Let $\mathcal{S}(G)_{H,\chi}$ denote the space of χ -coinvariants in $\mathcal{S}(G)$ under the action of H by right multiplications (see (5) below).

Theorem B (§3). *Let H , χ , and $Fp_{\mathfrak{h}}^{-1}(d\zeta)$ be as above. Then*

$$G \cdot p_{\mathfrak{h}}^{-1}(d\zeta) \cap \mathcal{N} \subset \text{WO}(\mathcal{S}(G)_{H,\chi}) \subset \overline{G \cdot Fp_{\mathfrak{h}}^{-1}(d\zeta) \cap \mathcal{N}}.$$

By Frobenius reciprocity for small induction (see *e.g.* [BZ76, Proposition 2.29] and [GGS17, Lemma 2.3.4]), $\pi \in \text{Irr}(G)$ is (H, χ) -distinguished if and only if the contragredient representation $\tilde{\pi}$ is a quotient of $\mathcal{S}(G)_{H,\chi\Delta_H}$, where Δ_H denotes the modular function. Since Whittaker quotient is a right-exact functor, Theorem B implies that

$$(2) \quad \text{for any } \pi \in \text{Irr}(G)_{H,\chi}, \text{WO}(\pi) \subset \overline{G \cdot \mathfrak{h}^\perp}.$$

For the case $G = \text{GL}_{n+k}(F)$ and $H = \text{GL}_n(F) \times \text{GL}_k(F)$, this statement is equivalent to [SV20, Theorem B], which is proven for all local or finite fields F of characteristic different from 2.

The inclusion (2) is especially useful for non-Archimedean F , since for them the top nilpotent orbits in $\text{WO}(\pi)$ coincide with the top nilpotent orbits in the wave-front set $\overline{\text{WF}}(\pi)$ by [MW87]. In §4 below we recall this notion and deduce from Theorem B the following corollary.

Corollary C (§4). *Suppose that F is non-Archimedean. Let χ be a character of H that is trivial on its unipotent radical, and let $\pi \in \text{Irr}(G)_{H,\chi}$. Then $\overline{\text{WF}}(\pi) \subset \overline{G \cdot \mathfrak{h}^\perp}$.*

This corollary was our starting point for this paper. It was inspired by the paper [HW17] that shows that the wave-front set of $L^2(G/H)$ in the Archimedean case is $\overline{G \cdot \mathfrak{h}^\perp}$.

Another powerful Archimedean analogue of Corollary C is proven in [GS]. Namely, in [GS] we show that if H is a spherical subgroup of G then the Zariski closure of $\overline{\text{WF}}$ coincides with the Zariski closure of a nilpotent \mathbf{G} -orbit in $\mathfrak{g}^*(\mathbb{C})$ that intersects \mathfrak{h}^\perp . We conjecture that the same holds in the p -adic case. We refer the reader to [GS, §9] for more details on this conjecture and its generalization, as well as some evidence and applications.

Next we ask the following question: given $\varphi \in \text{WO}(\mathcal{S}(G)_{H,\chi})$, does there exist $\pi \in \text{Irr}(G)_{H,\chi}$ with $\varphi \in \text{WO}(\pi)$? We can answer this question for certain φ , and for (absolutely) spherical subgroups H , *i.e.* subgroups $H = \mathbf{H}(F)$ with \mathbf{H} acting on the flag variety of \mathbf{G} with an open orbit. Let P be an adapted parabolic for G/H - see §5 below for this notion. For any character χ of H , denote by χ'_P the character of $P \cap H$ given by

$$(3) \quad \chi'_P := \chi|_{P \cap H} \Delta_P^{-1/2} \Delta_H^{-1},$$

where Δ denotes the modular functions.

Proposition D (§6). *Let P be an adapted parabolic of G/H such that PH is open in G and let $\pi = \mathcal{S}(G)_{P,\Delta_P^{-1/2}}$ be the space of smooth vectors of the normalized induction $\text{Ind}_P^G 1$. Then*

- (i) π is irreducible and unitarizable.
- (ii) $\text{WO}(\pi) = G \cdot \mathfrak{p}^\perp$
- (iii) Assume H is spherical, and χ is a character of H . If χ is trivial on the unipotent radical of H and $\chi'_P = 1$ then $\pi \in \text{Irr}(G)_{H,\chi}$.

We note that if H is unimodular then $\Delta_P = \Delta_H = 1$, and $\chi'_P = \chi|_P$ (see Lemma 6.1 below).

Remark 1.1. *In case X is a unimodular spherical homogeneous space and if in addition $\mathcal{M} \cap \mathcal{N} \subset G \cdot \mathfrak{p}^\perp$ then Theorem A and Proposition D imply the exact equality*

$$(4) \quad \text{WO}(\mathcal{S}(X)) = \mathcal{M} \cap \mathcal{N} = G \cdot \mathfrak{p}^\perp$$

We note that if G is quasi-split then [Kno90] and [PrSa19, Appendix A] imply the weaker inclusion $\mathcal{M} \cap \mathcal{N} \subset G \cdot \mathfrak{p}^\perp$ - see Corollary 5.8 below.

