

VANISHING OF CERTAIN EQUIVARIANT DISTRIBUTIONS ON SPHERICAL SPACES

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ABSTRACT. We prove vanishing of \mathfrak{z} -eigen distributions on a split real reductive group which change according to a non-degenerate character under the left action of the unipotent radical of the Borel subgroup, and are equivariant under the right action of a spherical subgroup.

This is a generalization of a result by Shalika, that concerned the group case. Shalika's result was crucial in the proof of his multiplicity one theorem. We view our result as a step in the study of multiplicities of quasi-regular representations on spherical varieties.

As an application we prove non-vanishing of spherical Bessel functions.

1. INTRODUCTION

1.1. Main results. In this paper we prove the following generalization of Shalika's result [Sha74, §2, Proposition 2.10].

Theorem A. *Let G be a split real reductive group and H be its spherical subgroup. Let U be the unipotent radical of a Borel subgroup B of G . Let ψ be a non-degenerate character of U and χ be a character of H . Let Z be the complement to the union of open $B \times H$ -double cosets in G . Let \mathfrak{z} be the center of the universal enveloping algebra of the Lie algebra \mathfrak{g} of G .*

Then there are no non-zero \mathfrak{z} -eigen $(U \times H, \psi \times \chi)$ -equivariant distributions supported on Z .

This result in the group case ([Sha74, §2, Proposition 2.10]) was crucial in the proof of Shalika's multiplicity one theorem.

Our proof begins by applying the technique used by Shalika. However, this technique was not enough for this generality and we had to complement it by using integrability of the singular support, as in [AG09].

Theorem A provides a new tool for the study of the multiplicities of the irreducible quotients of the quasi-regular representation of G on Schwartz functions on G/H , see §§1.4 below for more details.

1.2. Non-vanishing of spherical Bessel functions. Another application of Theorem A is to the study of spherical Bessel distributions and functions.

Definition 1.2.1. *Let G be a split real reductive group, and $H \subset G$ be a spherical subgroup. Let (π, V) be a (smooth) irreducible admissible representation of G . Let ϕ be an (H, χ) -equivariant continuous functional on V and v be a (U, ψ) -equivariant continuous functional on the contragredient representation \tilde{V} . Define the spherical Bessel distribution by*

$$\xi_{v, \phi}(f) := \langle v, \pi^*(f)\phi \rangle.$$

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Let Z be the complement to the union of open $B \times H$ -double cosets in G . Define the spherical Bessel function to be the restriction $j_{v,\phi} := \xi_{v,\phi}|_{G-Z}$.

It is well-known that $j_{v,\phi}$ is a smooth function. One can also deduce this from Lemma 3.2.3 below, since it implies that the singular support of $j_{v,\phi}$ is the zero section, which implies (by [Hör90, Theorem 8.5]) that $j_{v,\phi}$ is smooth.

Theorem A easily implies the following corollary.

Corollary B. *Suppose that v and ϕ are non-zero. Then $j_{v,\phi} \neq 0$.*

1.3. Non-archimedean analogs.

Over non-archimedean fields, the universal enveloping algebra does not act on the distributions. However, the Bernstein's center $\text{End}_{G \times G}(\mathcal{S}(G))$ does act. In [AGS] we study this action in details. In [AGK] we prove, using [AGS, Theorem A], analogs of Theorem A and Corollary B for non-archimedean fields of characteristic zero. These analogs are somewhat weaker for general spherical pairs, but are of the same strength for the group case and for Galois symmetric pairs. The group case of the non-archimedean counterpart of Corollary B was proven before in [LM, Appendix A].

1.4. Relation with multiplicities in regular representations of symmetric spaces.

Let (G, H) be a symmetric pair of real reductive groups. Suppose that G is quasi-split and let $B \subset G$ be a Borel subgroup. Let k be the number of open B -orbits on G/H .

Theorem A can be used to study the following conjecture.

Conjecture C. *Let (π, V) be a (smooth) irreducible admissible representation of G . Then the dimension of the space $(V^*)^H$ of H -invariant continuous functionals on V is at most k . In particular, any complex reductive symmetric pair is a Gelfand pair.*

We suggest to divide this conjecture into two cases

- π is non-degenerate, i.e. π has a non-zero continuous (U, ψ) -equivariant functional for some non-degenerate character ψ of U .
- π is degenerate.

