ز-FINITE DISTRIBUTIONS ON p-ADIC GROUPS

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ABSTRACT. For a real reductive group G, the center $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ of the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G acts on the space of distributions on G. This action proved to be very useful (see e.g. [HC63, HC65, Sha74, Bar03]).

Over non-Archimedean local fields, one can replace the action of $\mathfrak{z}(\mathcal{U}(\mathfrak{g}))$ by the action of the Bernstein center $\mathfrak{z}$ of G, i.e. the center of the category of smooth representations. However, this action is not well studied. In this paper we provide some tools to work with this action and prove the following results.

- The wavefront set of any $\mathfrak{z}$-finite distribution ξ on G over any point $g \in G$ lies inside the nilpotent cone of $T_g^* G \cong \mathfrak{g}$.
- Let $H_1, H_2 \subset G$ be symmetric subgroups. Consider the space $\mathcal{J}$ of $H_1 \times H_2$-invariant distributions on G. We prove that the $\mathfrak{z}$-finite distributions in $\mathcal{J}$ form a dense subspace. In fact we prove this result in wider generality, where the groups H_i are spherical subgroups of certain type and the invariance condition is replaced by equivariance.

Further we apply those results to density and regularity of relative characters.

The first result can be viewed as a version of Howe's expansion of characters.

The second result can be viewed as a spherical space analog of a classical theorem on density of characters of finite length representations. It can also be viewed as a spectral version of Bernstein's localization principle.

In the Archimedean case, the first result is well-known and the second remains open.

CONTENTS

1. Introduction

Let $\mathbf{G}$ be a reductive group defined over a non-Archimedean local field F. Let $G :=$ $\mathbf{G}(F)$ be the corresponding l-group and let $\mathcal{S}(G)$ be the space of locally constant compactly supported functions on G. Let $\mathfrak{z} := \mathfrak{z}(G) := \mathrm{End}_{G \times G}(\mathcal{S}(G))$ denote the Bernstein center (see Section 2.2). The action of $\mathfrak{z}$ on $\mathcal{S}(G)$ gives rise to the dual action on the space of distributions $\mathcal{S}^*(G)$.

In this paper we study $\mathfrak{z}$-finite distributions, i.e. distributions ξ such that $\dim(\mathfrak{z} \cdot \xi) < \infty$.

1.1. **Wavefront set of $\mathfrak{z}$-finite distributions.** Our first result concerns the wavefront set of such distributions. For $x \in G$ let $WF_x(\xi)$ denote the intersection of the wavefront set of ξ with the cotangent space $T_x^* G$ (see Section 5.1).

In Section 5 we prove

Theorem A. *Suppose that F has characteristic zero. Let $\xi \in \mathcal{S}^*(G)$ be a $\mathfrak{z}$-finite distribution. Then for any $x \in G$ we have*

$$(1) \qquad\qquad\qquad\qquad WF_x(\xi) \subset \mathcal{N}$$

where $\mathcal{N} \subset \mathfrak{g}^*$ is the nilpotent cone, and we identify the Lie algebra $\mathfrak{g}$ with $T_x G$ using the right action[1].

Our main tool is the theory of special balls. This theory was developed for $\mathbf{G} = \mathrm{GL}_n$ in [Say02] (where these balls were called fuzzy balls), using some ideas from [How74, How77]. In Section 5.2 and appendix A we recall the relevant part of this theory and adapt it to general reductive groups.

Remark B. *We need the characteristic zero assumption since we use the exponentiation map in order to identify a neighborhood of zero in $\mathfrak{g}$ with a neighborhood of the unit element in G. For $G = \mathrm{GL}_n$ one can use the map $X \mapsto \mathrm{Id} + X$ (as in [Say02]) and drop the assumption on the characteristic. It is likely that for other classical groups one can use the Cayley map, and considerably weaken the assumption on the characteristic. The general case can be possibly treated using [AS09, Appendix A].*

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[1]Since $\mathcal{N}$ is invariant by conjugation it does not matter whether we use the right or the left action.

1.2. **Density of ʒ-finite distributions.** The next results of this paper depend on closed subgroups of G. We will require some conditions on these subgroups. We will describe those conditions in Definition H below. If a subgroup $H \subset G$ satisfies these conditions we will call the pair (G, H) *a pair of finite type.* Conjecturally, if F has characteristic zero then this holds for all spherical pairs. As explained below in Section 1.5.1, many cases of this conjecture follow from Appendix B and [AAG12, Theorem A], based on [SV, Theorem 5.1.5], and [Del10]. Those cases include all symmetric pairs of reductive groups.

Theorem C (see Section 4 below). *Let $H_1, H_2 \subset G$ be two (closed) subgroups and χ_i be characters of H_i. Consider the two-sided action of $H_1 \times H_2$ on G and let*

$$\mathcal{I} := \mathcal{S}^*(G)^{(H_1 \times H_2, \chi_1 \times \chi_2)}$$

be the space of $(H_1 \times H_2, \chi_1 \times \chi_2)$-equivariant distributions on G. Note that the Bernstein center $\mathfrak{z}$ acts on $\mathcal{I}$. Assume that the pairs (G, H_i) are of finite type.
    Then the space of $\mathfrak{z}$-finite distributions in $\mathcal{I}$ is dense in $\mathcal{I}$.

1.3. **Applications.** In this subsection we continue to work in the notation and assumptions of Theorem C.

Important examples of $\mathfrak{z}$-finite distributions in $\mathcal{I}$ are $(H_1 \times H_2, \chi_1 \times \chi_2)$-relative characters of finite length representations (see Definition 3.2.1). It turns out that those examples are exhaustive. Namely, we have the following proposition.

Proposition D (see Section 3 below). *Any $\mathfrak{z}$-finite distribution in $\mathcal{I}$ is an $(H_1 \times H_2, \chi_1 \times \chi_2)$-relative character of some finite length representation of G.*

Together with Theorem C it implies

Corollary E. *The space of $(H_1 \times H_2, \chi_1 \times \chi_2)$-relative characters of finite length representations of G is dense in $\mathcal{I}$.*

Theorem A provides a simple proof of the easy part of Harish-Chandra's regularity theorem [HC81, How77], namely the regularity of the character on the set of regular semisimple elements. In Section 5.6 we generalize this result to the realm of spherical pairs. For that, we introduce the notion of $H_1 \times H_2$-cotoric elements and prove the following result.

Corollary F. *Suppose that F has characteristic zero, and H_i are F-points of algebraic groups $\mathbf{H}_i \subset \mathbf{G}$. Let $\xi \in \mathcal{I}$ be a $\mathfrak{z}$-finite distribution. Then ξ is smooth in the neighborhood of any $H_1 \times H_2$-cotoric element.*

This results generalizes the main result of [RR96, §5], since if $H := H_1 = H_2$ is a symmetric subgroup then the H-regular semisimple elements are cotoric (see Lemma 5.6.2).

Combining Theorems A and C we obtain the following tool to study invariant distributions:

Corollary G. *The subspace of distributions in $\mathcal{I}$ whose wavefront set at any point is contained in the nilpotent cone in the dual Lie algebra $\mathfrak{g}^*$ is dense in $\mathcal{I}$.*

1.4. **Related results.** The germ at the unit element of the character of an irreducible representation of G can be presented as a linear combination of Fourier transforms of invariant measures of nilpotent orbits. This was shown in [How74] for $\mathbf{G} = \mathrm{GL}_n$ and in [HC78] for general $\mathbf{G}$. This cannot be naively generalized to the case of symmetric pairs, since the nilpotent orbital integrals are not defined for symmetric spaces in general. However, in [RR96, §7] it is shown that the germ at the unit element of a relative character is a Fourier transform of a distribution supported on the nilpotent cone.

Theorem A can be viewed as a version of these results, which gives less information but works in wider generality. Namely, it implies that the germ of any relative character of any finite length representation is a Fourier transform of a distribution supported near the nilpotent cone.

Distributions arising in representation theory are often $\mathfrak{z}$-finite. In the Archimedean case (where $\mathfrak{z}$ means the center of the universal enveloping algebra of the Lie algebra) this was widely exploited. For example it was used to prove the Harish-Chandra regularity theorem ([HC63, HC65]), uniqueness of Whittaker models ([Sha74]) and Kirillov's conjecture ([Bar03]). Recently, it was used in [JSZ11] to prove uniqueness of Ginzburg-Rallis models and in [AG15] to show non-vanishing of Bessel-like functions. However, in the non-Archimedean case there were no tools that use finiteness of distributions under the Bernstein center. This work provides such a tool.

A classical result (see [DKV84, §A.2] and [Kaz86, Appendix]) says that characters of finite length representations span a dense subspace of the space of conjugation-invariant distributions on G. One can view Corollary E as the relative counterpart of this result.

One can attempt to generalize Theorem C in the following direction. Let an l-group G act on an l-space X, and let $\mathcal{E}$ be a G-equivariant sheaf on X. Let a complex commutative algebra A act on $\mathcal{S}(X, \mathcal{E})$. Let $\mathcal{I} := \mathcal{S}^*(X, \mathcal{E})^G$ be the space of invariant distributional sections of $\mathcal{E}$. Assume that A preserves $\mathcal{I}$. Is the space of A-finite distributions in $\mathcal{I}$ dense in $\mathcal{I}$? Another important special case of this question is the case when $A = \mathcal{S}(Y)$ where Y is some l-space and the action of A on $\mathcal{S}(X)$ is given by a map from X to Y and G acts on the fibers of this map. In this case the positive answer is given by Bernstein's localization principle [Ber84, §1.4]. Thus, one can view Theorem C as a spectral version of Bernstein's localization principle.

The Archimedean analogs of Theorem C as well as of Bernstein's localization principle are not known in general.

1.5. **Tools developed in this paper.**

1.5.1. *Pairs of finite type.*

Definition H. *Let $H < G$ be a closed subgroup and χ be its character. We say that the pair (G, H) has* finite type *if for any character χ of H and any compact open subgroup $K < G$, the module $(\mathrm{ind}_H^G \chi)^K$ over the Hecke algebra $\mathcal{H}_K(G)$ is finitely generated. Here, $\mathrm{ind}_H^G \chi$ denotes the compact induction.*

In Appendix B we give the following criterion for pairs to be of finite type.

Theorem I (cf. Theorem B.0.2). *Let H be a closed subgroup of G. Let $\mathbf{P}$ be a minimal parabolic subgroup of $\mathbf{G}$ and $P = \mathbf{P}(F)$. Suppose that H has finitely many orbits on G/P. Suppose that for any irreducible smooth representation ρ of G and any character χ of H we have*

$$(2) \qquad\qquad \dim \mathrm{Hom}_H(\rho|_H, \chi) < \infty.$$

Then the pair (G, H) is of finite type.

Remark J.

(1) In fact, Theorem B.0.2 gives a more precise statement, which deduces finite generation of $(\mathrm{ind}_H^G \chi)^K$ from formula (2) for a specific character derived from χ. One can strengthen other results of this paper in a similar way. However, this will require longer bookkeeping that we chose to avoid.

(2) An incomplete version of Theorem B.0.2 appeared in [AAG12], see Remark B.0.3 for more details.

(3) The condition (2) is proven in [Del10] and [SV, Theorem 5.1.5] for many cases, including arbitrary symmetric pairs over a field with characteristic different from 2.

1.5.2. *Representations generated by ɜ-finite distributions.* In order to prove Proposition D we prove the following lemma:

Lemma K (see Section 3). *Let (G, H) be a pair of finite type. Let χ be a character of H. Consider the left action of H on G and let $\xi \in \mathcal{S}^*(G)^{(H,\chi)}$ be an (H, χ)-equivariant ɜ-finite distribution. Then both $\mathcal{S}(G) * \xi$ and $\xi * \mathcal{S}(G)$ are finite length representations of G.*

This lemma implies the following corollary:

Corollary L. *Let $\xi \in \mathcal{S}^*(G)$ be a ɜ-finite distribution. Then $\mathcal{S}(G) * \xi * \mathcal{S}(G)$ is a finite length representation of $G \times G$.*

1.5.3. *Fuzzy balls.* The theory of special balls was developed in [Say02] based on [How74, How77] for $\mathbf{G} = GL_n$. This theory implies that any irreducible representation is annihilated by a certain collection of elements of the Hecke algebra. In Section 5.2 and appendix A we adapt this statement to representations of a general reductive group (see Theorem 5.2.6).

1.5.4. *Relations between convolution and exponentiation.* The exponentiation maps an open neighborhood U of zero in the Lie algebra of G to G. This gives rise to a map of the algebra $\mathcal{S}(U)$ of smooth compactly supported functions on U (with respect to convolution) to the Hecke algebra of G. Unfortunately, this map is not a homomorphism. In Proposition 5.3.4 we show that it does behave as a homomorphism on certain pairs of functions.

1.6. **Idea of the proof.**

1.6.1. *Sketch of the proof of Theorem A.* We first analyze the representation generated by ξ under the two-sided action of the Hecke algebra $\mathcal{H}(G)$, which has finite length by Corollary L. Then we use the theory of special balls, that produces, for any finite length representation, a large collection of elements in the Hecke algebra $\mathcal{H}(G)$ that annihilate it. Those elements will also annihilate ξ. In other words, for certain $e_B \in \mathcal{H}(G)$ we have the following vanishing of convolutions

$$(3) \qquad\qquad e_B * \xi = 0$$

Next we want to linearize this information. For this we use the exponentiation map and Proposition 5.3.4. Unfortunately, Proposition 5.3.4 is not directly applicable to the pair (e_B, ξ). However, we use the vanishing (3) to construct other vanishing convolutions, to which Proposition 5.3.4 is applicable. Thus we get that certain convolutions on the Lie algebra vanish. Those vanishings imply the desired restriction on the wave front set.