Corollary E (§6). *Let $H \subset G$ be an algebraic subgroup, and let χ be a character of H , trivial on its unipotent radical. Let P be an adapted parabolic of G/H .*

- (i) *If $P = G$ then every $\pi \in \text{Irr}(G)_{(H,\chi)}$ is finite-dimensional.*
- (ii) *If $P \neq G$, PH is open in G , H is (absolutely) spherical and $\chi'_P = 1$ then there exists an infinite-dimensional unitarizable $\pi \in \text{Irr}(G)_{(H,\chi)}$.*
- (iii) *If P is a Borel subgroup of G (in particular G is quasi-split), and $\chi'_P = 1$ then there exists a generic unitarizable $\pi \in \text{Irr}(G)_{(H,\chi)}$.*
- (iv) *If P is not a Borel subgroup of G then no $\pi \in \text{Irr}(G)_{(H,\chi)}$ is generic.*

For quasi-split G and spherical H , the adapted parabolic P is a Borel subgroup if and only if the dimension of the flag variety equals $\dim H - \dim(H \cap Z)$, where Z is the center of G . This follows from the local structure theorem (Theorem 5.3 below). In this case H is called a minimal spherical subgroup.

Over Archimedean F , Parts (i) and (iv) follow from [KS14, Theorem 4.2]. Over non-Archimedean F , Part (iv) was essentially shown in [PrSa19]. The emphasis in [PrSa19] is on symmetric pairs, for which Prasad showed that some properties of distinguished representations are governed by the properties of the real reductive group given by the root system of the symmetric pair. In support of this principle we derive the following corollary.

Corollary F (§6). *Suppose that G is quasi-split and let $H \subset G$ be a symmetric subgroup. Let $G_{\mathbb{R}}^H$ be the real reductive group corresponding to the symmetric pair (G, H) . Then*

- (i) *There exists a generic $\pi \in \text{Irr}(G)_H$ if and only if $G_{\mathbb{R}}^H$ is quasi-split.*
- (ii) *There exists an infinite-dimensional $\pi \in \text{Irr}(G)_H$ if and only if $G_{\mathbb{R}}^H$ is not a quotient of the product of a compact group and a torus.*

Part (i) was proven in [PrSa19] for non-Archimedean local fields of arbitrary characteristic, as well as a certain analogue for finite fields.

In [GS, §6] we apply this corollary in order to answer [PrSa19, Question 1] in many cases.

1.1. Acknowledgements. We thank Avraham Aizenbud, Itay Glazer, and Michal Zydor for fruitful discussions. D. G. was partially supported by ERC StG 637912 and ISF 249/17.

2. PRELIMINARIES

2.1. Smooth representations and generalized Whittaker quotients.

Notation 2.1. *If F is Archimedean, we denote by $\text{Rep}^\infty(G)$ the category of smooth representations of G in complex vector spaces (see e.g. [BZ76]).*

If F is Archimedean, we denote by $\text{Rep}^\infty(G)$ the category of smooth Fréchet representations of G of moderate growth, as in [dC191, §1.4].

For any algebraic subgroup $H \subset G$ and $\pi \in \text{Rep}^\infty(G)$, denote by π_H the space of coinvariants, i.e. quotient of π by the intersection of kernels of all G -invariant functionals. Explicitly,

$$(5) \quad \pi_H = \pi / \overline{\{\pi(g)v - v \mid v \in \pi, g \in H\}}$$

where the closure is needed only for Archimedean F . In the latter case, for connected H we have $\pi_H = \pi / \overline{\mathfrak{h}_\mathbb{C} \pi}$ which in turn is equal to the quotient of $H_0(\mathfrak{h}, \pi)$ by the closure of zero, where \mathfrak{h} denotes the Lie algebra of H . For any character χ of H , denote $\pi_{H, \chi} := (\pi \otimes \chi)_H$.

Definition 2.2. Let $\varphi \in \mathfrak{g}^*$ be a nilpotent element, and let $\pi \in \text{Rep}^\infty(G)$. We define the generalized Whittaker quotient π_φ in the following way.