In the first case, Conjecture C follows from the next one.

Conjecture D. *Let U be the unipotent radical of B and let ψ be a non-degenerate character of U . Let \mathfrak{z} be the center of the universal enveloping algebra of the Lie algebra \mathfrak{g} of G . Let λ be a character of \mathfrak{z} .*

Then the dimension of the space of (\mathfrak{z}, λ) -eigen (U, ψ) -equivariant distributions on G/H does not exceed k .

We believe that Theorem A can be useful in approaching this conjecture, since it allows to reduce the study of distributions to the union of open B -orbits.

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2. PRELIMINARIES

2.1. Conventions.

- By an algebraic manifold we mean a smooth real algebraic variety.
- We will use capital Latin letters to denote Lie groups and the corresponding Gothic letters to denote their Lie algebras.

- Let a Lie group G act on a smooth manifold M . For a vector $v \in \mathfrak{g}$ and a point $x \in M$ we will denote by $vx \in T_x M$ the image of v under the differential of the action map $g \mapsto gx$. Similarly, we will use the notation $\mathfrak{h}x$, for any subspace $\mathfrak{h} \subset \mathfrak{g}$.
- We denote by G_x the stabilizer of x in G and by \mathfrak{g}_x its Lie algebra.

2.2. Tangential and transversal differential operators. In this subsection we shortly review the method of [Sha74, §2]. For a more detailed description see [JSZ11, §§2.1].

Definition 2.2.1. *Let M be a smooth manifold and N be a smooth submanifold.*

- A vector field v on M is called *tangential to N* if for every point $p \in N$, $v_p \in T_p N$ and *transversal to N* if for every point $p \in N$, $v_p \notin T_p N$.
- A differential operator D is called *tangential to N* if every point $p \in N$ has an open neighborhood $U_x \subset N$ such that $D|_{U_x}$ is a finite sum of differential operators of the form $\phi V_1 \cdots V_r$, where ϕ is a smooth function on U_x , $r \geq 0$, and V_i are vector fields on U_x tangential to $U_x \cap N$.

Lemma 2.2.2 (cf. the proof of [Sha74, Proposition 2.10]). *Let M be a smooth manifold and N be a smooth submanifold. Let D be a differential operator on M tangential to N and V be a vector field on M transversal to N . Let η be a distribution on M supported in N such that $D\eta = V\eta$. Then $\eta = 0$.*

2.3. Singular support. Let M be an algebraic manifold and η be a distribution on M . The *singular support* of η is defined to be the singular support of the D-module generated by η and denoted $SS(\eta) \subset T^*M$.

We will shortly review the properties of the singular support that are most important for this paper. For more detailed overview we refer the reader to [AG09, §§2.3 and Appendix B].

Notation 2.3.1. *For a point $x \in M$*

- we denote by $SS_x(\eta)$ the fiber of x under the natural projection $SS(\eta) \rightarrow M$,
- for a submanifold $N \subset M$ we denote by $CN_N^M \subset T^*M$ the conormal bundle to N in M , and by $CN_{N,x}^M$ the conormal space at x to N in M .

Lemma 2.3.2 (See e.g. [AG09, Fact 2.3.9 and Appendix B]). *Let an algebraic group G act on M . Suppose that η is G -equivariant. Then*

$$SS(\eta) \subset \{(x, \phi) \in T^*M \mid \forall \alpha \in \mathfrak{g}, \phi(\alpha(x)) = 0\} = \bigcup_{x \in M} CN_{Gx}^M.$$

We will need the following well-known lemma

Lemma 2.3.3. *Let G be a real reductive group, $\mathcal{N} \subset \mathfrak{g}^*$ be the nilpotent cone and \mathfrak{z} be the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Let ξ be a \mathfrak{z} -eigen distribution on G . Identify T^*G with $G \times \mathfrak{g}^*$ using the right action. Then $SS(\xi) \subset G \times \mathcal{N}$.*

Theorem 2.3.4 (Integrability theorem, cf. [Gab81, GQS71, KKS73, Mal78]). *The $SS(\eta)$ is a coisotropic subvariety of T^*M .*

This theorem implies the following corollary (see [Aiz13, §3] for more details).