1.6.2. *Sketch of the proof of Theorem C.* Let us assume for simplicity that χ_i are trivial and H_i are unimodular. To prove Theorem C we first note that $\mathcal{I}$ is dual to the space $\mathcal{S}(G)_{H_1 \times H_2}$ of $(H_1 \times H_2)$-coinvariants of $\mathcal{S}(G)$. We can decompose $\mathcal{S}(G)$ to a direct sum with respect to Bernstein blocks. This leads to a decomposition of $\mathcal{S}(G)_{H_1 \times H_2}$. The finite type assumption implies that each summand is finitely generated over ɜ. Thus Artin-Rees Lemma and Hilbert's Nullstellensatz imply that the space of ɜ-finite functionals on those summands is dense in the space of arbitrary functionals.

For technical reasons, it is more convenient to work with unions of Bernstein blocks which correspond to compact open subgroups of G than with individual Bernstein blocks.

1.7. **Future applications.** We believe that Corollary G can be used in order to prove the following analog of Harish-Chandra's density theorem [HC78, Theorem 3.1].

Conjecture M. *Suppose that G is quasisplit. Let B be a Borel subgroup of G, U be the nilradical of B, ψ be a non-degenerate character of U, $H \subset G$ be a reductive spherical subgroup and $X = G/H$. Let $\mathcal{O}$ be the union of all open B-orbits in X.*

Then the sum of the one-dimensional spaces $\mathcal{S}^*(Ux)^{U,\psi}$, where x ranges over $\mathcal{O}$, is dense in $\mathcal{S}^*(X)^{U,\psi}$.

In the subsequent paper [AGK14] we prove a non-Archimedean analog of [AG15], which we consider as a step towards this conjecture. Namely, we use Theorem A in order to prove that under certain conditions on H any $\mathfrak{z}$-finite distribution $\xi \in \mathcal{S}^*(X)^{U,\psi}$ which is supported in the complement to $\mathcal{O}$ vanishes.

In the subsequent work [AGM] we prove that the set of cotoric elements is open and dense in G if H_1, H_2 are spherical subgroups. By Corollary F this implies that $H_1 \times H_2$-relative characters are smooth almost everywhere. In fact, in [AGM] we show that the dimension of the variety

$$\mathfrak{S} = \{(g,\alpha) \in G \times \mathfrak{g}^* \mid \alpha \text{ is nilpotent}, \ \langle \alpha, \mathfrak{h}_1 \rangle = 0, \ \langle \alpha, Ad^*(g)(\mathfrak{h}_2) \rangle = 0\} \subset T^*G$$

equals the dimension of G. Theorem A implies that the wavefront set of any $H_1 \times H_2$-relative character lies in $\mathfrak{S}$. Thus we obtain a certain version of holonomicity for relative characters.

1.8. **Structure of the paper.** In Section 2 we give the necessary preliminaries on the Bernstein center.

In Section 3 we prove Lemma K and deduce Proposition D.

In Section 4 we prove Theorem C.

In Section 5 we prove Theorem A. In Sections 5.1 and 5.2 we give the necessary preliminaries on wavefront set and on special balls. In Section 5.3 we deduce Theorem A from two main ingredients, which we prove in Sections 5.4 and 5.5. In Section 5.4 we prove the vanishing (3). In Section 5.5 we prove Proposition 5.3.4 that states that exponentiation commutes with convolution in certain cases. In Section 5.6 we prove Corollary F and Lemma 5.6.2, which allows to specialize Corollary F to the symmetric pair case and thus obtain a generalization of [RR96, §5].

In Appendix A we prove the statements on special balls that were formulated without proof in Section 5.2.

In Appendix B we prove Theorem I.

2. Preliminaries

2.1. **Conventions.** The following conventions will be used throughout the paper.
- Fix a non-Archimedean local field F.
- All the algebraic groups and algebraic varieties that we consider are defined over F. We will use bold letters, e.g. $\mathbf{G}, \mathbf{X}$ to denote algebraic groups and varieties defined over F, and their non-bold versions to denote the F-points of these varieties, considered as l-spaces or F-analytic manifolds (in the sense of [Ser64]).
- We will use capital Latin letters to denote F-analytic groups and algebraic groups, and the corresponding Gothic letters to denote their Lie algebras.
- For an l-group H

- let $\mathcal{M}(H)$ denote the category of smooth representations of H.
- Let Δ_H denote the modular character of H, i.e. the quotient of the right Haar measure by the left one.
- If H acts on an l-space X and $x \in X$, we denote by H_x the stabilizer of x.
- If V is a representation of H we denote by V_H the space of coinvariants

$$V_H := V/(\mathrm{Span}\{v - hv \,|\, v \in V, \, h \in H\}).$$

- Fix a reductive group $\mathbf{G}$.
- For a sheaf $\mathcal{F}$ of $\mathbb{C}$-vector spaces on an l-space X we denote by $\mathcal{S}(X, \mathcal{F})$ the space of compactly-supported sections and by $\mathcal{S}^*(X, \mathcal{F})$ the dual space.
- For a compact open subgroup $K < G$ we denote by $\mathcal{H}_K(G)$ the corresponding Hecke algebra.

2.2. Bernstein center.
In this subsection we review the basics of the theory of the Bernstein center from [BD84].

Definition 2.2.1. *The* Bernstein center $\mathfrak{z} := \mathfrak{z}(G)$ *is the algebra of endomorphisms of the identity functor of the category $\mathcal{M}(G)$ of smooth representations of G.*

Definition 2.2.2. *Let $K < G$ be a compact open subgroup. For $V \in \mathcal{M}(G)$ denote by $V^{(K)}$ the subrepresentation generated by its K-fixed vectors. Denote also*

$$\mathcal{M}_K(G) := \{V \in \mathcal{M}(G) \,|\, V = V^{(K)}\}$$

and

$$\mathcal{M}_K(G)^\perp := \{V \in \mathcal{M}(G) \,|\, V^K = 0\}.$$

We have a functor $\mathcal{P}_K(V) := V^K$ from $\mathcal{M}(G)$ to the category $\mathcal{M}(\mathcal{H}_K(G))$ of modules over $\mathcal{H}_K(G)$. We call K a splitting subgroup if the category $\mathcal{M}(G)$ is the direct sum of the categories $\mathcal{M}_K(G)$ and $\mathcal{M}_K(G)^\perp$, and the functor $\mathcal{P}_K : \mathcal{M}_K(G) \to \mathcal{M}(\mathcal{H}_K(G))$ is an equivalence of categories.

Remark 2.2.3. *Recall that an abelian category $\mathcal{A}$ is a direct sum of two abelian subcategories $\mathcal{B}$ and $\mathcal{C}$, if every object of $\mathcal{A}$ is isomorphic to a direct sum of an object in $\mathcal{B}$ and an object in $\mathcal{C}$, and, furthermore, that there are no non-trivial morphisms between objects of $\mathcal{B}$ and $\mathcal{C}$.*

Theorem 2.2.4 ([BD84])**.**

(1) *The center of the algebra $\mathrm{End}_G(\mathcal{S}(G))$ of G-endomorphisms of $\mathcal{S}(G)$ is the algebra $\mathrm{End}_{G \times G}(\mathcal{S}(G))$ and the natural morphism from $\mathfrak{z}$ to this center is an isomorphism.*

(2) *The set of splitting subgroups defines a basis at 1 for the topology of G.*

(3) *For any splitting open compact subgroup $K \subset G$ we have*

    (a) *The center $\mathfrak{z}(\mathcal{H}_K(G))$ of the K-Hecke algebra is a direct summand of the Bernstein center $\mathfrak{z}$. In particular, the natural map $\mathfrak{z} \to \mathfrak{z}(\mathcal{H}_K(G))$ is onto.*

    (b) *The algebra $\mathcal{H}_K(G)$ is finitely generated as a module over its center $\mathfrak{z}(\mathcal{H}_K(G))$, and thus also over $\mathfrak{z}$.*

    (c) *The algebra $\mathfrak{z}(\mathcal{H}_K(G))$ is finitely generated over $\mathbb{C}$ and has no nilpotents.*

3. ʒ-FINITE DISTRIBUTIONS AND RELATIVE CHARACTERS

3.1. Finite length representations, ʒ-finite distributions and proof of Lemma K.
We start with several criteria for admissibility of smooth representations. For these criteria we will need the following definition.

Definition 3.1.1. *We say that a smooth representation π of G is*

- locally finitely generated *if for any compact open subgroup $K \subset G$ the module π^K is finitely generated over the Hecke algebra $\mathcal{H}_K(G)$,*
- $\mathfrak{z}$-finite *if there exists an ideal $I \subset \mathfrak{z}$ of finite codimension that acts by zero on π.*

Lemma 3.1.2. *Let $\pi \in \mathcal{M}(G)$ be a $\mathfrak{z}$-finite smooth representation. Assume that for any compact open subgroup $K \subset G$ the space π^K is finite-dimensional. Then π has finite length.*

Proof. It is enough to show that $\pi \subset \mathcal{M}_K(G)$ for some splitting subgroup $K \subset G$. Let $I \subset \mathfrak{z}$ be an ideal of finite codimension that acts by zero on π. For any splitting K denote by $i_K \subset \mathfrak{z}$ the idempotent that acts by identity on $\mathcal{M}_K(G)$ and by zero on $\mathcal{M}_K(G)^\perp$. Let j_K be the image of i_K in $\mathfrak{z}/I$. Since $\mathfrak{z}/I$ is finite-dimensional there exists a splitting K such that $j_K = j_{K'}$ for any splitting subgroup $K' \subset K$, thus $\pi^{K'} \subset \pi^{(K')} = \pi^{(K)}$ and thus, by Theorem 2.2.4(2), $\pi = \pi^{(K)}$. $\square$

Corollary 3.1.3. *Any $\mathfrak{z}$-finite locally finitely generated $\pi \in \mathcal{M}(G)$ has finite length.*

Proof. By Lemma 3.1.2 and Theorem 2.2.4(2) it is enough to show that π^K is finite-dimensional for any splitting subgroup $K \subset G$. This follows from Theorem 2.2.4(3a,3b). $\square$

Proposition 3.1.4. *Let $\pi \in \mathcal{M}(G)$ be locally finitely generated. Then*
  (i) any $\mathfrak{z}$-finite quotient ρ of π has finite length,
  (ii) any $\mathfrak{z}$-finite subrepresentation ρ of $\widetilde{\pi}$ is has finite length.

Proof. Part (i) follows from Corollary 3.1.3. To prove part (ii) denote by $\rho_\perp \subset \pi$ the joint kernel of all the functionals in ρ. Then $\rho \subset (\rho_\perp)^\perp \cong \widetilde{\pi/\rho_\perp}$. Since $\pi/\rho_\perp$ has finite length by part (i), we get that ρ has finite length. $\square$

Proof of Lemma K.
  (i) *Proof that $\mathcal{S}(G) * \xi$ has finite length.*
      Consider the natural epimorphism $\mathcal{S}(G) \twoheadrightarrow \mathcal{S}(G) * \xi$. It is easy to see that there exists a character χ' of H such that this epimorphism factors through $\operatorname{ind}_H^G(\chi')$. Since (G, H) has finite type, $\operatorname{ind}_H^G(\chi')$ is locally finitely generated and thus, by Proposition 3.1.4(i), $\mathcal{S}(G) * \xi$ has finite length.
  (ii) *Proof that $\xi * \mathcal{S}(G)$ has finite length.*
      Let G act on itself by $g \cdot x = xg^{-1}$. This gives rise to an action of G on $\mathcal{S}^*(G)^{(H,\chi)}$. Let $\mathcal{F}$ be the natural equivariant sheaf on $X = G/H$ such that $\mathcal{S}^*(G)^{(H,\chi)} \cong \mathcal{S}^*(X, \mathcal{F})$. Consider ξ as an element in $\mathcal{S}^*(X, \mathcal{F})$. Then

$$\xi * \mathcal{S}(G) \hookrightarrow \widetilde{\mathcal{S}(X, \mathcal{F})} = \widetilde{\operatorname{ind}_H^G(\chi'')}$$

for some character χ'' of H, and Proposition 3.1.4(ii) implies that $\xi * \mathcal{S}(G)$ has finite length.

$\square$

3.2. **Relative characters and proof of Proposition D.**
Let us recall the definition of relative character.

Definition 3.2.1. *Let (π, V) be a finite length representation of G. Let $(\widetilde{\pi}, \widetilde{V})$ be its smooth contragredient. Let $H_1, H_2 \subset G$ be subgroups and χ_1, χ_2 be their characters. Let $l_1 \in (V^*)^{H_1, \chi_1^{-1}}$ and $l_2 \in (\widetilde{V}^*)^{H_2, \chi_2^{-1}}$ be equivariant functionals. The* relative character *$\Xi_{l_1, l_2}^\pi \in \mathcal{H}(G)^*$ is the generalized function on G given by*

(4) $$\Xi_{l_1, l_2}^\pi(f) := \langle l_2, \pi^*(f) l_1 \rangle.$$

We refer to such relative characters as $(H_1 \times H_2, \chi_1 \times \chi_2)$-relative characters of π.

Since we can identify $\mathcal{I}$ with the space $(\mathcal{H}(G)^*)^{H_1 \times H_2, \chi_1 \times \chi_2}$ of invariant generalized functions, we can consider the relative character as an element in $\mathcal{I}$.