Choose an \mathfrak{sl}_2 -triple (e, h, f) such that φ is given by the Killing form pairing with f . Now, let \mathfrak{g}_1^h denote the eigenspace of the adjoint action of h on \mathfrak{g} corresponding to eigenvalue 1, and $\mathfrak{g}_{\geq 2}^h$ denote the sum of the eigenspaces with eigenvalues 2 and higher. Consider the symplectic form ω_φ on \mathfrak{g}_1^h given by $\omega_\varphi(X, Y) := \varphi([X, Y])$ and choose a Lagrangian \mathfrak{l} for this form. Let \mathfrak{n} be the nilpotent Lie algebra $\mathfrak{l} \oplus \mathfrak{g}_{\geq 2}^h$ and $N \subset G$ be the corresponding unipotent subgroup. Let η_φ denote the unitary character of N given by φ . Then we define $\pi_\varphi := \pi_{N, \eta_\varphi}$.

For more details about this definition, and the proof that it is independent of choices, we refer the reader to [GGS, §2.5].

Definition 2.3. Let $\pi \in \text{Rep}^\infty(G)$. Define $\text{WO}(\pi)$ to be the set of all nilpotent elements $\varphi \in \mathfrak{g}^*$ satisfying $\pi_\varphi \neq 0$.

2.2. Invariant distributions. Let X be the manifold of F -points of a smooth algebraic variety defined over F . Let $\mathcal{S}(X)$ denote the space of Schwartz functions on X . For non-Archimedean F , this means the space of locally-constant compactly supported functions, see [BZ76]. For Archimedean X , Schwartz functions are functions that decay rapidly together with all their derivatives, see *e.g.* [AG08] for the precise definition. Let $\mathcal{S}^*(X)$ denote the linear dual space if F is non-Archimedean, and continuous linear dual space if F is Archimedean. We refer to elements of $\mathcal{S}^*(X)$ as tempered distributions. For a group H acting on X and its character χ , we denote by $\mathcal{S}^*(X)^{H, \chi}$ the space of tempered distributions ξ satisfying $h\xi = \chi(h)\xi$. This space is dual to $\mathcal{S}(G)_{H, \chi}$.

Theorem 2.4. Let an F -algebraic group H act algebraically on X . Let χ be a character of H of the form (1). Suppose that the stabilizer H_x of any point $x \in X$ is unipotent.

- (i) If $\mathcal{S}^*(X)^{H, \chi} \neq 0$ then there exists $x \in X$ such that $\chi|_{H_x} = 1$.
- (ii) If there exists $x \in X$ such that $\chi|_{H_x} = 1$ and one of the following holds
 - (a) F is non-Archimedean
 - (b) H is solvable and X is quasi-projective
Then $\mathcal{S}^*(X)^{H, \chi} \neq 0$.

Proof. Since all algebraic characters of unipotent groups are trivial, we have $\Delta_H|_{H_x} = \Delta_x = 1$ for any $x \in X$. Now, (i) follows from [Ber83, §1.5] and [BZ76, §6] for non-Archimedean F , and from [AG13, Theorem 2.2.15 and the proof of Corollary 2.2.16] for Archimedean F .

For (ii), note that the condition $\chi|_{H_x} = 1$ implies that the orbit Hx of x has an H -invariant measure. The Archimedean case under the condition (iib) follows now from [GSS19, Theorem A]. Suppose now that F is non-Archimedean, and let \mathcal{O} be an orbit of minimal dimension among those that have an (H, χ) -equivariant measure. Since all the stabilizers are unipotent,

and thus Zariski connected, [HS17, Theorem 1.4] implies that $\mathcal{S}^*(\overline{\mathcal{O}})^{H,\chi} \neq 0$. This in turn implies $\mathcal{S}^*(X)^{H,\chi} \neq 0$. \square

To complement this theorem in the Archimedean case we will need the following one.

Theorem 2.5 (cf. [GSS19, Theorem B]). *Let G be a real reductive group, and let $H, N \subset G$ be real algebraic subgroups, such that N is unipotent. Let χ be a character of H as in (1), and let η be a unitary character of N . Assume that for some $g \in G$ we have*

$$\chi|_{H \cap N^g} = \eta^g|_{H \cap N^g},$$

where $N^g = gNg^{-1}$ and η^g is the character of N^g given by $\eta^g(gxg^{-1}) = \eta(x)$. Then

$$\mathcal{S}^*(G)^{N \times H, \eta \times \chi} \neq 0.$$

3. PROOF OF THEOREMS A AND B

Let $\varphi \in \mathfrak{g}^*$ be a nilpotent element. As in Definition 2.2, let (e, h, f) be an \mathfrak{sl}_2 -triple in \mathfrak{g} such that φ is given by the Killing form pairing with f . Let $\mathfrak{v} := \mathfrak{g}_{\geq 2}^h$, and let $\mathfrak{n} \supset \mathfrak{v}$ be as in Definition 2.2. Let $V := \text{Exp}(\mathfrak{v}) \subset N := \text{Exp}(\mathfrak{n})$ be the corresponding nilpotent subgroups of G . Let η_φ be the unitary character of N corresponding to φ .