Corollary 2.3.5. *Let $N \subset M$ be a closed algebraic submanifold. Suppose that ξ is supported in N . Suppose further that for any $x \in N$, we have $CN_{N,x}^M \not\subset SS_x(\eta)$. Then $\eta = 0$.*

3. PROOF OF THEOREM A

3.1. Sketch of the proof. We decompose G into $B \times H$ -double cosets. Each double coset \mathcal{O} we decompose $\mathcal{O} = \mathcal{O}_s \cup \mathcal{O}_c$ in a certain way. We prove the required vanishing coset by coset, using Shalika's method (see §§2.2) for \mathcal{O}_s and singular support analysis (see §§2.3) for \mathcal{O}_c .

3.2. Notation and lemmas.

Notation 3.2.1.

- Fix a maximal torus $T \subset B$ and let $\mathfrak{t} \subset \mathfrak{b}$ denote the corresponding Lie algebras. Let Φ denote the root system, Φ^+ denote the set of positive roots and $\Delta \subset \Phi^+$ denote the set of simple roots. For $\alpha \in \Phi$ let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be the corresponding root space.
- Let $C \in \mathfrak{z}$ denote the Casimir element.
- We choose $E_\alpha \in \mathfrak{g}_\alpha$, for any $\alpha \in \Phi$ such that $C = \sum_{\alpha \in \Phi^+} E_{-\alpha} E_\alpha + D$, where D is in the universal enveloping algebra of the Cartan subalgebra \mathfrak{t} .
- Let $V := \sum_{\alpha \in \Delta} d\psi(E_\alpha) E_{-\alpha} \in \mathfrak{g}$. Note that $C\xi = (V + D)\xi$ for any (U, ψ) -equivariant distribution ξ on G .
- Let $\mathcal{O} \subset G$ be a stable $B \times H$ -double coset, i.e the set of real points of the complexification of a double coset. Consider the left action of G on itself and define

$$\mathcal{O}_c := \{x \in \mathcal{O} \mid Vx \in T_x \mathcal{O}\} \text{ and } \mathcal{O}_s := \mathcal{O} \setminus \mathcal{O}_c.$$

Remark 3.2.2. By definition \mathcal{O}_c is a Zariski closed real algebraic subvariety of \mathcal{O} , as such we can stratify it by smooth locally closed real algebraic subvarieties of \mathcal{O} .

We will need the following lemmas, that will be proved in subsequent subsections.

Lemma 3.2.3. Let $x \in G$. Let ξ be a \mathfrak{z} -eigen $(U \times H, \psi \times \chi)$ equivariant distribution on G . Then $SS_x(\xi) \subset CN_{BxH, x}$.

Lemma 3.2.4. Let $\mathcal{O} \subset Z$ be a $B \times H$ -double coset. Then $\mathcal{O}_s \neq \emptyset$.

3.3. Proof of Theorem A. Suppose that there exists a non-zero \mathfrak{z} -eigen (U, ψ) -equivariant distribution ξ supported on Z .

For any stable $B \times H$ -double coset $\mathcal{O} \subset G$, stratify \mathcal{O}_c to a union of smooth locally closed varieties \mathcal{O}_c^i .

The collection

$$\{\mathcal{O}_c^i \mid \mathcal{O} \text{ is a } B \times H \text{ - double coset}\} \cup \{\mathcal{O}_s \mid \mathcal{O} \text{ is a } B \times H \text{ - double coset}\}$$

is a stratification of G . Reorder this collection to a sequence $\{S_i\}_{i=1}^N$ of smooth locally closed subvarieties of G such that $U_k := \bigcup_{i=1}^k S_i$ is open in G for any $1 \leq k \leq N$.

Let k be the maximal integer such that $\xi|_{U_{k-1}} = 0$. Let $\eta := \xi|_{U_k}$. We will now show that $\eta = 0$, which leads to a contradiction.

Case 1. $S_k = \mathcal{O}_s$ for some double coset \mathcal{O} .

Recall that we have the following decomposition of the Casimir element

$$C = \sum_{\alpha \in \Phi^+} E_{-\alpha} E_\alpha + D$$

Since η is \mathfrak{z} -eigen and (U, ψ) -equivariant, we have, for some scalar λ ,

$$\lambda\eta = C\eta = \sum_{\alpha \in \Phi^+} E_{-\alpha} E_\alpha \eta + D\eta = \sum_{\alpha \in \Phi^+} E_{-\alpha} d\psi(E_\alpha) \eta + D\eta = V\eta + D\eta$$

Let $D' := \lambda Id - D$. We have $V\eta = D'\eta$, and it is easy to see that D' is tangential to \mathcal{O}_s , and V is transversal to \mathcal{O}_s . Now, Lemma 2.2.2 implies $\eta = 0$ which is a contradiction.

