MULTIPLICITY ONE THEOREMS IN REPRESENTATION THEORY

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ABSTRACT. An important question of Representation Theory is "what happens to an irreducible representation of a group when one restricts it to a subgroup?". Usually it stops being irreducible and it is interesting to know whether it decomposes to distinct irreducible representations.

This report contains an exposition on this question and a short description of a recent proof that in the case of the pair (GL_{n+1}, GL_n) over local fields the answer is positive.

The authors would like to mention that they were guided in this project by Eitan Sayag and Joseph Bernstein.

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1. REPRESENTATION THEORY OF FINITE GROUPS

Among most important notions in mathematics are the notions of a group and of a linear space. Representation theory is an area of mathematics that combines these notions.

A *representation* π of a group G is a linear action of G on a linear space V. This means that to any element g we attach a linear operator $\pi(g)$ from V to V such that the operator $\pi(gh)$ attached to the product gh is equal to the composition $\pi(g) \circ \pi(h)$. If V is n-dimensional linear space, operators can be viewed as $n \times n$ matrices and composition of operators as product of matrices.

Let us first discuss the case when the group G is finite.

Examples. *Let S_n denote the group of permutations of $\{1...n\}$. This group has $n!$ elements.*

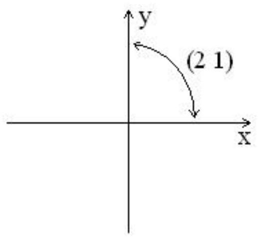
(1) *S_2 acts on $\mathbb{R}^2$ permuting the coordinates. Namely, for the trivial permutation $(1\,2)$ we attach the matrix*

$$\pi(1\,2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and to the non-trivial permutation $(2\,1)$ we attach the matrix

$$\pi(2\,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Date: December 26, 2008.



(2) *S_3 acts on $\mathbb{R}^3$ permuting the coordinates. Namely,*

$$\pi(1\,2\,3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \pi(1\,3\,2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \pi(2\,1\,3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

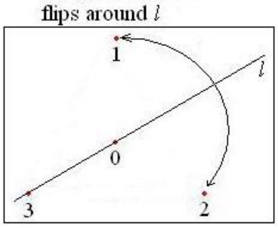
$$\pi(2\,3\,1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \pi(3\,1\,2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \pi(3\,2\,1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The main task of representation theory is to classify all the representations of a given group. The first notion that was introduced for dealing with this task was the notion of *irreducible representation.* Irreducible representations are the building blocks for all representations. A representation is called irreducible if it has no subrepresentations except 0 and itself.

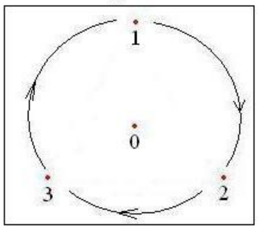
Examples.

(1) *Any representation on a one-dimensional space is an irreducible representation.*
(2) *Consider the 3-dimensional representation of S_3 described before. Consider the plain in $\mathbb{R}^3$ defined by $\{x+y+z = 0\}$. This is a subrepresentation and it is irreducible. One can view this representation in the following way: consider an equilateral triangle in the plain with center at 0. Let S_3 act on it by permuting the vertices. This action extends to a linear action on the whole plain.*

The permutation (2 1 3) flips around l

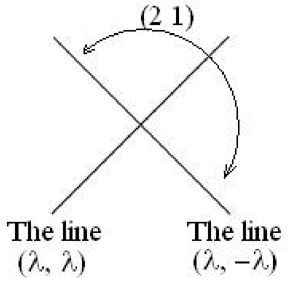


The permutation (2 3 1) rotates by 120°



A fundamental theorem in representation theory of finite groups says that every irreducible representation of a finite group is finite dimensional and every representation of a finite group decomposes to a direct sum of irreducible ones.

Example. *The action of S_2 on $\mathbb{R}^2$ by permuting the coordinates decomposes to the direct sum of the line (λ, λ) and the line $(\lambda, -\lambda)$.*



Because of this theorem, the task of representation theory of finite groups divides into two parts:

- Classify all irreducible representations of a given group G.
- Understand how a given representation of G decomposes into irreducible ones.

For a representation π and an irreducible representation τ one can ask with what multiplicity τ occurs in the decomposition of π to irreducible representations. We denote this number by $\langle \pi, \tau \rangle$ and call it the *intertwining number*. If τ does not occur in the decomposition of π then $\langle \pi, \tau \rangle = 0$.