Proof of Theorem A. Suppose that $\varphi \in \text{WO}(\mathcal{S}(X))$. Then by definition of $\text{WO}(\mathcal{S}(X))$, we have $\mathcal{S}^*(X)^{N, \eta_\varphi} \neq 0$ and thus $\mathcal{S}^*(X)^{V, \eta_\varphi} \neq 0$. By Theorem 2.4, this implies that there exists $x \in X$ such that φ vanishes on the Lie algebra \mathfrak{v}_x of the stabilizer of x in V . Thus, the restriction $\varphi|_{\mathfrak{v}}$ lies in the image of the moment map $\mu_V : T_x^*X \rightarrow \mathfrak{v}^*$ which is the dual of the map $\mathfrak{v} \rightarrow T_x X$ given by the differential of the action of V on X . But μ_V equals the composition of $\mu : T_x^*X \rightarrow \mathfrak{g}^*$ with the restriction $\mathfrak{g}^* \rightarrow \mathfrak{v}^*$:

$$(6) \quad T_x^*X \xrightarrow{\mu} \mathfrak{g}^* \twoheadrightarrow \mathfrak{v}^*$$

Thus there exists $\psi \in \mathcal{M}$ such that $\psi|_{\mathfrak{v}} = \varphi$. Since the kernel of the restriction to \mathfrak{v} is $(\mathfrak{g}^*)_{\geq -1}^h$, we have $\psi = \varphi + \psi'$, where $\psi' \in (\mathfrak{g}^*)_{\geq -1}^h$. Let $T \subset G$ be the one-dimensional torus with Lie algebra spanned by h , and let $t \in T$ with $|t| < 1$. Then the coadjoint action $Ad^*(t)$ of t is given on $(\mathfrak{g}^*)_{\geq -1}^h$ by t^i . Define a sequence $\psi_n \in \mathfrak{g}^*$ by $\psi_n := t^{2n} Ad^*(t^n)\psi$. Then ψ_n converges to φ . Since \mathcal{M} is conic and G -invariant, $\psi_n \in \mathcal{M}$. Since \mathcal{M} is closed, we have $\varphi \in \mathcal{M}$.

Conversely, if φ lies in the image of the moment map then φ vanishes on \mathfrak{g}_x for some $x \in X$ and thus η_φ is trivial on N_x . By Theorem 2.4 this implies $\mathcal{S}^*(X)^{N, \eta_\varphi} \neq 0$, which by definition means $\varphi \in \text{WO}(\mathcal{S}(X))$. \square

Let us now prove Theorem B in a similar way.

Proof of Theorem B. Suppose that $\varphi \in \text{WO}(\mathcal{S}(G)_{H,\chi})$. Then by definition we have $\mathcal{S}^*(G)^{N \times H, \eta_\varphi \times \chi} \neq 0$ and thus $\mathcal{S}^*(G)^{V \times H, \eta_\varphi \times \chi} \neq 0$. By Theorem 2.4, this implies that there exists $g \in G$ such that $\eta_\varphi \times \chi$ vanishes on the stabilizer in $V \times H$ of g . Replacing φ by $g^{-1} \cdot \varphi$ we can assume that g is the unit element. Then the stabilizer can be identified with $V \cap H$, and we have $\varphi|_{\mathfrak{v} \cap \mathfrak{h}} = d\zeta|_{\mathfrak{v} \cap \mathfrak{h}}$. Thus there exists $\psi \in \mathfrak{g}^*$ such that $\psi|_{\mathfrak{v}} = \varphi|_{\mathfrak{v}}$ and $\psi|_{\mathfrak{h}} = d\zeta$.

Since the kernel of the restriction to \mathfrak{v} is $(\mathfrak{g}^*)_{\geq -1}^h$, we have $\psi = \varphi + \psi'$, where $\psi' \in (\mathfrak{g}^*)_{\geq -1}^h$. Let $T \subset G$ be the one-dimensional torus with Lie algebra spanned by h , and let $t \in T$ with $|t| < 1$. Define a sequence $\psi_n := t^{2n} Ad^*(t^n)\psi \in \mathfrak{g}^*$. Then $\psi_n \rightarrow \varphi$ and thus $\varphi \in \overline{G \cdot Fp_{\mathfrak{h}}^{-1}(d\zeta)} \cap \mathcal{N}$.