Lemma 3.2.2 (see Appendix B.2). *Let (G, H) be a pair of finite type. Let V be a finite length representation of G and χ be a character of H. Then $\dim V_{(H,\chi)} < \infty$.*

Proof of Proposition D. Let $\xi \in \mathcal{I}$. Consider the pullback of ξ to $G \times G$ under the multiplication map. This gives us a G-invariant bilinear form B on $\mathcal{H}(G)$. Let $\mathcal{L}$ be its left kernel and $\mathcal{R}$ be its right kernel, $M := \mathcal{L} \backslash \mathcal{H}(G)$ and $N := \mathcal{H}(G)/\mathcal{R}$. We consider the right G-module M as a left one using the anti-involution $g \mapsto g^{-1}$. We get a non-degenerate pairing between M and N. Lemma K implies that M and N have finite length and thus $M = \tilde{N}$. We can consider the form B as an element in $(M_{H_1,\chi_1} \otimes N_{H_2,\chi_2})^*$. Since the pairs (G, H_i) are of finite type, Lemma 3.2.2 implies that M_{H_1,χ_1} and N_{H_2,χ_2} are finite-dimensional and thus

$$(M_{H_1,\chi_1} \otimes N_{H_2,\chi_2})^* \cong (M_{H_1,\chi_1})^* \otimes (N_{H_2,\chi_2})^* \cong (M^*)^{H_1,\chi_1^{-1}} \otimes (N^*)^{H_2,\chi_2^{-1}}.$$

Therefore B defines an element in $(M^*)^{H_1,\chi_1^{-1}} \otimes (N^*)^{H_2,\chi_2^{-1}}$ which can be written in the form $B = \sum_{i=1}^{k} l_1^i \otimes l_2^i$. Let

$$l_1 := (l_1^1, \ldots, l_1^k) \in ((M^k)^*)^{H_1,\chi_1^{-1}}, \; l_2 := (l_2^1, \ldots, l_2^k) \in ((N^k)^*)^{H_2,\chi_2^{-1}}.$$

It is easy to see that

$$\xi = \sum_{i=1}^{k} \Xi_{l_1^i, l_2^i}^M = \Xi_{l_1, l_2}^{M^k}.$$

$\square$

4. Density of ʒ-finite distributions

For the proof of Theorem C we will need the following lemma.

Lemma 4.0.1. *Let $H < G$ be a closed subgroup and χ be a character of H. Then there exists a character χ' of H such that for any $V \in \mathcal{M}(G)$ and any splitting subgroup $K \subset G$ (see Definition 2.2.2) we have*

$$(V^{(K)})_{(H,\chi)} \cong (\mathrm{ind}_H^G \chi')^K \otimes_{\mathcal{H}_K(G)} V^K.$$

Here we consider the left $\mathcal{H}_K(G)$-module $(\mathrm{ind}_H^G \chi')^K$ as a right one using the anti-involution $g \mapsto g^{-1}$.

Proof. First note that $V^{(K)} \cong \mathcal{H}(G) \otimes_{\mathcal{H}_K(G)} V^K$, where the action of G is given by the left action on $\mathcal{H}(G)$. Let H act on G from the left and G act on itself by $g \cdot x = xg^{-1}$. This gives an action of G on $\mathcal{H}(G)_{(H,\chi)}$. It is easy to see that we have an isomorphism $\mathcal{H}(G)_{(H,\chi)} \cong (\mathrm{ind}_H^G \chi')$ for some character χ' of H. Now

$$(V^{(K)})_{(H,\chi)} \cong \mathcal{H}(G)_{(H,\chi)} \otimes_{\mathcal{H}_K(G)} V^K \cong \mathrm{ind}_H^G \chi' \otimes_{\mathcal{H}_K(G)} V^K \cong (\mathrm{ind}_H^G \chi')^K \otimes_{\mathcal{H}_K(G)} V^K.$$

$\square$

Lemma 4.0.2. *Let A be a unital commutative algebra finitely generated over $\mathbb{C}$. Let M be a finitely generated A-module, and M^* denote the space of all $\mathbb{C}$-linear functionals on M. Then the space of A-finite vectors in M^* is dense in M^*.*

Proof. It is enough to show that the intersection of the kernels of all A-finite functionals on M is zero. Let v be an element of this intersection, $\mathfrak{m} \subset A$ be any maximal ideal and i be any integer. Then $M/\mathfrak{m}^i M$ is finite-dimensional over $\mathbb{C}$ and thus any functional on it defines an A-finite functional on M. Such a functional vanishes on v, and thus the image of v in $M/\mathfrak{m}^i M$ is zero. We conclude that v belongs to the space $\bigcap_{\mathfrak{m}} \bigcap_i (\mathfrak{m}^i M)$, which is zero by the Artin-Rees lemma. $\square$

Proof of Theorem C. Denote $X_i := G/H_i$. For some line bundle $\mathcal{F}_1$ on X_1 we have

$$\mathcal{I} \cong \mathcal{S}^*(X_1, \mathcal{F}_1)^{(H_2, \chi_2)} \cong (\mathcal{S}(X_1, \mathcal{F}_1)_{(H_2, \chi_2^{-1})})^*.$$

Thus it is enough to show that for any $f \in \mathcal{S}(X_1, \mathcal{F}_1)_{(H_2, \chi_2^{-1})}$ such that $\langle \xi, f \rangle = 0$ for any ʒ-finite distribution $\xi \in \mathcal{I}$, we have $f = 0$. Let f be like that. Let $K < G$ be a

splitting open compact subgroup that fixes a representative of f in $\mathcal{S}(X_1, \mathcal{F}_1)$. Note that $V := \mathcal{S}(X_1, \mathcal{F}_1) = \mathrm{ind}_{H_1}^G \chi_1'$ for some character χ_1' of H_1 . Since K is a splitting subgroup, $V^{(K)}$ is a direct summand of V as a G-representation. Hence $M := (V^{(K)})_{(H_2, \chi_2^{-1})}$ is a direct summand of $V_{(H_2, \chi_2^{-1})}$ as a $\mathfrak{z}$-module which contains f. Therefore it is enough to show that the space of $\mathfrak{z}$-finite vectors in M^* (which by Theorem 2.2.4(3a) equals the space of $\mathfrak{z}(\mathcal{H}_K(G))$-finite vectors in M^*) is dense in M^*. By Lemma 4.0.1, there exists a character χ_2' of H_2 such that

$$M = (\mathrm{ind}_{H_2}^G \chi_2')^K \otimes_{\mathcal{H}_K(G)} V^K = (\mathrm{ind}_{H_2}^G \chi_2')^K \otimes_{\mathcal{H}_K} (\mathrm{ind}_{H_1}^G \chi_1')^K,$$

where we consider the left $\mathcal{H}_K(G)$-module $(\mathrm{ind}_{H_2}^G \chi_2')^K$ as a right one using the anti-involution $g \mapsto g^{-1}$. The assumption implies that $(\mathrm{ind}_{H_i}^G \chi_i')^K$ are finitely generated over $\mathcal{H}_K(G)$. By Theorem 2.2.4(3b) this implies that they are also finitely generated over $\mathfrak{z}(\mathcal{H}_K(G))$. Thus M is also finitely generated over $\mathfrak{z}(\mathcal{H}_K(G))$. The assertion follows now from Lemma 4.0.2 in view of Theorem 2.2.4(3c). $\qquad\square$

5. Wavefront set of $\mathfrak{z}$-finite distributions and the proof of Theorem A

In this section we assume that F has characteristic zero.

5.1. Preliminaries on wave front set.
In this section we give an overview of the theory of the wave front set as developed by D. Heifetz [Hei85], following L. Hörmander (see [Hör90, §8]). For simplicity we ignore here the difference between distributions and generalized functions.

Definition 5.1.1.
  (1) *Let V be a finite-dimensional vector space over F. Let $f \in C^\infty(V^*)$ and $w_0 \in V^*$. We say that f vanishes asymptotically in the direction of w_0 if there exists $\rho \in \mathcal{S}(V^*)$ with $\rho(w_0) \neq 0$ such that the function $\phi \in C^\infty(V^* \times F)$ defined by $\phi(w, \lambda) := f(\lambda w) \cdot \rho(w)$ is compactly supported.*
  (2) *Let $U \subset V$ be an open set and $\xi \in \mathcal{S}^*(U)$. Let $x_0 \in U$ and $w_0 \in V^*$. We say that ξ is smooth at (x_0, w_0) if there exists a compactly supported non-negative function $\rho \in \mathcal{S}(V)$ with $\rho(x_0) \neq 0$ such that the Fourier transform $\mathcal{F}^*(\rho \cdot \xi)$ vanishes asymptotically in the direction of w_0.*
  (3) *The complement in T^*U of the set of smooth pairs (x_0, w_0) of ξ is called the* wave front set *of ξ and denoted by $WF(\xi)$.*
  (4) *For a point $x \in U$ we denote $WF_x(\xi) := WF(\xi) \cap T_x^*U$.*

Remark 5.1.2. *Heifetz defined $WF_\Lambda(\xi)$ for any open subgroup Λ of $F^\times$ of finite index. Our definition above is slightly different from the definition in [Hei85]. They relate by*

$$WF(\xi) - (U \times \{0\}) = WF_{F^\times}(\xi).$$

Proposition 5.1.3 (see [Hör90, Theorem 8.2.4] and [Hei85, Theorem 2.8])**.** *Let $U \subset F^m$ and $V \subset F^n$ be open subsets. Suppose that $f : U \to V$ is an analytic submersion[2]. Then for any $\xi \in \mathcal{S}^*(V)$, we have*

$$WF(f^*(\xi)) \subset f^*(WF(\xi)) := \left\{ (x, v) \in T^*U \,\middle|\, \exists w \in WF_{f(x)}(\xi) \text{ s.t. } d_{f(x)}^* f(w) = v \right\}.$$

Corollary 5.1.4. *Let $V, U \subset F^n$ be open subsets. Let $f : V \to U$ be an analytic isomorphism. Then for any $\xi \in \mathcal{S}^*(V)$ we have $WF(f^*(\xi)) = f^*(WF(\xi))$.*

Corollary 5.1.5. *Let X be an F-analytic manifold[3]. We can define the wave front set of any distribution in $\mathcal{S}^*(X)$, as a subset of the cotangent bundle T^*X.*

We will need the following standard properties of the wavefront set.

---

[2]*i.e. the differential of f is surjective.*

[3]In the classical sense of [Ser64] and not in the sense of rigid geometry or Berkovich geometry.

Lemma 5.1.6. *Let X be an F-analytic manifold and $\xi \in \mathcal{S}^*(X)$ be a distribution on it.*

    (1) Let $x \in X$. Assume that $WF_x(\xi) = \{0\}$. Then ξ is smooth at x, i.e. there exists an analytic embedding $\phi : U \hookrightarrow X$ from an open neighborhood U of the origin in F^n to X such that $\phi(0) = x$ and $\phi^*(\xi)$ coincides with a Haar measure.

    (2) [Aiz13, Theorem 4.1.5] Let an F-analytic group H act analytically on X. Suppose that ξ changes under the action of H by some character of H. Then

$$WF(\xi) \subset \{(x, v) \in T^*X \,|\, v(\mathfrak{h}x) = 0\},$$

where $\mathfrak{h}x$ denotes the image of the differential of the action map $h \mapsto hx$.

5.2. **Preliminaries on special balls.** The notions of special balls and admissible balls were defined in [Say02] (under the name *fuzzy balls*) for $\mathbf{G} = GL_n$. Here we generalize them to arbitrary reductive groups, using the standard theory of exponentiation.

Notation 5.2.1. *Let $\mathcal{O}$ be the ring of integers in F. Fix a uniformizer $\varpi \in \mathcal{O}$ and denote $q := |\varpi|^{-1}$.*

We start with the following standard lemma on exponentiation.

Lemma 5.2.2. *There exists a lattice (i.e. a free finitely-generated $\mathcal{O}$-submodule of full rank) $\mathcal{L} \subset \mathfrak{g}$, a compact open subgroup $K \subset G$ and an analytic diffeomorphism $\exp : \mathcal{L} \to K$ such that*

    (1) For any $x \in \mathcal{L}$, $\exp|_{\mathcal{O}\cdot x}$ is a group homomorphism.

    (2) $\frac{d}{dt}\exp(tx)|_{t=0} = x$.

    (3) For any $X \in \varpi^m \mathcal{L}, Y \in \varpi^n \mathcal{L}$ we have

$$\exp^{-1}(\exp(X)\exp(Y)) - X - Y \in \varpi^{m+n}\mathcal{L}.$$

For completeness we will indicate the proof of this lemma in Appendix A.1.

Remark 5.2.3. *The conditions* (1) *and* (2) *define the map* $\exp$ *uniquely.*

We fix such an $\mathcal{L}$. Fix also an additive character ψ of F that is trivial on $\mathcal{O}$ and non-trivial on $\varpi^{-1}\mathcal{O}$.

Definition 5.2.4.

- *For a vector space V over F and a lattice $\Lambda \subset V$ denote $\Lambda^\perp := \{y \in V^* \,|\, \forall x \in \Lambda, \langle x, y \rangle \in \mathcal{O}\} \subset V^*$.*
- *For a set $B = a + \Lambda \subset \mathfrak{g}^*$ define a subset $K_B := exp(\Lambda^\perp) \subset G$. Define also a function η_B of K_B by $\eta_B(exp(x)) = \psi(\langle a, x \rangle)$. Note that K_B and η_B depend only on the set B and not on its presentation as $a + \Lambda$.*
- *An* admissible ball *is a set $B \subset \mathfrak{g}^*$ of the form $a + \Lambda$, where $\Lambda \supset \mathcal{L}^\perp$ is a lattice such that K_B is a group and η_B is its character.*

    Define $e_B \in \mathcal{H}(G)$ to be the measure $\eta_B e_{K_B}$, where e_{K_B} is the normalized Haar measure on K_B.

- *An admissible ball B is called* nilpotent *if it intersects the nilpotent cone $\mathcal{N} \subset \mathfrak{g}^*$.*
- *For an element $x \in \mathfrak{g}^*$ we define $|x| := \min\{|\alpha| \,|\, x \in \alpha\mathcal{L}^\perp, \alpha \in F\}$.*
- *A* special ball *of radius $r \geq 1$ is a set $B \subset \mathfrak{g}^*$ of the form $c + \alpha\mathcal{L}^\perp$, where $\alpha \in F, c \in \mathfrak{g}^*, |\alpha| = r$ and either $|c| = r^2$ or $|c\varpi| = r^2$. It is easy to see that any special ball is an admissible ball.*
- *For $Y \in \mathfrak{g}^*$ we denote by $B(Y)$ the unique special ball containing Y (see Lemma A.2.1).*
- *Denote the set of all special balls by $\mathfrak{F}$.*
- *$\mathcal{L}_n := \varpi^n \mathcal{L}, K_n := \exp(\mathcal{L}_n)$ for $n \geq 0$.*

In Appendix A we give more details about admissible and special balls and prove the following fundamental statements.