It is interesting to know what happens if we consider an irreducible representation of G as a representation of a subgroup $H \subset G$. Consider its decomposition to a direct sum of irreducible representations of H. It is said that the pair (G, H) has *multiplicity one* property if all those irreducible representations appear in the decomposition with multiplicity one. More precisely, the pair (G, H) has multiplicity one property if for any irreducible representations π of G and τ of H, $\langle \pi|_H, \tau \rangle \leq 1$, where $\pi|_H$ means π considered as a representation of H. Such pairs are also called *strong Gelfand pairs*.

In the 1950s the following trick for proving this property was introduced by I.M. Gelfand. Suppose that there exists a map $T : G \to G$ such that for any g in G we have

(1) $T(T(g)) = g$
(2) for any g' in G: $T(gg') = T(g')T(g)$
(3) there exists h in H such that $T(g) = hgh^{-1}$.

Then the pair (G, H) has multiplicity one property.

Example. *One can show that the pair (S_{n+1}, S_n) has multiplicity one property using Gelfand trick and taking T to be the inversion: $T(g) = g^{-1}$.*

The condition (3) can be rephrased in the following way.

Consider the space of functions from G to $\mathbb{R}$. The group H acts on this space in the following way: $\pi(h)(\delta_g) = \delta_{hgh^{-1}}$, where δ_g is the function defined by

$$\delta_g(g') = \begin{cases} 1, & \text{if } g = g'; \\ 0, & \text{if } g \neq g'. \end{cases}$$

A function f is called H*-invariant* if $\pi(h)f = f$ for any h in H.

Now the condition (3) is equivalent to the condition

(3') any H-invariant function on G is T-invariant.

2. Representation theory of infinite groups

Now let us go on to infinite groups. When we consider infinite groups we want them to have some geometric structure. Usually people consider topological groups. One can think of a topological group as a group that is also a geometric object. For example: a strait line, a circle, a torus (i.e. the surface of a bagel).

A nice class of topological groups are compact groups. Informally, a topological group is compact if it is bounded as a figure. For example, the circle is compact while the strait line is not.

The representation theory of compact groups is similar to the representation theory of finite groups. In particular all irreducible representations are finite dimensional and all representations decompose to a direct sum of irreducible ones. Also, questions of multiplicity one behave similarly and in particular Gelfand trick works.

Example. *Let $O_n(\mathbb{R})$ denote the group of orthogonal $n \times n$ matrices with real entries. This is a compact group. Using Gelfand trick with the map $T(g) = g^{-1}$ one can show that the pair $(O_{n+1}(\mathbb{R}), O_n(\mathbb{R}))$ has multiplicity one property.*

However, with non-compact groups the situation is much more difficult.

We will restrict ourselves to *reductive groups* over *local fields*. The best example of a reductive group is the group GL_n of invertible $n \times n$ matrices. An example of a local field is the field $\mathbb{R}$ of real numbers. Other examples are the fields $\mathbb{Q}_p$ of $p - adic$ numbers. We will not define them here, but they behave similarly to the field $\mathbb{R}$ and in fact in many aspects they are much simpler to work with. A reductive group over a local field F is a reductive group with parameters from the field F. For example, the group $GL_n(F)$ is the group of invertible $n \times n$ matrices with entries from F.

The first problem we encounter with non-compact groups is that irreducible representations may be infinite dimensional.

The second problem is that a representation does not always decompose to irreducible ones. Still, it is possible to define the intertwining numbers $\langle \pi, \tau \rangle$.

The third problem that we encounter is with Gelfand trick. We do not have it literally, but we have a modification by I.M. Gelfand and D. Kazhdan. We still need a map $T : G \to G$ which satisfies the conditions (1) and (2). But in condition (3') we will have to replace functions by "generalized functions", or *distributions.* So, we need condition

(3") any H-invariant distribution on G is T-invariant.

Let us now explain what a distribution is.

3. Distributions

The first distribution considered in history was Dirac's δ-function. It was introduced by the physicists P. Dirac in the 1920s. According to his definition, it is

"a function equal to zero outside 0, but the value at 0 is infinite and such that

$$\int_{-\infty}^{\infty} \delta_0 dx = 1".$$

He said also that for any continuous function ϕ,

$$\int_{-\infty}^{\infty} \delta_0(x)\phi(x)dx = \phi(0).$$

The mathematicians, of course, said that it is impossible. Really, if a function is zero at every point except one then its integral is zero. Still, physicists have performed meaningful computations using δ_0 and obtained important results.