Conversely, if $\varphi \in G \cdot p_{\mathfrak{h}}^{-1}(d\zeta) \cap \mathcal{N}$ then $\chi|_{H \cap N^g} = \eta_{\varphi}^g|_{H \cap N^g}$ for some $g \in G$. By Theorem 2.5 for Archimedean F , and Theorem 2.4 for non-Archimedean F , this implies $\mathcal{S}^*(G)^{N \times H, \eta_{\varphi} \times \chi} \neq 0$ and thus $\varphi \in \text{WO}(\mathcal{S}(G)_{H, \chi})$. \square

4. WAVE-FRONT SETS AND THE PROOF OF COROLLARY C

In this section only we assume that F is non-Archimedean. Let $\pi \in \text{Irr}(G)$. Let χ_{π} be the character of π . It is a generalized function on G and it defines a generalized function ξ_{π} on a neighborhood of zero in \mathfrak{g}_n , by restriction to a neighborhood of $1 \in G$ and applying logarithm. By [How74] and [HC77, p. 180], ξ_{π} is a combination of Fourier transforms of G -invariant measures of nilpotent coadjoint orbits. The measures extend to \mathfrak{g}^* by [RR72]. Denote by $\overline{WF}(\pi)$ the closure of the union of all the orbits that appear in the decomposition of ξ_{π} with non-zero coefficients.

Theorem 4.1 ([MW87, Proposition I.11, Theorem I.16 and Corollary I.17], and [Var14]).

For any $\pi \in \text{Irr}(G)$ we have $\overline{WF}(\pi) = \overline{\text{WO}(\pi)}$.

Proof of Corollary C. Since the Whittaker quotient is a space of coinvariants, the functor $\pi \mapsto \pi_{\varphi}$ is right-exact. If π is H -distinguished then $\tilde{\pi}$ is a quotient of $\mathcal{S}(G)_{H, \Delta_{H\chi}}$ and thus $\text{WO}(\tilde{\pi}) \subset \text{WO}(\mathcal{S}(G)_{H, \Delta_{H\chi}})$. Note that the image of the moment map of G/H is $G \cdot \mathfrak{h}^{\perp}$. Thus Theorem B implies $\text{WO}(\mathcal{S}(G)_{H, \Delta_{H\chi}}) \subset \overline{G \cdot \mathfrak{h}^{\perp}}$. By Theorem 4.1 we have $\overline{WF}(\tilde{\pi}) = \overline{\text{WO}(\tilde{\pi})}$. It is easy to see that $\overline{WF}(\tilde{\pi}) = -\overline{WF}(\pi)$. Altogether we obtain

$$\overline{WF}(\pi) = -\overline{WF}(\tilde{\pi}) = -\overline{\text{WO}(\tilde{\pi})} \subset -\overline{\text{WO}(\mathcal{S}(G)_{H, \Delta_{H\chi}})} \subset \overline{G \cdot \mathfrak{h}^{\perp}}$$

\square

5. PRELIMINARIES ON THE GEOMETRY OF X

In this section we will identify the algebraic groups and algebraic varieties with their points over the algebraic closure \overline{F} of F . We will view the F -points of the varieties as subsets invariant under the absolute Galois group $\text{Gal}(\overline{F}/F)$. As for the Lie algebra, we will use the notation \mathfrak{g} to denote its F -points (as in the rest of the paper), and $\mathfrak{g}(\overline{F})$ will denote the points over the algebraic closure. We start with several definitions and a theorem from [KnKr].

Definition 5.1. An F -group is called elementary if it is connected and all F -rational elements are semi-simple. An elementary radical is the subgroup generated by F -rational unipotent elements. It is the smallest normal subgroup with elementary quotient. If $\mathbf{H} = \mathbf{L}\mathbf{U}$ is a Levi decomposition of an F -group \mathbf{H} then $\mathbf{H}_{el} = \mathbf{L}_{el}\mathbf{U}$, where $\mathbf{L}_{el} \subset \mathbf{L}$ is the product of all non-anisotropic simple factors.

Note that a group over an algebraically closed field is elementary if and only if it is a torus.

Definition 5.2. Fix a minimal F -parabolic subgroup \mathbf{P}_0 of \mathbf{G} . The F -adapted parabolic \mathbf{P} of \mathbf{X} that includes \mathbf{P}_0 is defined by

$$(7) \quad \mathbf{P} := \{g \in \mathbf{G} \mid g\mathbf{P}_0x = \mathbf{P}_0x \text{ for } x \text{ in a dense open subset of } \mathbf{X}\}$$

We denote the F -points of \mathbf{P} by P . We will call P an adapted parabolic of X .

Most of the statements in [KnKr] deal with F -dense varieties. They are applicable to our case, since we assume \mathbf{X} to be irreducible, X to be non-empty, and F to be a local field. Under these assumptions, [Pop14, §1.A.2 and Proposition 2.6] implies that \mathbf{X} is F -dense, *i.e.* X is Zariski dense in \mathbf{X} .