Theorem 5.2.5. *Let (π, V) be a smooth representation. Then $\{\pi(e_B)\}_{B \in \mathfrak{F}}$ form a full family of mutually orthogonal projectors, i.e.*

(1) for any $B, C \in \mathfrak{F}$ we have

$$e_B e_C = \begin{cases} e_B & B = C, \\ 0 & B \neq C. \end{cases}$$

(2)

$$V = \bigoplus_{B \in \mathfrak{F}} \pi(e_B) V.$$

Theorem 5.2.6. *For any finitely generated smooth representation π there exist only finitely many non-nilpotent special balls B such that $\pi(e_B) \neq 0$.*

Lemma 5.2.7. *Let B be an admissible ball and let $1_B \in \mathcal{S}(\mathfrak{g}^*)$ denote the characteristic function of B. Let $\mathcal{F}(1_B)$ denote the Fourier transform of 1_B, considered as a measure on $\mathfrak{g}$. Then $\mathcal{F}(1_B) = \exp^*(e_B)$.*

5.3. **Proof of Theorem A.** We will need some preparations.

Proposition 5.3.1 (see Section 5.4). *Let $\xi \in \mathcal{S}^*(G)$ be a $\mathfrak{z}$-finite distribution. Then there exists a compact subset $D \subset \mathfrak{g}^*$ such that for any non-nilpotent special ball $B \subset \mathfrak{g}^* \setminus D$ we have $e_B * \xi = 0$.*

The following is a straightforward computation.

Lemma 5.3.2. *Let $B := a + \alpha \mathcal{L}^\perp$ be an admissible ball, with $|\alpha|^2 > |a|$. Let S be the set of all special balls contained in B. Then $e_B = \sum_{C \in S} e_C$.*

The last 2 statements give us the following corollary.

Corollary 5.3.3. *Let $\xi \in \mathcal{S}^*(G)$ be a $\mathfrak{z}$-finite distribution. Then there exists a compact subset $D \subset \mathfrak{g}^*$ s.t. for any non-nilpotent admissible ball of the form $B := a + \alpha \mathcal{L}^\perp \subset \mathfrak{g}^* \setminus D$, with $|\alpha|^2 > |a|$, we have $e_B * \xi = 0$.*

Proposition 5.3.4 (See Section 5.5). *Let $n, l > 0$ and let $B = a + \varpi^{-n} \mathcal{L}^\perp$ be an admissible ball. Assume that $|a| = q^{n+l}$. Then for any $\xi \in \mathcal{S}^*(\exp(\varpi^l \mathcal{L}))$ we have*

$$\exp^*(e_B * \xi) = \exp^*(e_B) * \exp^*(\xi)$$

Proof of Theorem A. Since any shift of ξ is also $\mathfrak{z}$-finite, we can assume that x is the unit element $1 \in G$. Thus it is enough to show that $WF_0(exp^*(\xi)) \subset \mathcal{N}$.

Let $Y \in \mathfrak{g}^*$ be non-nilpotent. Then there exists m such that for all big enough $\alpha \in F$ the set $\alpha(Y + \varpi^m \mathcal{L}^\perp)$ is a non-nilpotent admissible ball. Let $B := Y + \varpi^m \mathcal{L}^\perp$. There exists l such that $\varpi^{l-m} Y \in \mathcal{L}^\perp$. Let ϕ be the characteristic function of $\varpi^l \mathcal{L}$ and μ be the characteristic function of $K_l := exp(\varpi^l \mathcal{L})$. Let $\zeta := \mu\xi$ and $\eta := \phi exp^*(\xi) = exp^*(\zeta)$. We have to show that for all big enough $\alpha \in F$ we have $\mathcal{F}(\eta)|_{\alpha B} = 0$. By Corollary 5.3.3, for all big enough $\alpha \in F$ we have

$$e_{\alpha B} * \zeta = 0.$$

By Proposition 5.3.4 for all big enough $\alpha \in F$ we have:

$$exp^*(e_{\alpha B} * \zeta) = exp^*(e_{\alpha B}) * \eta = \mathcal{F}^{-1}(\mathcal{F}(\exp^*(e_{\alpha B}))\mathcal{F}(\eta)).$$

Lemma 5.2.7 implies now that $\mathcal{F}(\eta)|_{\alpha B} = 0$ for all big enough $\alpha \in F$. $\qquad\square$

5.4. **Proof of Proposition 5.3.1.**

Proof of Proposition 5.3.1. Let $\pi := \mathcal{S}(G) * \xi * \mathcal{S}(G)$. By Corollary L, π is a finite length representation of $G \times G$ and thus, by Theorem 5.2.6, there exists a finite set X of special balls of $G \times G$ such that $\pi(e_{\mathcal{B}}) = 0$ for a non-nilpotent $\mathcal{B} \notin X$. Let D be the union of the projections of the balls in X to the first coordinate. It is easy to see that for any non-nilpotent special ball $B \subset \mathfrak{g}^* \setminus D$ and any special ball C, $B \times C \notin X$ and thus for any $f \in \mathcal{S}(G)$ we have

$$e_B * \xi * f * e_C = e_B * e_B * \xi * f * e_C = \pi(e_{B \times C})(e_B * \xi * f) = 0$$

By Theorem 5.2.5,

$$e_B * \xi * f = \sum_{C,C' \in \mathfrak{F}} e_{C'} * e_B * \xi * f * e_C = \sum_{C \in \mathfrak{F}} e_B * \xi * f * e_C,$$

where the sum goes over all special balls in $\mathfrak{g}^*$. This implies $e_B * \xi * f = 0$. Since this holds for any $f \in \mathcal{S}(G)$, $e_B * \xi$ vanishes. $\qquad\square$

5.5. Proof of Proposition 5.3.4.

From standard properties of the exponential map (see Lemma 5.2.2(3)) we obtain the following Corollary.

Corollary 5.5.1.

(i) *For any natural number n and any $a \in \mathcal{L}$ we have*

$$\exp(a + \mathcal{L}_n) = \exp(a)\exp(\mathcal{L}_n) = \exp(a)K_n$$

(ii) *Let e_{K_0} be the Haar probability measure on K_0. Then $\exp^*(e_{K_0})$ is the Haar probability measure on $\mathcal{L}$.*

(iii) *Let n and l be natural numbers. By (i) we can define $\overline{\exp} : \mathcal{L}_0/\mathcal{L}_{n+l} \to K_0/K_{n+l}$. Let α, β be measures on K_0/K_{n+l} such that α is supported on K_l/K_{n+l} and β is supported on K_n/K_{n+l}. Then*

$$\overline{\exp}^*(\alpha * \beta) = \overline{\exp}^*(\alpha) * \overline{\exp}^*(\beta)$$

Proof of Proposition 5.3.4.

Step 1. Proof for the case $l = 0$.

In this case for any $b \in \mathcal{L}_0$ we have

$$(e_B * \xi)|_{\exp(b)K_n} = (e_{K_n} * \xi)|_{\exp(b)K_n} = \left(\int_{\exp(b)K_n} \xi\right)(\#K_0/K_n)e_{K_0}|_{\exp(b)K_n}.$$

Also,

$$(\exp^*(e_B) * \exp^*(\xi))|_{b+\mathcal{L}_n} = \left(\int_{b+\mathcal{L}_n} \exp^*(\xi)\right)(\#\mathcal{L}_0/\mathcal{L}_n)\mu_{\mathcal{L}_0}|_{b+\mathcal{L}_n},$$

where $\mu_{\mathcal{L}_0}$ is the Haar probability measure on $\mathcal{L}_0$. By Corollary 5.5.1(i),

$$\exp^{-1}(\exp(b)K_n) = b + \mathcal{L}_n \quad \text{and} \quad \int_{\exp(b)K_n} \xi = \int_{b+\mathcal{L}_n} \exp^*(\xi).$$

Thus, by Corollary 5.5.1(ii),

$$exp^*(e_{K_0}|_{\exp(b)K_n}) = \mu_{\mathcal{L}_0}|_{b+\mathcal{L}_n}.$$

We get

$$\exp^*(e_B * \xi)|_{(b+\mathcal{L}_n)} = \exp^*((e_B * \xi)|_{(\exp(b)K_n)}) =$$

$$= \exp^*\left(\left(\int_{\exp(b)K_n} \xi\right)(\#K_0/K_n)e_{K_0}|_{\exp(b)K_n}\right) =$$

$$= \left(\int_{\exp(b)K_n} \xi\right)(\#K_0/K_n)\exp^*(e_{K_0}|_{\exp(b)K_n}) =$$

$$= \left(\int_{\exp(b)K_n} \xi\right)(\#K_0/K_n)\mu_{\mathcal{L}_0}|_{b+\mathcal{L}_n} = \left(\int_{b+\mathcal{L}_n} \exp^*(\xi)\right)(\#\mathcal{L}_0/\mathcal{L}_n)\mu_{\mathcal{L}_0}|_{b+\mathcal{L}_n} =$$

$$= (\exp^*(e_B) * \exp^*(\xi))|_{b+\mathcal{L}_n}.$$

Step 2. Proof for the general case.

Denote by $p_{\mathcal{L}}$ and p_K the natural projections $\mathcal{L}_0 \to \mathcal{L}_0/\mathcal{L}_{n+l}$ and $K_0 \to K_0/K_{n+l}$. There exist measures β and α on K_0/K_{n+l} such that $e_{K_{n+l}} * \xi = p_K^*(\beta)$ and $e_B = p_K^*(\alpha)$. Clearly $\mathrm{Supp}(\beta) \subset K_l/K_{n+l}$ and $\mathrm{Supp}(\alpha) \subset K_n/K_{n+l}$. We have

$$(5) \qquad \exp^*(e_B * \xi) = \exp^*(e_B * e_{K_{n+l}} * \xi) = \exp^*(p_K^*(\alpha) * p_K^*(\beta)) = \exp^*(p_K^*(\alpha * \beta)).$$

From the commutative diagram

$$(6) \qquad \begin{array}{ccc} \mathcal{L}_0 & \xrightarrow{\ \exp\ } & K_0 \\ \downarrow{\scriptstyle p_{\mathcal{L}}} & & \downarrow{\scriptstyle p_K} \\ \mathcal{L}_0/\mathcal{L}_{n+l} & \xrightarrow{\ \overline{\exp}\ } & K_0/K_{n+l} \end{array}$$

we have

$$(7) \qquad \exp^*(p_K^*(\alpha * \beta)) = p_{\mathcal{L}}^*(\overline{\exp}^*(\alpha * \beta)).$$

By Corollary 5.5.1(iii) we have

$$(8) \qquad p_{\mathcal{L}}^*(\overline{\exp}^*(\alpha * \beta)) = p_{\mathcal{L}}^*(\overline{\exp}^*(\alpha) * \overline{\exp}^*(\beta)) = p_{\mathcal{L}}^*(\overline{\exp}^*(\alpha)) * p_{\mathcal{L}}^*(\overline{\exp}^*(\beta)).$$

Applying the diagram (6) again we get

$$(9) \quad p_{\mathcal{L}}^*(\overline{\exp}^*(\alpha)) * p_{\mathcal{L}}^*(\overline{\exp}^*(\beta)) = \exp^*(p_K^*(\alpha)) * \exp^*(p_K^*(\beta)) = \exp^*(e_B) * \exp^*(e_{K_{n+l}} * \xi).$$

Applying Step 1 twice we have

$$(10) \quad \exp^*(e_B) * \exp^*(e_{K_{n+l}} * \xi) = \exp^*(e_B) * \exp^*(e_{K_{n+l}}) * \exp^*(\xi) =$$
$$= \exp^*(e_{K_{n+l}}) * \exp^*(e_B) * \exp^*(\xi) = \exp^*(e_{K_{n+l}} * e_B) * \exp^*(\xi) = \exp^*(e_B) * \exp^*(\xi).$$

Combining (5,7-10) we get $\exp^*(e_B * \xi) = \exp^*(e_B) * \exp^*(\xi)$. $\qquad \square$

5.6. **Regularity of invariant $\mathfrak{z}$-finite distributions at cotoric elements and proof of Corollary F.** In this section we prove a generalization of Corollary F. We will need the following notion.

Definition 5.6.1. *Let $\mathbf{H_1}, \mathbf{H_2} < \mathbf{G}$ be algebraic subgroups. We say that an element $g \in G$ is $H_1 \times H_2$-cotoric if the conormal space to H_1gH_2 at g intersects trivially the nilpotent cone of $\mathfrak{g}^*$.*

Lemma 5.6.2. *Let $\mathbf{H}$ be an open subgroup the group of fixed points of an involution θ of $\mathbf{G}$. Let $g \in G$ be an $\mathbf{H} \times \mathbf{H}$-regular semisimple element, i.e. an element such that the double coset $\mathbf{H}g\mathbf{H}$ is closed and of maximal dimension. Then g is $\mathbf{H} \times \mathbf{H}$-cotoric.*

In particular, the set of cotoric elements contains an open dense subset of $\mathbf{H} \times \mathbf{H}$.

Proof. Let σ be the anti-involution given by $\sigma(g) := \theta(g^{-1})$. Let $(\mathbf{H} \times \mathbf{H})_g$ be the stabilizer of g with respect to the two-sided action of $\mathbf{H} \times \mathbf{H}$, and $N_{\mathbf{H}g\mathbf{H},g}^{\mathbf{G}}$ be the normal space to the double coset $\mathbf{H}g\mathbf{H}$ at g in $\mathbf{G}$. Since g is $\mathbf{H} \times \mathbf{H}$-regular semisimple, the Luna slice theorem (see e.g. [Dre00, Theorem 5.4]) implies that $(\mathbf{H} \times \mathbf{H})_g$ acts trivially on $N_{\mathbf{H}g\mathbf{H},g}^{\mathbf{G}}$.