Finally, mathematicians have obtained a rigorous definition of a kind of object that δ_0 could be. They called it *distribution.* They defined a distribution to be a continuous functional on the space of smooth functions which are equal to zero outside an interval. In more simple words, it is something that assigns a number to every smooth function which is equal to zero outside an interval.

Examples.

(1) $\delta_0(\phi) = \phi(0)$
(2) *Any continuous function f can become a distribution by*

$$f(\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)dx.$$

4. Back to multiplicity one

Let F be a local field (for example $F = \mathbb{R}$). Consider the group $GL_n(F)$ of invertible $n \times n$ matrices with entries from F.

In the late 1980s J. Bernstein and S. Rallis conjectured that the pair $(GL_{n+1}(F), GL_n(F))$ has multiplicity one property.

In summer 2007 this was proven by A. Aizenbud, D. Gourevitch, S. Rallis and G. Schiffmann for p-adic fields.

In summer 2008 this was proven for $F = \mathbb{R}$ and $F = \mathbb{C}$ by A. Aizenbud and D. Gourevitch and independently by B. Sun and C.B. Zhu, thus completing the proof for all local fields (of characteristic zero).

A. Aizenbud and D. Gourevitch are PhD students at the Weizmann Institute of Science and were visitors at the Max Planck Institute for Mathematics and the Hausdorff Center for Mathematics in both summers.

5. A couple of words about the proof

By Gelfand - Kazhdan criterion it is enough to prove that every $GL_n(F)$-invariant distribution ξ on $GL_{n+1}(F)$ is transposition invariant.

The proof of this was complicated by the following fact: not for every matrix g in $GL_{n+1}(F)$ there exists a matrix h in $GL_n(F)$ such that $hgh^{-1} = g^t$.

However, it exists for most matrices g. More precisely, such g form an open dense set. This implies that every continuous $GL_n(F)$-invariant function on $GL_{n+1}(F)$ is transposition invariant. However, this does not imply that every $GL_n(F)$-invariant distribution on $GL_{n+1}(F)$ is transposition invariant.

Still, one can use this fact to show that every "bad" distribution ξ (i.e. a distribution ξ which is $GL_n(F)$-invariant but not transposition invariant) "lives" on a certain small closed subset S of $GL_{n+1}(F)$. Now one has to prove that there are no "bad" distributions that "live" on S.

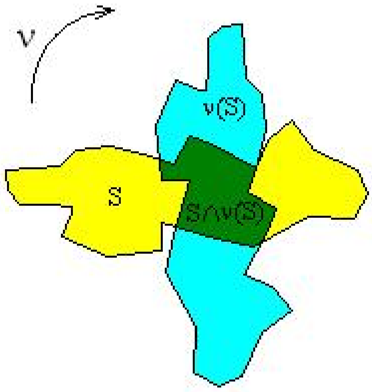
There are several ways to fight distributions that "live" on small subsets. Most of them are based on Fourier transform. Let us consider Fourier transform as a black box that takes a distribution on a vector space and produces a new distribution on the same space.

An important phenomenon connected to Fourier transform is the so-called "uncertainty principle". It says that if a distribution "lives" on a small set then its Fourier transform "lives" on a large set. For example, the Dirac's δ_0 "lives" on the point 0, while its Fourier transform is the constant function 1, which "lives" on the whole real line.

Due to the uncertainty principle there are very few distributions that "live" on S together with their Fourier transform. Since Fourier transform of a "bad" distribution is also "bad", it gives us a very strong condition on "bad" distributions. In some similar problems this condition is already enough to show that there are no "bad" distributions.

Unfortunately, in the current case this condition was not sufficient and this was the reason that the problem was stuck for so many years. However, this condition implied that every "bad" distribution "lives" on a much smaller set.

Finally, the problem was solved using the following simple minded observation. Suppose we have an invertible transformation $\nu : GL_{n+1}(F) \to GL_{n+1}(F)$ which preserves the situation, i.e. $\nu(g^t) = \nu(g)^t$ for every g in $GL_{n+1}(F)$ and $\nu(hgh^{-1}) = h\nu(g)h^{-1}$ for every g in $GL_{n+1}(F)$ and h in $GL_n(F)$. Then ν maps every "bad" distribution to a "bad" distribution. Therefore every "bad" distribution "lives" on the intersection of S and $\nu(S)$.



Iterative use of this observation and Fourier transform completed the proof.