Theorem 5.3 (Local structure theorem, [KnKr, Corollary 4.6]). *Let $\mathbf{P} = \mathbf{L}\mathbf{U}$ be a Levi decomposition of an adapted parabolic of \mathbf{X} . Then there exists a smooth affine \mathbf{L} -subvariety $\mathbf{X}_{el} \subset \mathbf{X}$ such that*

- (i) \mathbf{L}_{el} acts trivially on \mathbf{X}_{el} , all the \mathbf{L} -orbits on \mathbf{X}_{el} are closed, and the categorical quotient $\mathbf{X}_{el} \rightarrow \mathbf{X}_{el}/\mathbf{L}$ is a locally trivial bundle in etale topology.
- (ii) The natural morphism

$$(8) \quad \mathbf{U} \times \mathbf{X}_{el} = \mathbf{P} \times^{\mathbf{L}} \mathbf{X}_{el} \rightarrow \mathbf{X}$$

is an open embedding.

Recall that we denote by \mathcal{N} the nilpotent cone of \mathfrak{g}^* and by \mathcal{M} the closure of the image of the moment map of X .

Corollary 5.4. *If $P = G$ then $\mathcal{M} \cap \mathcal{N} = \{0\}$.*

It is easy to see from the definition, that any subgroup of an elementary group is reductive. Thus Theorem 5.3 implies the following corollary.

Corollary 5.5. *There exists an open dense P -invariant subset X^0 of X such that the stabilizer in P of any point in X^0 is reductive.*

Theorem 5.3 holds over any field of characteristic zero. It was first proven for algebraically closed fields in [BLV86]. We can therefore consider the adapted parabolic of \mathbf{X} over the algebraic closure \overline{F} and compare it to \mathbf{P} .

Proposition 5.6 ([KnKr, Proposition 9.1]). *Let $\mathbf{B} \subset \mathbf{P}_0$ be a Borel subgroup of \mathbf{G} , and let \mathbf{Q} be the adapted parabolic subgroup of \mathbf{X} that includes \mathbf{B} . Then $\mathbf{P} = \mathbf{P}_0\mathbf{Q}$.*

Let $\mathbf{Q}^- = \mathbf{M}\mathbf{U}^-$ be a parabolic subgroup of \mathbf{G} opposite to \mathbf{Q} (where $\mathbf{M} = \mathbf{Q} \cap \mathbf{Q}^-$), let $\mathbf{M}_{\mathfrak{s}}$ be the stabilizer in \mathbf{M} of a point in \mathbf{X}_{el} (where \mathbf{X}_{el} comes from Theorem 5.3 applied over \overline{F}), and let \mathbf{S} be the preimage of $\mathbf{M}_{\mathfrak{s}}$ under the projection $\mathbf{Q}^- \rightarrow \mathbf{M}$. Let us now consider the moment map over the algebraic closure $\mu : T^*\mathbf{X} \rightarrow \mathfrak{g}(\overline{F})^*$. Let \mathfrak{s} and \mathfrak{q}^- denote the Lie algebras of \mathbf{S} and \mathbf{Q}^- .

Theorem 5.7 ([PrSa19, Appendix A] using [Kno90, Satz 5.4]).

$$\overline{\mu(T^*\mathbf{X})} = \mathbf{G} \cdot \mathfrak{s}^{\perp}$$

Corollary 5.8. *We have*

$$\mathcal{M} \cap \mathcal{N} \subset \overline{\mu(T^*\mathbf{X})}(F) \cap \mathcal{N} = (\mathbf{G} \cdot (\mathfrak{q}^-)^{\perp})(F)$$

Proof. Since $\mathbf{M}/\mathbf{M}_{\mathfrak{s}}$ is elementary over \overline{F} , it is a torus. Thus the nilpotent elements of \mathfrak{s}^{\perp} lie in $(\mathfrak{q}^-)^{\perp}$. The corollary follows now from Theorem 5.7. \square

Note that the Killing form identifies $(\mathfrak{q}^-)^{\perp}$ with \mathfrak{u}^- .

6. PROOF OF PROPOSITION D AND COROLLARIES E AND F

Proof of Proposition D. (i) is a well-known consequence of Bruhat theory, see [KV96, §4] for the Archimedean case. The non-Archimedean case is proven analogously.

(ii) follows from Theorem B. Indeed, we take ζ to be trivial and $\xi_m \circ \psi$ to be $\Delta_P^{-1/2}$. Then $p_{\mathfrak{h}}^{-1}(d\zeta) = \mathfrak{p}^\perp$. Since $G \cdot \mathfrak{p}^\perp$ is closed and lies in \mathcal{N} , we obtain that

$$G \cdot p_{\mathfrak{h}}^{-1}(d\zeta) \cap \mathcal{N} = G \cdot \mathfrak{p}^\perp = \overline{G \cdot Fp_{\mathfrak{h}}^{-1}(d\zeta)} \cap \mathcal{N}.$$

By Theorem B we obtain $G \cdot \mathfrak{p}^\perp = \text{WO}(\mathcal{S}(G)_{P, \Delta_P^{-1/2}}) = \text{WO}(\pi)$.