Let $x = g\sigma(g)$. By [AG09, Proposition 7.2.1(ii)], the pair consisting of the group $(\mathbf{H} \times \mathbf{H})_g$ and its action on $N_{\mathbf{H}g\mathbf{H},g}^{\mathbf{G}}$ is isomorphic to the pair consisting of the centralizer $\mathbf{H}_x$ and its adjoint action on the centralizer $\mathfrak{g}_x^\sigma$ of x in the space $\mathfrak{g}^\sigma$ of fixed points of σ in $\mathfrak{g}$. Since g is $\mathbf{H} \times \mathbf{H}$-semisimple, [AG09, Proposition 7.2.1(i)] shows that x is a semisimple element of $\mathbf{G}$. Thus $\mathbf{G}_x$ is a reductive group.

Now, assume that x is not cotoric. Then, using a non-degenerate θ-invariant and $\mathbf{G}$-invariant quadratic form on $\mathfrak{g}$ (see e.g. [AG09, Lemma 7.1.9]), we can find a nilpotent element $\alpha \in \mathfrak{g}_x^\sigma$. Using the Jacobson-Morozov theorem for symmetric pairs (see e.g. [AG09, Lemma 7.1.11]), for some $t \neq 1 \in F$ we can find an element $h \in H_x$ such that $ad(h)(\alpha) = t\alpha$. This contradicts the fact that H_x acts trivially $\mathfrak{g}_x^\sigma$. $\qquad \square$

In view of Lemma 5.1.6, Theorem A gives us the following corollary.

Corollary 5.6.3. *Let* $\mathbf{H_1}, \mathbf{H_2} < \mathbf{G}$ *be algebraic subgroups. Let* χ_i *be characters of* H_i, *and let* ξ *be an* $(H_1 \times H_2, \chi_1 \times \chi_2)$-*equivariant* ʒ-*finite distribution. Let* $x \in G$ *be an* $H_1 \times H_2$-*cotoric element. Then* ξ *is smooth at* x.

In view of Lemma 5.6.2 this corollary implies Corollary F.

APPENDIX A. FUZZY BALLS (JOINT WITH ALEXANDER KEMARSKY)

In this section we prove the statements on admissible balls and special balls formulated in Section 5.2. We follow [Say02, §4 and §5.1]. Throughout the section we assume that F has characteristic zero.

A.1. **The exponential map and proof of Lemma 5.2.2.** It is enough to prove Lemma 5.2.2 for $\mathbf{G} = \mathrm{GL}_n$. Consider the power series

$$\mathcal{E}xp(X) := \sum_{k=0}^{\infty} a_k X^k := \sum_{k=0}^{\infty} X^k/k! \text{ and } \mathcal{L}og(X) := \sum_{k=0}^{\infty} b_k (X-1)^k := \sum_{k=1}^{\infty} (-1)^{k-1}(X-1)^k/k,$$

where $X \in \mathrm{Mat}_{n \times n}(F)$. The Baker-Campbell-Hausdorff formula is the following equality of power series

(11) $\quad \mathcal{L}og(\mathcal{E}xp(X+Y)) - X - Y =$

$$= \sum_{n=1}^{\infty} \sum_{|i|+|j|=n} \left(c_{ij} ad_X^{i_1} \circ ad_Y^{j_1} \cdots ad_X^{i_k} \circ ad_Y^{j_k}(X) + d_{ij} ad_Y^{i_1} \circ ad_X^{j_1} \cdots ad_Y^{i_k} \circ ad_X^{j_k}(Y) \right),$$

where $i = (i_1, \ldots, i_k)$ and $j = (i_1, \ldots, i_k)$ are multi-indices and $c_{ij}, d_{ij} \in \mathbb{Q} \subset F$ are certain constants. Let $\alpha_n := \max(|a_n|, |b_n|, \max_{|i|+|j|=n}(|c_{ij}|), \max_{|i|+|j|=n}(|d_{ij}|))$. It is well known for some constant $C > 1$ and all $n \geq 1$ we have $\alpha_n \leq C^n$. Define $\mathcal{L} := \{X \in \mathrm{Mat}_{n \times n}(F) \,|\, |X_{ij}| < C^{-1}\}$. It is easy to see that the power series $\mathcal{E}xp$ converge on $\mathcal{L}$. We define exp to be the corresponding analytic map, and K to be $\exp(\mathcal{L})$. Finally, it follows from (11) that $(\mathcal{L}, K, \exp)$ satisfy the requirements (1)-(3). $\qquad\square$

A.2. **Proof of Theorem 5.2.5.** We start with the following easy lemma.

Lemma A.2.1. *Let* $\mathcal{B}$ *denote the collection of all special balls. Then* $\mathfrak{g}^*$ *decomposes as a disjoint union*

$$\mathfrak{g}^* = \coprod_{B \in \mathcal{B}} B.$$

Proof. Let $X \in \mathfrak{g}^*$. If $|X| \leq 1$, then $X \in \mathcal{L}_1^{\perp}$. If $|X| = q^m > 1$, then $X \in X + \mathcal{L}_{[\frac{m}{2}]}^{\perp}$. Thus $\mathfrak{g}^* = \bigcup B$. Let $B_1 = X + \mathcal{L}_m^{\perp}$, $B_2 = Y + \mathcal{L}_n^{\perp}$ be special balls and suppose that $z = X + l_1 = Y + l_2 \in B_1 \cap B_2$. Then $|z| = |X| = |Y|$, so $m = n$. Let $Y + l' \in B_2$. We can rewrite this element as

$$Y + l' = Y + l_2 + l' - l_2 = z + (l' - l_2) \in X + \mathcal{L}_m^{\perp} = B_1.$$

We have obtained $B_2 \subset B_1$ and clearly by the same argument applied to B_1, we obtain $B_1 \subset B_2$. Therefore, $B_1 = B_2$. $\qquad\square$

Let (π, V) be a smooth representation. The following lemma is straightforward.

Lemma A.2.2. *For an admissible ball* B, *the image of* $\pi(e_B)$ *consists of* (η_B^{-1}, K_B)-*equivariant vectors in* V, *i.e.*

$$\pi(e_B)V = \left\{ v \in V : \pi(k)v = \eta_B^{-1}(k)v \;\; \forall k \in K_B \right\}.$$

Moreover, e_B *is a projection, that is* $e_B = e_B^2$.

Lemma A.2.3. *Let* B_1, B_2 *be distinct special balls. Then* $e_{B_1} e_{B_2} = 0$

Proof. Suppose $B_1 \neq B_2$ and $e_{B_1}e_{B_2} \neq 0$. Then for any $a \in K := K_{B_1} \cap K_{B_2}$ we have

$$\eta_{B_1}(a^{-1})e_{B_1}e_{B_2} = ae_{B_1}e_{B_2} = e_{B_1}ae_{B_2} = \eta_{B_2}(a^{-1})e_{B_1}e_{B_2}$$

We get $\eta_{B_1}|_K = \eta_{B_2}|_K$. Now, if $K_{B_1} = K_{B_2}$, then $B_1 = B_2$, a contradiction. Otherwise we can assume $K_{B_1} \subset K_{B_2}$, but then the character η_{B_1} is a restriction of η_{B_2} from the bigger group K_{B_2}, thus B_1 and B_2 intersect and thus by Lemma A.2.1 they coincide, which again is a contradiction. $\qquad\square$

Lemma A.2.4. *Let $v \in V^{K_N}$ and B be a special ball with radius bigger than q^N. Then $\pi(e_B)v = 0$.*

Proof.

$$\pi(e_B)v = \int_{K_B} \eta_B(k)\pi(k)v\,dk = \left(\int_{K_B} \eta_B(k)\,dk\right)v = 0.$$

$\qquad\square$

Lemma A.2.5. *For every $0 \neq v \in V$, there exists a special ball B, such that $\pi_B(v) \neq 0$.*

Proof. Let $0 \neq v \in V$. If $v \in V^{K_0}$ then $v \in V(B)$ for $B = 0 + \mathcal{L}^{\perp}$.

Suppose $v \notin V^{K_0}$. Let $n \geq 1$ be the minimal n such that $v \in V^{K_{2n}}, v \notin V^{K_{2n-2}}$. Thus the group $A = K_n/K_{2n}$ acts on the finite dimensional space W generated in V by the orbit $K_n v$. Note that $K_n/K_{2n} \simeq \mathcal{L}_n/\mathcal{L}_{2n}$ and by standard properties of the exponential map (see Lemma 5.2.2(3)) the group $\mathcal{L}_n/\mathcal{L}_{2n}$ is commutative. Thus, the group A is a commutative finite group. The space W can be decomposed as a direct sum of one-dimensional characters of A. For a character χ of A and $w \in W$, let $w(\chi) \in W$ be the projection of w to the χ-isotypic component of W. Then $v = \sum_{\chi \in A^*} v(\chi)$, and thus there exists a character χ of A with $v(\chi) \neq 0$ and $\chi|_{K_{2n-2}/K_{2n}} \neq 1$. Lift χ to a character η of K_n and note that there exists a unique special ball B with $K_B = K_n$ and $\eta_B = \eta$. Then $\pi(B)v = v(\chi) \neq 0$. $\qquad\square$

Proof of Theorem 5.2.5. Part (1) follows from Lemmas A.2.2 and A.2.3. To prove part (2) take $0 \neq v \in V$, and let $w = \sum_{B \in \mathfrak{F}} \pi(e_B)v$. By Lemma A.2.4 we know that the sum is well-defined. By Lemmas A.2.2 and A.2.3, $\pi(e_B)(v - w) = 0$ for all special balls B. By Lemma A.2.5, it follows that $v = w$. $\qquad\square$

A.3. Proof of Theorem 5.2.6.

Definition A.3.1. *A special set is a finite union of special balls. For a special set $T = \cup B_i$, denote $e_T := \sum e_{B_i}$.*

Lemma A.3.2. *Let T and S be two special sets in $\mathfrak{g}^*$ and let $g \in G$. Let (π, V) be a smooth representation of G. Suppose that $\pi(e_T)\pi(g)\pi(e_S) \neq 0$. Then $ad(g)S \cap T \neq \emptyset$.*

Proof. By linearity we reduce to the case where

$$T = B = X + \mathcal{L}_m^{\perp} \text{ and } S = B' = X' + \mathcal{L}_{m'}^{\perp}$$

are special balls. Let

$$K = K_B = \exp(\mathcal{L}_m),\ K' = K_{B'} = \exp(\mathcal{L}_{m'}),\ \eta = \eta_B, \eta' = \eta_{B'}$$

We first check that

$$\eta(a) = \eta'(g^{-1}ag)$$

for all $a \in K \cap gK'g^{-1}$. Indeed, let $v \in V$ be such that $\pi(e_B)\pi(g)\pi(e'_B)(v) \neq 0$. Then for all $a \in K \cap gK'g^{-1}$ we have

(12) $\eta(a)^{-1}\pi(e_B)\pi(g)\pi(e_{B'})(v) = \pi(a)\pi(e_B)\pi(g)\pi(e_{B'})(v) = \pi(e_B)\pi(a)\pi(g)\pi(e_{B'})(v),$

since $\pi(e_B)\pi(a) = \pi(a)\pi(e_B)$ for $a \in K$. On the other hand,
(13)
$\pi(e_B)\pi(a)\pi(g)\pi(e_{B'})(v) = \pi(e_B)\pi(g)\pi(g^{-1}ag)\pi(e_{B'})(v) = \eta'^{-1}(g^{-1}ag)\pi(e_B)\pi(g)\pi(e_{B'})(v).$

Combining equations (12) and (13), we obtain $\eta(a) = \eta'(g^{-1}ag)$ for every $a \in K \cap gK'g^{-1}$. Therefore,

$$\psi_0\left(X(\log(a))\right) = \psi_0\left(X'(Ad(g^{-1})\log(a))\right) = \psi_0\left((Ad(g)X')(\log(a))\right).$$

We see that for $b \in \mathcal{L}_m \cap Ad(g)\mathcal{L}_{m'}$,

$$(Ad(g)X' - X)(b) \in \mathcal{O}.$$

Thus,

$$Ad(g)X' - X \in (\mathcal{L}_m \cap Ad(g)\mathcal{L}_{m'})^{\perp} = \mathcal{L}_m^{\perp} + Ad(g)\mathcal{L}_{m'}^{\perp},$$

that is, there exist $u \in \mathcal{L}_m^{\perp}$ and $v \in \mathcal{L}_{m'}^{\perp}$ such that

$$Ad(g)X' - X = u + Ad(g)v.$$

Hence $Ad(g)(X' - v) = X + u \in X + \mathcal{L}_m^{\perp} = B$. $\qquad\square$

The following Lemma due to Howe plays a central role in the proof of Theorem 5.2.6.

Lemma A.3.3 ([HCDBS99, Lemma 12.2]). *Let $S \subset \mathfrak{g}$ be compact. There exists a compact subset S_1 such that*

$$Ad(G)S \subset S_1 + \mathcal{N}.$$

Proof of Theorem 5.2.6. Suppose V is generated by $v_1, ..., v_n$ and for each i pick all special balls B_{ij} such that $\pi(e_{B_{ij}})(v_i) \neq 0$. Note that by Lemma A.2.4 for every $v \in V$, there are only finitely many special ball B such that $\pi(e_B)v \neq 0$. Let $S = \cup B_{ij}$. By Theorem 5.2.5 $\pi(S)v_i = v_i$. Since S is compact, Lemma A.3.3 implies

$$Ad(G)S \subset \mathcal{L}_m^{\perp} + \mathcal{N}$$

for some large m. Let B be a special ball such that $\pi(e_B) \neq 0$. Let us show that there exists $g \in G$ such that $\pi(e_B)\pi(g)\pi(e_S) \neq 0$. Indeed, suppose on the contrary that $\pi(e_B)\pi(g)\pi(e_S) = 0$ for every $g \in G$. Let v be such that $\pi(e_B)v \neq 0$ and write

$$v = \sum_{j=1, 1 \leq i_j \leq n}^{k} c_i \pi(g_i) v_{i_j}.$$

Then

$$\pi(e_B)(v) = \sum_{j=1, 1 \leq i_j \leq n}^{k} c_i \pi(e_B)\pi(g_i) v_{i_j} = \sum_{j=1, 1 \leq i_j \leq n}^{k} c_i \pi(e_B)\pi(g_i)\pi(e_S) v_{i_j} = 0,$$

and we obtain a contradiction! By Lemma A.3.2

$$Ad(g)S \cap B \neq \emptyset.$$

In particular $B \cap (\mathcal{L}_m^{\perp} + \mathcal{N}) \neq \emptyset$. Suppose $B = X + \mathcal{L}_n^{\perp}$ with $n \geq m$ and let $Y \in B \cap (\mathcal{L}_m^{\perp} + \mathcal{N})$. Then $Y = X + l = l' + n$, $l \in \mathcal{L}_m^{\perp}, l' \in \mathcal{L}_n^{\perp}$, and $n \in \mathcal{N}$. In particular, $n = X + (l - l') \in X + \mathcal{L}_n^{\perp}$, so $n \in B \cap \mathcal{N}$.

We have obtained that every special ball B that acts on V as non-zero and has big enough radius is a nilpotent special ball. Since the number of special balls with a bounded radius is finite, we obtain that all except of finitely many non-nilpotent balls act on V as zero. $\qquad\square$

A.4. **Proof of Lemma 5.2.7.** We follow [Say02, §5.1]. Assume that $B = B(X, L)$. Note that $\exp^*(e_B) = f\mu$, where μ is the Haar measure on L, normalized such that $\mu(L) = 1$ and f is a function given by $f(y) = \psi_0(\langle y, X \rangle)1_L(y)$. Then

$$\mathcal{F}(\mu)(Z) = \int_{y \in L} \psi_0(\langle y, X + Z \rangle)d\mu(y).$$

The last integral is an integral of an additive character on an additive group. Such an integral is zero, unless the character is trivial. In our case this means that the integral is zero, unless $X + Z \in L^\perp$, which happens if and only if $-Z \in X + L^\perp$ and in that case the integral equals 1. Therefore, $\mathcal{F}(\mu)(Z) = 1_{X+L^\perp}(-Z)$. As $\mathcal{F} \circ \mathcal{F} = -Id$ (under the identification $\mathfrak{g} \simeq \mathfrak{g}^{**}$), we get that $\mathcal{F}(1_B) = \exp^*(e_B)$, as claimed.

Appendix B. Finite Generation of Hecke Modules (by A. Aizenbud and D. Gourevitch)

In this section we prove a stronger version of Theorem I. For its formulation we will need the following definition.

Definition B.0.1. *Let $(G, (H, \chi))$ be a twisted pair, i.e. $H < G$ is a (closed) subgroup, and χ is its character.*

    (1) *Denote by $D_{G/H}$ the G-equivariant sheaf of smooth measures on G/H and by $\Delta_{G/H}$ its fiber at $[1] \in G/H$, considered as a character of H. Note that $\Delta_{G/H} = (\Delta_G)|_H \cdot \Delta_H^{-1} = \Delta_H^{-1}$.*

    (2) *We define the* dual *of the twisted pair $(G, (H, \chi))$ to be the pair $(G, (H, \hat{\chi}))$, where $\hat{\chi} = \Delta_{G/H}\chi^{-1}$. Note that $\hat{\hat{\chi}} = \chi$.*

The following theorem clearly implies Theorem I.

Theorem B.0.2. *Let $(G, (H, \chi))$ be a twisted pair. Let $\mathbf{P}$ be a minimal parabolic subgroup of $\mathbf{G}$ and $P = \mathbf{P}(F)$. Suppose that H has finitely many orbits on G/P. Suppose that for any irreducible smooth representation ρ of G we have*

(14) $$\dim \mathrm{Hom}_H(\rho|_H, \hat{\chi}) < \infty.$$

Then for any open compact subgroup K of G the module $\mathrm{ind}_H^G(\chi)^K$ over the Hecke algebra $\mathcal{H}_K(G)$ is finitely generated.

Let us now give an overview of the argument. In Lemma B.1.9 we present a criterion, due to Bernstein, for the finite generation of spaces of K-invariants. The proof of the criterion uses the theory of Bernstein Center. Using this criterion we introduce a notion of twisted pairs of finite type (see Definition B.2.1(2)) which is equivalent to the local finite generation of $\mathrm{ind}_H^G(\chi)$. Bernstein's criterion is given in terms of all parabolic subgroups of G. This allows us to define an intermediate notion of finite cuspidal type (see Definition B.2.1(3)), which means that the criterion holds for the group G as a parabolic subgroup of itself.

Then we introduce duality between twisted pairs. We prove that condition (14) implies that the dual pair $(G, (H, \hat{\chi}))$ is of finite cuspidal type (see Appendix B.4). We use a simple trick (Corollary B.4.6) to imply that the pair $(G, (H, \chi))$ is itself of finite cuspidal type.

In order to analyze the condition of Lemma B.1.9 for all parabolic subgroups of G we introduce the notion of a descendant of the pair $(G, (H, \chi))$ and prove that if all the descendants are of finite cuspidal type then the pair itself is of finite type (see Appendix B.5).

Thus it remains to show that if the conditions of Theorem I hold for a twisted pair then they hold for all its descendants. This we do in Appendix B.3, using a homological algebra argument.

Remark B.0.3. *The argument here is an adaptation of a similar argument in [AAG12] that dealt with the case of trivial χ. However, the argument in [AAG12] did not take into account the modular characters of various groups that appear along the way. As a result, it is not valid for non-unimodular H and even for unimodular H it is not complete. The gap in the original argument is filled mainly by the proof of Corollary B.4.6.*

B.1. **Preliminaries.**

Notation B.1.1. *For a subgroup $H < G$ we denote by $\mathrm{ind}_H^G : \mathcal{M}(H) \to \mathcal{M}(G)$ the compactly supported induction functor and by $\mathrm{Ind}_H^G : \mathcal{M}(H) \to \mathcal{M}(G)$ the full induction functor. For $\pi \in \mathcal{M}(G)$ denote by $\widetilde{\pi} := (\pi^*)^\infty$ the smooth contragredient representation. Note that for any character χ of H we have $\widetilde{\mathrm{ind}_H^G(\chi)} = \mathrm{Ind}_H^G(\hat\chi)$.*

Definition B.1.2. *Let $\mathbf{P} < \mathbf{G}$ be a parabolic subgroup with unipotent radical $\mathbf{U}$, and let $\mathbf{M} := \mathbf{P}/\mathbf{U}$. Such $\mathbf{M}$ is called a Levi subquotient of G. Note that every representation of M can be considered as a representation of P using the quotient morphism $P \twoheadrightarrow M$. Define:*

(1) *The Jacquet functor $r_M^G : \mathcal{M}(G) \to \mathcal{M}(M)$ by $r_M^G(\pi) := (\Delta_{G/P}^{\frac{1}{2}} \cdot \pi|_P)_U$. Note that r_M^G is defined for any representation of P.*

(2) *The parabolic induction functor $i_M^G : \mathcal{M}(M) \to \mathcal{M}(G)$ by $i_M^G(\tau) := \mathrm{ind}_P^G(\Delta_{G/P}^{-\frac{1}{2}}\tau)$.*

Note that i_M^G is right adjoint to r_M^G. A representation π of G is called cuspidal if $r_M^G(\pi) = 0$ for any Levi subquotient $\mathbf{M}$ of $\mathbf{G}$.

It is well-known that i_M^G and r_M^G are exact functors.

Definition B.1.3. *A smooth representation V of G is called* compact *if for any $v \in V$ and $\xi \in \widetilde{V}$ the matrix coefficient function defined by $m_{v,\xi}(g) := \xi(gv)$ is a compactly supported function on G.*

Theorem B.1.4 (Bernstein-Zelevinsky)**.** *Any compact representation of G is a projective object in the category $\mathcal{M}(G)$.*

Definition B.1.5.

(1) *Denote by G^1 the preimage in G of the maximal compact subgroup of $G/[G,G]$.*

(2) *Denote by $Z(G)$ the center of G and denote $G_0 := G^1 Z(G)$.*

(3) *We call a complex character of G unramified if it is trivial on G^1. We denote the set of all unramified characters by Ψ_G. Note that G/G^1 is a lattice and therefore we can identify Ψ_G with $(\mathbb{C}^\times)^n$. This defines a structure of algebraic variety on Ψ_G.*

(4) *For any smooth representation ρ of G we denote $\Psi(\rho) := \mathrm{ind}_{G^1}^G(\rho|_{G^1})$. Note that $\Psi(\rho) \simeq \rho \otimes \mathcal{O}(\Psi_G)$, where G acts only on the first factor, but this action depends on the second factor. This identification gives a structure of $\mathcal{O}(\Psi_G)$-module on $\Psi(\rho)$.*

Theorem B.1.6 (Harish-Chandra)**.** *Let V be a cuspidal representation of G. Then $V|_{G^1}$ is a compact representation of G^1.*

Corollary B.1.7. *Let ρ be a cuspidal representation of G. Then*
(i) $\rho|_{G^1}$ is a projective object in the category $\mathcal{M}(G^1)$.
(ii) $\Psi(\rho)$ is a projective object in the category $\mathcal{M}(G)$.

Proof. (i) Follows from Theorems B.1.4 and B.1.6.
(ii) note that

$$Hom_G(\Psi(\rho), \pi) \cong Hom_{G/G_1}(\mathcal{O}(\Psi_M), Hom_{G^1}(\rho, \pi)),$$

for any representation π. Therefore the functor $\pi \mapsto Hom_G(\Psi(\rho), \pi)$ is a composition of two exact functors and hence is exact. $\qquad\square$

We will use Bernstein's second adjointness theorem.

Theorem B.1.8 ([Ber87] or [Bus01, Theorem 3])**.** *Let* $\mathbf{P} \subset \mathbf{G}$ *be a parabolic subgroup and let* $\overline{\mathbf{P}}$ *be an opposite parabolic subgroup. Let* $\mathbf{M}$ *be the Levi quotient of* $\mathbf{P}$ *and let* $\overline{r}_M^G : \mathcal{M}(G) \to \mathcal{M}(M)$ *denote the Jacquet functor defined using* $\overline{P}$. *Then* $\overline{r}_M^G$ *is right adjoint to* i_M^G. *In particular,* i_M^G *maps projective objects to projective ones and hence for any irreducible cuspidal representation* ρ *of* M, $i_M^G(\Psi(\rho))$ *is a projective object of* $\mathcal{M}(G)$.

We now present a criterion, due to Bernstein, for local finite generation.

Lemma B.1.9 ([AAG12, Lemma 2.1.10])**.** *Let* $V \in \mathcal{M}(G)$. *Suppose that for any parabolic* $P < G$ *and any irreducible cuspidal representation* ρ *of* M *(where* M *denotes the reductive quotient of* P)*,* $\mathrm{Hom}_G(i_M^G(\Psi(\rho)), V)$ *is a finitely generated module over* $\mathcal{O}(\Psi_M)$. *Then* V^K *is a finitely generated module over* $\mathfrak{z}(\mathcal{H}_K(G))$, *for any compact open subgroup* $K < G$.

The theory of Bernstein center gives us the following Lemma:

Lemma B.1.10. *Let* V *be a smooth finitely generated representation of* G. *Let* $W \subset V$ *be a subrepresentation. Then* W *is finitely generated.*

Proof. Let $v_1, \ldots, v_n$ be the generators of V. By Theorem 2.2.4(2) we can choose a splitting subgroup $K \subset G$ s.t. $v_i \in V^K$. Then $V \in \mathcal{M}_K(G)$ and V^K is finitely generated over $\mathcal{H}_K(G)$. Hence $W \in \mathcal{M}_K(G)$ and thus it is enough to show that W^K is finitely generated over $\mathcal{H}_K(G)$. By Theorem 2.2.4(3b) $\mathcal{H}_K(G)$ is finite over its center $\mathfrak{z}(\mathcal{H}_K(G))$. So V^K is finitely generated over $Z(\mathcal{H}_K(G))$. From Theorem 2.2.4(3c) it follows that $\mathfrak{z}(\mathcal{H}_K(G))$ is Notherian, and thus W^K is finitely generated generated over $\mathfrak{z}(\mathcal{H}_K(G))$. $\square$

B.2. **Finite multiplicity and duality of twisted pairs.** Let $(G, (H, \chi))$ be a twisted pair.

Definition B.2.1. *We say that the pair* $(G, (H, \chi))$
  (1) *has* finite multiplicities *(resp.* finite cuspidal multiplicities*) if for any irreducible (resp. cuspidal irreducible) representation* π *of* G,
$$\dim \mathrm{Hom}(\mathrm{ind}_H^G(\chi), \pi) < \infty.$$
  (2) *has* finite type *if for any parabolic* $\mathbf{P} < \mathbf{G}$ *and any irreducible cuspidal representation* ρ *of* M *(where* $\mathbf{M}$ *denotes the reductive quotient of* $\mathbf{P}$)*,*
$$\mathrm{Hom}_G(i_M^G(\Psi(\rho)), \mathrm{ind}_H^G(\chi))$$
  is a finitely generated module over $\mathcal{O}(\Psi_M)$.
  (3) *has* finite cuspidal type *if for any irreducible cuspidal representation* ρ *of* G, $\mathrm{Hom}_G(\Psi(\rho), \mathrm{ind}_H^G(\chi))$ *is a finitely generated module over* $\mathcal{O}(\Psi_M)$.
  (4) *is* F-spherical[4] *if for any parabolic subgroup* $\mathbf{P} \subset \mathbf{G}$, *there is a finite number of double cosets in* $P \backslash G / H$.