For (iii) note that $P \cap H$ is the stabilizer of the coset $[1] \in G/H$, that lies in an open P -orbit. Thus, by Corollary 5.5, $P \cap H$ is reductive, and thus $\Delta_{P \cap H} = 1$.

(iii) follows now from [GSS19, Proposition D]. More precisely, [GSS19, Proposition D] is the case of trivial χ , but the general case is proven in the same way. \square

Proof of Corollary E. (i) Let $\pi \in \text{Irr}(G)_H$. Present G as a finite quotient of $Z \times K \times M$, where Z, K, M are reductive groups with Z commutative, K compact and M generated by its unipotent elements. Then $\pi \in \text{Irr}(Z \times K \times M)$, and we can assume $G = Z \times K \times M$. Since $P = G$, Theorem B and Corollary 5.4 imply that $\text{WO}(\tilde{\pi}) = \{0\}$. Thus [GGS, Theorem C] implies that M acts locally finitely on π . Now, Z acts on π by scalars by the Schur-Dixmier lemma, and π has a K -finite-vector. Since π is irreducible, we obtain that it is finite-dimensional.

(ii) follows from Proposition D.

(iii) Let $\pi := \text{Ind}_P^G 1$. As explained in the proof of Proposition D(i,ii), π is irreducible and generic. As before, $P \cap H$ is reductive and thus is unimodular. By [GSS19, Corollary C], $\pi \in \text{Irr}(G)_{H, \chi}$. Again, [GSS19, Corollary C] is the case of trivial χ , but the general case is proven in the same way.

(iv) If G is not quasi-split then it has no generic representations. If G is quasi-split then by Theorem B, Proposition 5.6 and Corollary 5.8 we have $\text{WO}(\tilde{\pi}) \subset \mathcal{M} \cap \mathcal{N} \subset \mathbf{G} \cdot \mathfrak{p}^\perp$. If P is not a Borel subgroup then \mathfrak{p}^\perp has no regular nilpotent elements, and thus neither has $\text{WO}(\tilde{\pi})$. \square

Lemma 6.1. *If PH is open in G and $\Delta_H = 1$ then $\Delta_P|_{P \cap H} = 1$.*

Proof. Since $\Delta_H = 1$, there exists a G -invariant measure on G/H . The restriction of this measure to $PH/H \cong P/(P \cap H)$ is P -invariant. Thus $\Delta_P|_{P \cap H} = \Delta_{P \cap H}$. But $\Delta_{P \cap H} = 1$. \square

Proof of Corollary F. Let θ be the involution of G such that $H = G^\theta$. Let P be an adapted parabolic for G/H such that PH is open in G . Since H is reductive, $\Delta_H = 1$. By Lemma 6.1, $\Delta_P|_{P \cap H} = 1$. Thus, by Corollary E, the existence of generic $\pi \in \text{Irr}(G)_H$ is equivalent to P being a Borel, and the non-existence of infinite-dimensional $\pi \in \text{Irr}(G)_H$ is equivalent to P being G . Now, $P = G$ if and only if G/H is a torus, which in turn is equivalent to $G_{\mathbb{R}}^H$ being a quotient of the product of a compact group and a torus. This proves (ii). To prove (i), we use the fact that P is a Borel if and only if there exists a θ -split Borel. The existence of a θ -split Borel is equivalent to $G_{\mathbb{R}}^H$ being quasi-split by [PrSa19]. \square

REFERENCES

- [AG08] A. Aizenbud, D. Gourevitch: *Schwartz functions on Nash Manifolds*, International Mathematics Research Notices, (2008): rnm155-37 DOI: 10.1093/imrn/rnm155. See also arXiv:0704.2891 [math.AG].