The following lemma helps to connect multiplicities to duality.

Lemma B.2.2. *Let* $\pi, \rho \in \mathcal{M}(G)$ *and assume that* ρ *has finite length. Then the natural morphism* $\mathrm{Hom}(\pi, \rho) \to \mathrm{Hom}(\widetilde{\rho}, \widetilde{\pi})$ *is an isomorphism.*

Proof. By Theorem 2.2.4 we can choose a splitting subgroup K such that ρ^K generates ρ. Then
$$\mathrm{Hom}(\pi, \rho) \cong \mathrm{Hom}_{\mathcal{H}_K(G)}(\pi^K, \rho^K) \cong \{v \in (\pi^K)^* \otimes \rho^K \,|\, \forall a \in \mathcal{H}_K(G), \, (a \otimes 1 - 1 \otimes a)v = 0\}.$$

---

[4] If $char F = 0$ and G is quasi-split over F then (G, H) is an F-spherical pair if and only if it is a spherical pair of algebraic groups.

Here we consider the standard action of $\mathcal{H}_K(G)^{opposite} \otimes \mathcal{H}_K(G)$ on acts on $(\pi^K)^* \otimes \rho^K$. Also

$$\text{Hom}(\widetilde{\rho}, \widetilde{\pi}) \cong \text{Hom}_{\mathcal{H}_K}((\rho^*)^K, (\pi^*)^K) \cong \text{Hom}_{\mathcal{H}_K}((\rho^K)^*, (\pi^K)^*) \cong \rho^K \otimes (\pi^*)^K \cong$$
$$\cong \{v \in \rho^K \otimes (\pi^K)^* \mid \forall a \in \mathcal{H}_K(G), (a \otimes 1 - 1 \otimes a)v = 0\}$$

This easily implies the assertion. □

Using Frobenius reciprocity we obtain the following corollary.

Corollary B.2.3. *Let π be a smooth representation of G of finite length. Then*

$$\dim \text{Hom}_G(\text{ind}_H^G(\chi), \pi) = \dim \text{Hom}_H(\tilde{\pi}, \hat{\chi})$$

Lemma 3.2.2 follows from Corollary B.2.3 and the next lemma.

Lemma B.2.4. *If $(G, (H, \chi))$ has finite (cuspidal) type then it has finite (cuspidal) multiplicities.*

Proof. Let π be a irreducible (cuspidal) representation of G. By Theorem 2.2.4 we can choose a splitting compact open subgroup $K < G$ s.t. $\pi^K \neq 0$. Then

$$\text{Hom}_G(\text{ind}_H^G(\hat{\chi}), \pi) = \text{Hom}_{\mathcal{H}_K(G)}((\text{ind}_H^G(\hat{\chi}))^K, \pi^K).$$

Since $(\text{ind}_H^G(\hat{\chi}))^K$ is finitely generated, this implies that $\dim \text{Hom}_G(\text{ind}_H^G(\hat{\chi}), \pi) < \infty$. □

In view of Lemma B.1.9 and Corollary B.2.3, Theorem B.0.2 is equivalent to the following one.

Theorem B.2.5. *If $(G, (H, \chi))$ is F-spherical and has finite multiplicity then it has finite type.*

B.3. Descent Of Finite Multiplicity.

Notation B.3.1. *Let $(G, (H, \chi))$ be a twisted pair. Let $\mathbf{P} < \mathbf{G}$ be a parabolic subgroup and $\mathbf{M}$ be its Levi quotient. Let $\overline{\mathbf{P}} < \mathbf{G}$ be a parabolic subgroup opposite to $\mathbf{P}$ and $\overline{\mathbf{U}}$ be its unipotent radical. Let $X := G/H$. Let $\mathcal{F}$ be the natural G-equivariant sheaf on X such that the stalk at $[1]$ coincides with χ as a representation of H.*

Let $x \in X$. It is easy to see that there exists a geometric quotient $A^x = \overline{U}\backslash\overline{P}x$. Denote by $\mathcal{F}^x$ the natural M-equivariant sheaf on A^x such that $\overline{r}_M^G(\mathcal{S}(\overline{P}x, \mathcal{F}|_{\overline{P}_x})) = \mathcal{S}(A^x, \mathcal{F}^x)$. Suppose that $\mathcal{F}^x \neq 0$. Let y be the image of x in A^x. We denote its stabilizer in M by H_M^x, and we consider the fiber $(\mathcal{F}^x)_y$ as a character of H_M^x, and denote it by χ_M^x.

We say that the twisted pair $(M, (H_M^x, \chi_M^x))$ is a P-descendent of the twisted pair $(G, (H, \chi))$. We will say that descendants $(M, (H_M^x, \chi_M^x))$ and $(M, (H_M^{x'}, \chi_M^{x'}))$ are equivalent if x and x' belong to the same P-orbit.

The following version of the Bernstein-Zelevinsky geometric lemma follows from the exactness of $\overline{r}_M^G$.

Lemma B.3.2. *Let $\mathbf{P} < \mathbf{G}$ be a parabolic subgroup and $\mathbf{M}$ be its Levi quotient. Let $(G, (H, \chi))$ be an F-spherical twisted pair. Then $\text{ind}_H^G(\chi)$ has a finite filtration such that*

$$Gr(\overline{r}_M^G(\text{ind}_H^G(\chi))) \simeq \bigoplus_i \text{ind}_{H_i}^G(\chi_i),$$

where $(M, (H_i, \chi_i))$ ranges over all the P-descendants of $(G, (H, \chi))$ up to equivalence.

The goal of this subsection is to prove the following lemma.

Lemma B.3.3. *Let $(G, (H, \chi))$ be an F-spherical pair of finite multiplicity. Let $P < G$ be a parabolic subgroup and M be its Levi quotient. Let $(M, (H', \chi'))$ be a descendant of $(G, (H, \chi))$. Then $(M, (H', \chi'))$ is an F-spherical pair of finite cuspidal multiplicity.*

Remark B.3.4. *One can easily show that the converse statement to this lemma is also true. Namely, if all the descendants of the pair have finite cuspidal multiplicity then the pair has finite multiplicity. This also implies that in this case all the descendants have finite multiplicity. However we will not use these facts.*

We will need the following lemmas.

Lemma B.3.5. *Let M be an l-group and V, W be smooth representations of M such that that $\dim \operatorname{Hom}(V, W) < \infty$. Let $0 = F^0 V \subset ... \subset F^{n-1} V \subset F^n V = V$ be a finite filtration of V by subrepresentations. Suppose that for any i, either*

$$\dim \operatorname{Hom}(F^i V / F^{i-1} V, W) = \infty$$

or

$$\text{both } \dim \operatorname{Hom}(F^i V / F^{i-1} V, W) < \infty \text{ and } \dim \operatorname{Ext}^1(F^i V / F^{i-1} V, W) < \infty.$$

Then $\dim \operatorname{Hom}(F^i V / F^{i-1} V, W) < \infty$ for any i.

Proof. We prove by a decreasing induction on i that $\dim \operatorname{Hom}(F^i V, W) < \infty$, and, therefore, $\dim \operatorname{Hom}(F^i V / F^{i-1} V, W) < \infty$ and by the conditions of the lemma

$$\dim \operatorname{Ext}^1(F^i V / F^{i-1} V, W) < \infty.$$

Consider the short exact sequence

$$0 \to F^{i-1} V \to F^i V \to F^i V / F^{i-1} V \to 0,$$

and the corresponding long exact sequence

$$... \leftarrow \operatorname{Ext}^1(F^i V / F^{i-1} V, W) \leftarrow \operatorname{Hom}(F^{i-1} V, W) \leftarrow \operatorname{Hom}(F^i V, W) \leftarrow \operatorname{Hom}(F^i V / F^{i-1} V, W) \leftarrow 0.$$

In this sequence $\dim \operatorname{Ext}^1(F^i V / F^{i-1} V, W) < \infty$ and $\dim \operatorname{Hom}(F^i V, W) < \infty$, and hence $\dim \operatorname{Hom}(F^{i-1} V, W) < \infty$. $\square$

Lemma B.3.6. *Let $(G, (H, \chi))$ be a twisted pair. Let ρ be an irreducible cuspidal representation of G. Suppose that $\dim \operatorname{Hom}(\operatorname{ind}_H^G(\chi), \rho) < \infty$. Then $\dim \operatorname{Ext}^1(\operatorname{ind}_H^G(\chi), \rho) < \infty$.*

For the proof we will need the following straightforward lemma.

Lemma B.3.7. *Let L be a lattice. Let V be a linear space. Let L act on V by a character. Then*

$$\mathrm{H}_1(L, V) = \mathrm{H}_0(L, V) \otimes_{\mathbb{C}} (L \otimes_{\mathbb{Z}} \mathbb{C}).$$

Proof of Lemma B.3.6. By Lemma B.2.2

$$\operatorname{Ext}^i(\operatorname{ind}_H^G(\chi), \rho) \cong \operatorname{Ext}^i(\widetilde{\rho}, \operatorname{Ind}_H^G(\hat{\chi})).$$

By Frobenius reciprocity

$$\operatorname{Ext}^i(\widetilde{\rho}, \operatorname{Ind}_H^G(\hat{\chi})) \cong \operatorname{Ext}_H^i(\widetilde{\rho}, \hat{\chi}).$$

Let $I := H \cap G^1$ and $J := H \cap G_0$. Note that

$$\operatorname{Ext}_H^i(\widetilde{\rho}, \hat{\chi}) \cong \operatorname{Ext}_H^i(\widetilde{\rho} \otimes \hat{\chi}^{-1}, \mathbb{C}) \cong \operatorname{Ext}_{H/I}^i((\widetilde{\rho} \otimes \hat{\chi}^{-1})_I, \mathbb{C}),$$

where the last isomorphism follows from Corollary B.1.7. Now, since H/J is finite,

$$\operatorname{Ext}_{H/I}^i((\widetilde{\rho} \otimes \hat{\chi}^{-1})_I, \mathbb{C}) \cong \operatorname{Hom}_{H/J}(\mathrm{H}_i(J/I, (\widetilde{\rho} \otimes \hat{\chi}^{-1})_I), \mathbb{C}),$$

which implies the assertion by Lemma B.3.7. $\square$

Now we are ready to prove Lemma B.3.3.

Proof of Lemma B.3.3. Clearly (M, H') is F-spherical. It remains to prove that

$$\dim \operatorname{Hom}(\operatorname{ind}_{H'}^M(\chi'), \tau) < \infty, \tag{15}$$

for any irreducible cuspidal representation τ of M.

Since $(G, (H, \chi))$ has finite multiplicity, we have $\dim \operatorname{Hom}(\operatorname{ind}_H^G(\chi), \pi) < \infty$ for any irreducible $\pi \in \mathcal{M}(G)$. Thus for any irreducible $\tau \in \mathcal{M}(M)$ we have

$$\dim \operatorname{Hom}_G(\operatorname{ind}_H^G(\chi), \bar{i}_M^G(\tau))) < \infty.$$

Thus

$$\dim \operatorname{Hom}_M(\bar{r}_M^G(\operatorname{ind}_H^G(\chi)), \tau) < \infty.$$

By Lemma B.3.2, there exists a filtration on $\bar{r}_M^G(\operatorname{ind}_H^G(\chi))$ such that $Gr^i(\bar{r}_M^G(\operatorname{ind}_H^G(\chi))) = \operatorname{ind}_{H_i}^G(\chi_i)$ where $(M, (H_i, \chi_i))$ ranges over all the descendants of $(G, (H, \chi))$ up to equivalence. In particular we can assume that for some i_0, we have $H_{i_0} = H'$, $\chi_{i_0} = \chi'$. By Lemma B.3.6, this filtration satisfies the conditions of Lemma B.3.5 and thus (15) holds. $\square$

B.4. **Finite cuspidal type.**
Let us now prove the following cuspidal analogue of theorem B.2.5

Theorem B.4.1. *If $(G, (H, \hat{\chi}))$ has finite cuspidal multiplicity, then $(G, (H, \chi))$ has finite cuspidal type.*

We will need several lemmas.

Lemma B.4.2. *Let A be a locally compact group and B be a closed subgroup. Suppose that $A = BZ(A)$. Then any character of B can be lifted to A.*

Proof. Taking quotient by the kernel of the character we reduce to the case of abelian A. In this case the statement is [Dix57, Theorem 5]. $\square$

Lemma B.4.3. *Let (G, H) be an F-spherical pair, and denote $\widetilde{H} = HZ(G) \cap G^1$. Let χ be a character of H. Suppose that for any smooth (respectively cuspidal) irreducible representation ρ of G we have*

$$\dim \operatorname{Hom}_H(\rho|_H, \chi) < \infty$$

Then for any smooth (respectively cuspidal) irreducible representation ρ of G and for every character ψ of $\widetilde{H}$ whose restriction to $H \cap G^1$ coincides with χ, we have

$$\dim \operatorname{Hom}_{\widetilde{H}}(\rho|_{\widetilde{H}}, \psi) < \infty.$$

Proof. Let ρ be a smooth (respectively cuspidal) irreducible representation of G. Using Lemma B.4.2 extend χ to a character χ' of $HZ(G)$. Let ϕ be a character of $\widetilde{H}$ whose restriction to $H \cap G^1$ is trivial. We have to show that

$$\dim \operatorname{Hom}_{\widetilde{H}}\left(\rho|_{\widetilde{H}}, \phi\chi'\right) < \infty.$$

We have

$$\operatorname{Hom}_{\widetilde{H}}\left(\rho|_{\widetilde{H}}, \phi\chi'\right) = \operatorname{Hom}_{HZ(G) \cap G_0}\left(\rho|_{(HZ(G)) \cap G_0}, \operatorname{Ind}_{\widetilde{H}}^{HZ(G) \cap G_0} \phi\chi'\right).$$

Since

$$HZ(G) \cap G_0 = \widetilde{H}Z(G) \cap G_0 = \widetilde{H}Z(G),$$

the subspace of $\operatorname{Ind}_{\widetilde{H}}^{(HZ(G)) \cap G_0} \phi\chi'$ that transforms under $Z(G)$ according to the central character of ρ is at most one dimensional. If this subspace is 0, then the lemma is clear.

Otherwise, denote it by τ. Since $H \cap G^1$ is normal in $HZ(G)$, we get that the restriction of $\mathrm{Ind}_{\widetilde{H}}^{(HZ(G)) \cap G_0} \phi$ to $H \cap G^1$ is trivial, and thus that $\tau|_{H \cap G^1} = \chi'_{H \cap G^1}$. Hence

$$\mathrm{Hom}_{\widetilde{H}} \left(\rho|_{\widetilde{H}}, \phi\chi' \right) = \mathrm{Hom}_{(HZ(G)) \cap G_0} \left(\rho|_{(HZ(G)) \cap G_0}, \tau \right) =$$
$$= \mathrm{Hom}_{H \cap G_0} \left(\rho|_{H \cap G_0}, \tau|_{H \cap G_0} \right) = \mathrm{Hom}_H \left(\rho|_H, \mathrm{Ind}_{H \cap G_0}^H \tau|_{H \cap G_0} \right).$$

Since $H/H \cap G_0$ is finite and abelian, we have

$$\mathrm{Ind}_{H \cap G_0}^H (\tau|_{H \cap G_0}) = \chi \left(\bigoplus_{i=1}^N \chi_i \right)$$

where χ_i are characters of H, s.t. $(\chi_i)|_{H \cap G^1} = 1$. By Lemma B.4.2 the characters χ_i can be extended to characters of G, because $H/(H \cap G^1)$ is a sub-lattice of G/G^1. Denoting the extensions by Θ_i, we get that

$$\mathrm{Hom}_H \left(\rho|_H, \chi\chi_i \right) = \mathrm{Hom}_H \left((\rho \otimes \Theta_i^{-1})|_H, \chi \right),$$

but $\rho \otimes \Theta_i^{-1}$ is again smooth (respectively cuspidal) irreducible representation of G, so this last space is finite-dimensional. $\qquad\square$

Lemma B.4.4. *Let A be a commutative unital Noetherian algebra without zero divisors and let K be its field of fractions. Let $K^{\mathbb{N}}$ be the space of all sequences of elements of K. Let V be a finite dimensional subspace of $K^{\mathbb{N}}$ and let $M := V \cap A^{\mathbb{N}}$. Then M is finitely generated.*

Proof. Since A does not have zero divisors, M injects into $K^{\mathbb{N}}$. There is a number n such that the projection of V to $K^{\{1,\dots n\}}$ is injective. Therefore, M injects into $A^{\{1,\dots n\}}$, and, since A is Noetherian, M is finitely generated. $\qquad\square$

Lemma B.4.5. *Let L be an l-group, and let $L' \subset L$ be an open normal subgroup of L such that L/L' is a lattice. Let ρ be a smooth representation of L of countable dimension. Suppose that for any character χ of L whose restriction to L' is trivial we have*

$$\dim \mathrm{Hom}_L(\rho, \chi) < \infty.$$

Consider $\mathrm{Hom}_L(\rho, \mathcal{S}(L/L'))$ as a representation of L, where L acts by $((hf)(x))([y]) = (f(x))([yh])$. Then this representation is finitely generated.

Proof. By assumption, the action of L on $\mathrm{Hom}_L(\rho, \mathcal{S}(L/L'))$ factors through L/L'. Since L/L' is discrete, $\mathcal{S}(L/L')$ is the group algebra $\mathbb{C}[L/L']$. We want to show that $\mathrm{Hom}_L(\rho, \mathbb{C}[L/L'])$ is a finitely generated module over $\mathbb{C}[L/L']$.

Let $\mathbb{C}(L/L')$ be the fraction field of $\mathbb{C}[L/L']$. Choosing a countable basis for the vector space of ρ, we can identify any $\mathbb{C}$-linear map from ρ to $\mathbb{C}(L/L')$ with an element of $\mathbb{C}(L/L')^{\mathbb{N}}$. Moreover, the condition that the map intertwines the action of L/L' translates into a collection of linear equations that the tuple in $\mathbb{C}(L/L')^{\mathbb{N}}$ should satisfy. Hence, $\mathrm{Hom}_{L'}(\rho, \mathbb{C}[L/L'])$ is the intersection of the $\mathbb{C}(L/L')$-vector space $\mathrm{Hom}_L(\rho, \mathbb{C}(L/L'))$ and $\mathbb{C}[L/L']^{\mathbb{N}}$. By Lemma B.4.4, it suffices to prove that $\mathrm{Hom}_L(\rho, \mathbb{C}(L/L'))$ is finite dimensional over $\mathbb{C}(L/L')$.

Since L is separable, and ρ is smooth and of countable dimension, there are only countably many linear equations defining $\mathrm{Hom}_L(\rho, \mathbb{C}(L/L'))$; denote them by $\phi_1, \phi_2, \dots \in \left(\mathbb{C}(L/L')^{\mathbb{N}} \right)^*$. Choose a countable subfield $K \subset \mathbb{C}$ that contains all the coefficients of the elements of $\mathbb{C}(L/L')$ that appear in any of the ϕ_i's. If we define W as the $K(L/L')$-linear subspace of $K(L/L')^{\mathbb{N}}$ defined by the ϕ_i's, then $\mathrm{Hom}_L(\rho, \mathbb{C}(L/L')) = W \otimes_{K(L/L')} \mathbb{C}(L/L')$, so $\dim_{\mathbb{C}(L/L')} \mathrm{Hom}_L(\rho, \mathbb{C}(L/L')) = \dim_{K(L/L')} W$.

Since L/L' is a lattice generated by, say, $g_1, \dots, g_n$, we get that $K(L/L') = K(t_1^{\pm 1}, \dots, t_n^{\pm 1})$ $= K(t_1, \dots, t_n)$. Choosing elements $\pi_1, \dots, \pi_n \in \mathbb{C}$ such that $tr.deg_K(K(\pi_1, \dots, \pi_n)) = n$, we get an injection ι of $K(L/L')$ into $\mathbb{C}$. As before, we get that if we denote the $\mathbb{C}$-vector subspace of $\mathbb{C}^{\mathbb{N}}$ cut by the equations $\iota(\phi_i)$ by U, then $\dim_{K(L/L')} W = \dim_{\mathbb{C}} U$. However,

U is isomorphic to $\mathrm{Hom}_L(\rho, \chi)$, where χ is the character of L/L' such that $\chi(g_i) = \pi_i$. By assumption, this last vector space is finite dimensional. $\square$

Now we are ready to prove Theorem B.4.1.

Proof of Theorem B.4.1. Let ρ be an irreducible cuspidal representation of G. By Corollary B.2.3 we know that $\dim \mathrm{Hom}_H(\rho, \chi) < \infty$. We need to show that $\mathrm{Hom}_G(\Psi(\rho), \mathrm{ind}_H^G \chi)$ is finitely generated over $\mathcal{O}(\Psi_G)$. We have

$$\mathrm{Hom}_G(\Psi(\rho), \mathrm{ind}_H^G \chi) = \mathrm{Hom}_{G^1}(\rho, \mathrm{ind}_H^G \chi).$$

Here we consider the space $\Phi := \mathrm{Hom}_{G^1}(\rho, \mathrm{ind}_H^G \chi)$ with the natural action of G. Note that G^1 acts trivially and hence this action gives rise to an action of G/G^1, which gives the $\mathcal{O}(\Psi_G)$ - module structure. Let $\Theta := \mathrm{Hom}_{G^1}(\rho, \mathrm{Ind}_{HZ(G)}^G \mathrm{ind}_H^{HZ(G)} \chi)$. Clearly $\Phi \subset \Theta$. Thus, by Lemma B.1.10, it is enough to show that Θ is finitely generated over G.

Denote $H' := H \cap G^1$ and $H'' := (HZ(G)) \cap G^1$. Consider the subspace

$$V := \mathrm{Hom}_{G^1}(\rho, \mathrm{Ind}_{H''}^{G^1}(\mathrm{ind}_{H'}^{H''}(\chi|_{H'}))) \subset \Theta.$$

It generates Θ as a representation of G, and therefore also as an $\mathcal{O}(\Psi_G)$ - module. Note that V is H-invariant. Therefore it is enough to show that V is finitely generated over H.

By Frobenius reciprocity we have $V \cong \mathrm{Hom}_{H''}(\rho, \mathrm{ind}_{H'}^{H''}(\chi|_{H'}))$.

By Lemma B.4.2 χ can be extended to a character χ' of $HZ(G)$. Thus

$$\mathrm{ind}_{H'}^{H''}(\chi|_{H'}) \cong \chi'|_{H''} \mathcal{S}(H''/H').$$

Let $\rho' := \chi'|_H^{-1} \rho|_H$. Then

$$V \cong \mathrm{Hom}_{H''}(\rho', \mathcal{S}(H''/H')).$$

Under this isomorphism, the action Π of H on V is given by

$$((\Pi(h)(f))(v))([k]) = f(\rho'(h^{-1})v)([h^{-1}kh]),$$

where $h \in H$, $f \in \mathrm{Hom}_{H''}(\rho', \mathcal{S}(H''/H'))$, $v \in \rho'$, $k \in H''$, $[k] = kH' \in H''/H'$.

Let Ξ be the action of H'' on $\mathrm{Hom}_{H''}(\rho', \mathcal{S}(H''/H'))$ as described in Lemma B.4.5, i.e.

$$((\Xi(h)(f))(v))([k]) = f(v)([kh]).$$

Let us show that $\mathrm{Hom}_{H''}(\rho', \mathcal{S}(H''/H'))$ is finitely generated w.r.t. the action Ξ. By Lemma B.4.5 it is enough to show that

$$(16) \qquad\qquad \dim \mathrm{Hom}_{H''}(\rho', \theta) < \infty$$

for any character θ of H'' with trivial restriction to H'. Note that $\mathrm{Hom}_{H''}(\rho', \theta) \cong \mathrm{Hom}_{H''}(\rho, \chi'\theta)$. Thus (16) follows from the hypothesis $\dim \mathrm{Hom}_H(\rho, \chi) < \infty$ in view of Lemma B.4.3 and we have shown that $\mathrm{Hom}_{H''}(\rho', \mathcal{S}(H''/H'))$ is finitely generated w.r.t. the action Ξ.

Now it is enough to show that for any $h \in H''$ there exist an $h' \in H$ and a scalar α s.t.

$$\Xi(h) = \alpha \Pi(h').$$

In order to show this let us decompose h to a product $h = zh'$ where $h' \in H$ and $z \in Z(G)$. Now

$$((\Xi(h)(f))(v))([k]) = f(v)([kh]) = f(h^{-1}v)([h^{-1}kh]) = f(h'^{-1}z^{-1}v)([h'^{-1}kh']) =$$
$$= \alpha f(h'^{-1}v)([h'^{-1}kh']) = \alpha((\Pi(h')(f))(v))([k]),$$

where α is the scalar with which z^{-1} acts on ρ'.

Thus V is finitely generated over H, thus Φ and Θ are finitely generated over G and $\mathrm{Hom}_G(\Psi(\rho), \mathrm{ind}_H^G \chi)$ is finitely generated over $\mathcal{O}(\Psi_G)$. $\square$

Corollary B.4.6. *If $(G, (H, \chi))$ has finite cuspidal multiplicity, then $(G, (H, \chi))$ has finite cuspidal type.*

Proof. Assume that $(G, (H, \chi))$ has finite cuspidal multiplicity. By Theorem B.4.1 the twisted pair $(G, (H, \hat{\chi}))$ has finite cuspidal type. By Lemma B.2.4 the pair $(G, (H, \hat{\chi}))$ has finite cuspidal multiplicity. Applying Theorem B.4.1 again we obtain that $(G, (H, \chi))$ has finite cuspidal type. □

B.5. **Proof of Theorem B.2.5.**

Let $\mathbf{P} < \mathbf{G}$ be a parabolic subgroup and $\mathbf{M}$ be the Levi quotient of $\mathbf{P}$. Let ρ be a cuspidal representation of M. We have to show that $\mathrm{Hom}(i_M^G(\Psi(\rho)), \mathrm{ind}_H^G(\chi))$ is finitely generated over $\mathcal{O}(\Psi_M)$. By second adjointness theorem (Theorem B.1.8), we have

$$\mathrm{Hom}(i_M^G(\Psi(\rho)), \mathrm{ind}_H^G(\chi)) = \mathrm{Hom}((\Psi(\rho)), \bar{r}_M^G(\mathrm{ind}_H^G(\chi))).$$

By Lemma B.3.2 the representation $\bar{r}_M^G(\mathrm{ind}_H^G(\chi))$ has a filtration s.t.

$$Gr_i(\bar{r}_M^G(\mathrm{ind}_H^G(\chi))) = \mathrm{ind}_{H_i}^M(\chi_i)$$

where $(M, (H_i, \chi_i))$ are the descendants of $(G, (H, \chi))$. Since $i_M^G(\Psi(\rho))$ is a projective object (Theorem B.1.8), this gives us filtration on $\mathrm{Hom}((\Psi(\rho)), \bar{r}_M^G(\mathrm{ind}_H^G(\chi)))$ with

$$Gr_i \mathrm{Hom}((\Psi(\rho)), \bar{r}_M^G(\mathrm{ind}_H^G(\chi))) = \mathrm{Hom}((\Psi(\rho)), \mathrm{ind}_{H_i}^M(\chi_i)).$$

So it remains to show that $(M, (H_i, \chi_i))$ are of finite cuspidal type. This follows from Lemma B.3.3 and Corollary B.4.6.

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