- [AG13] A. Aizenbud, D. Gourevitch: *Smooth Transfer of Kloostermann Integrals*, American Journal of Mathematics **135**, 143-182 (2013). See also arXiv:1001.2490[math.RT].
- [Ber83] J. Bernstein, *P-invariant Distributions on $GL(N)$ and the classification of unitary representations of $GL(N)$ (non-Archimedean case)*, Lie group representations, II (College Park, Md., 1982/1983), 50–102, Lecture Notes in Math., **1041**, Springer, Berlin (1984).
- [BLV86] M. Brion, D. Luna, and T. Vust: *Espaces homogènes sphériques*, Invent. Math. **84** (1986), 617-632.
- [BZ76] I.N. Bernstein, A.V. Zelevinsky: *Representations Of The Group $GL(N,F)$, Where F Is A Non-Archimedean Local Field*, Uspekhi Mat. Nauk 31:3 , 5-70 (1976).
- [BZ77] I.N. Bernstein, A.V. Zelevinsky: *Induced representations of reductive p -adic groups. I*. Ann. Sci. Ec. Norm. Super, 4^eserie **10**, 441-472 (1977).
- [Cas89] W. Casselman: *Canonical extensions of Harish-Chandra modules to representations of G* , Can. J. Math., Vol. XLI, No. 3, pp. 385-438 (1989).
- [dCl91] F. du Cloux: *Sur les représentations différentiables des groupes de Lie algébriques*, Ann. Sci. École Norm. Sup. (4) **24**, no. 3, 257–318 (1991).
- [GGs17] R. Gomez, D. Gourevitch, S. Sahi: *Generalized and degenerate Whittaker models*, Compositio Mathematica **153**, n. 2, pp. 223-256 (2017). DOI:10.1112/S0010437X16007788.
- [GGs] R.Gomez, D. Gourevitch, S. Sahi: *Whittaker supports for representations of reductive groups*, arXiv:1610.00284v4. To appear in Annales de l’Institut Fourier.
- [GK75] I.M. Gelfand, D. Kazhdan: *Representations of the group $GL(n, K)$ where K is a local field*, Lie groups and their representations (Proc. Summer School, Bolyai Janos Math. Soc., Budapest, 1971), pp. 95-118. Halsted, New York (1975).
- [GS] D. Gourevitch and E. Sayag: *Annihilator varieties of distinguished modules of reductive Lie algebras*, arXiv:2001.11746.
- [GSS19] D. Gourevitch, S. Sahi, E. Sayag: *Analytic continuation of equivariant distributions*, International Mathematics Research Notices, **23**, pp 7160-7192 (2019).
- [HS17] J. Hong, B. Sun: *Generalized semi-invariant distributions on p -adic spaces*, Math. Ann. **367**, no. 3-4, 1727-1776 (2017).
- [HW17] B. Harris, T. Weich: *Wave front sets of reductive Lie group representations III*, Adv. Math. **313**, 176-236 (2017).
- [HC77] Harish-Chandra: *The characters of reductive p -adic groups. Contributions to algebra (collection of papers dedicated to Ellis Kolchin)*, pp. 175-182. Academic Press, New York (1977).
- [How74] R. Howe: *The Fourier transform and germs of characters (case of GL_n over a p -adic field)* . Math. Ann 208, 305-322 (1974).
- [HR90] Michael J. Heumos, Stephen Rallis: *Symplectic-Whittaker models for GL_n* . Pacific J. Math., 146(2):247279, 1990.
- [Kno90] F.Knop: *Weylgruppe und Momentabbildung*, Invent. Math. **99**(1) 1-23 (1990).
- [KnKr] F.Knop, B. Kroetz: *Reductive group actions*, arXiv:1604.01005v2.
- [KS14] B. Kroetz, H.Schlichtkrull: *Multiplicity bounds and the subrepresentation theorem for real spherical spaces.*, Transactions of the AMS **368**, n. 4, pp 2749-2762 (2016).
- [KV96] J. A.C. Kolk and V.S. Varadarajan, *On the transverse symbol of vectorial distributions and some applications to harmonic analysis*, Indag. Mathem., N.S., 7 (1), pp 67-96 (1996).
- [MW87] C. Moeglin, J.L. Waldspurger: *Modeles de Whittaker degeneres pour des groupes p -adiques*, Math. Z. **196**, no. 3, pp 427-452 (1987).
- [OS08] O. Offen, E. Sayag: *Uniqueness and disjointness of Klyachko models*, J. Funct. Anal. 254, no. 11, 28462865 (2008)
- [Pop14] F. Pop: *Little survey on large fields - old & new*, Valuation theory in interaction, 432-463, EMS Ser. Congr. Rep.,Eur. Math. Soc., Zurich (2014).
- [PrSa19] D. Prasad, with an appendix by Y. Sakellaridis: *Generic representations for symmetric spaces*, Advances in Mathematics **348** 378-411 (2019).
- [RR72] R. Ranga Rao: *Orbital Integrals in Reductive Groups*, Annals of Mathematics, **96**, no. 3, 505-510 (1972).

- [SV20] E. Sayag and M.K. Verma: *On disjointness of linear models and degenerate Whittaker models*, Journal of Number Theory **207**, 56-82 (2020).
- [Var14] S. Varma: *On a result of Moeglin and Waldspurger in residual characteristic 2*, Math. Z. **277**, no. 3-4, 1027–1048 (2014).
- [Wall92] N. Wallach: *Real Reductive groups I, II*, Pure and Applied Math. **132**, Academic Press, Boston, MA (1988, 1992).

DMITRY GOUREVITCH, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, WEIZMANN INSTITUTE OF SCIENCE, POB 26, REHOVOT 76100, ISRAEL

E-mail address: `dimagur@weizmann.ac.il`

URL: `http://www.wisdom.weizmann.ac.il/~dimagur`

EITAN SAYAG, DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, P.O.B. 653, BE'ER SHEVA 84105, ISRAEL

E-mail address: `eitan.sayag@gmail.com`