Case 2. $S_k \subset \mathcal{O}_c$ for some orbit \mathcal{O} .

By Corollary 2.3.5 it is enough to show that for any $x \in S_k$ we have

$$(1) \quad CN_{S_k, x}^G \not\subseteq SS_x(\eta).$$

By Lemma 3.2.3, $SS_x(\eta) \subset CN_{\mathcal{O}, x}^G$. By Lemma 3.2.4, $S_k \subsetneq \mathcal{O}$, thus $CN_{S_k, x}^G \not\supseteq CN_{\mathcal{O}, x}^G$ which implies (1). □

3.4. Proof of Lemma 3.2.3.

Proof. Let \mathfrak{h} denote the Lie algebra of H and $ad(x)\mathfrak{h}$ denote its conjugation by x . Identify T_x^*G with \mathfrak{g}^* using multiplication by x^{-1} on the right. Then

$$CN_{BxH,x}^G = (\mathfrak{t} + \mathfrak{u} + ad(x)\mathfrak{h})^\perp.$$

Since ξ is $\mathfrak{u} \times \mathfrak{h}$ -equivariant, Lemma 2.3.2 implies that $SS_x(\xi) \subset (\mathfrak{u} + ad(x)\mathfrak{h})^\perp$. Since ξ is also \mathfrak{z} -eigen, Lemma 2.3.3 implies that $SS_x(\xi) \subset \mathcal{N}$, where $\mathcal{N} \subset \mathfrak{g}^*$ is the nilpotent cone. Now we have

$$SS_x(\xi) \subset (\mathfrak{u} + ad(x)\mathfrak{h})^\perp \cap \mathcal{N} = (\mathfrak{t} + \mathfrak{u} + ad(x)\mathfrak{h})^\perp = CN_{BxH,x}^G.$$

□

3.5. Proof of Lemma 3.2.4.

First we need the following lemmas and notation.

Lemma 3.5.1. *Let $K \subset K_i \subset G$ for $i = 1, \dots, n$ be algebraic subgroups. Suppose that K_i generate G . Let Y be a transitive G space. Let $y \in Y$. Assume that Ky is Zariski dense in $K_i y$ for each i . Then Ky is Zariski dense in Y .*

Proof. By induction we may assume that $n = 2$. Let

$$O_l := \underbrace{K_1 K_2 \cdots K_1 K_2}_{l \text{ times}} y.$$

It is enough to prove that for any l the orbit Ky is dense in O_l . Let us prove it by induction on l . Suppose that we have already proven that Ky is dense in O_{l-1} . Then

$$\overline{Ky} = \overline{K_2 y} = \overline{K_2 Ky} = \overline{K_2 O_{l-1} y}.$$

Thus Ky is dense in $K_2 O_{l-1}$. Similarly, $K_2 O_{l-1}$ is dense in $K_1 K_2 O_{l-1} = O_l$. □

Notation 3.5.2.

- Let Y denote the spherical space G/H , and Z' denote the image of Z in Y .
- For a simple root $\alpha \in \Delta$, denote by $P_\alpha \subset G$ the parabolic subgroup whose Lie algebra is $\mathfrak{g}_{-\alpha} \oplus \mathfrak{b}$.

Lemma 3.5.3. *Let $x \in Z'$. Then there exists a simple root $\alpha \in \Delta$ such that $\mathfrak{g}_{-\alpha} x \not\subset \mathfrak{b}x$.*

Proof. Assume the contrary. Then for any $\alpha \in \Delta$, $T_x P_\alpha x = T_x Bx$. Thus Bx is Zariski dense in $P_\alpha x$. By Lemma 3.5.1 this implies that Bx is dense in Y , which contradicts the condition $x \in Z'$. □

Proof of Lemma 3.2.4. Note that \mathcal{O}_c is invariant with respect to the right action of H . Let \mathcal{O}' denote the image of \mathcal{O} in Y , and let \mathcal{O}'_c denote the image of \mathcal{O}_c in Y . Choose $x \in \mathcal{O}'$ and let $a : B \rightarrow \mathcal{O}'$ denote the action map. It is enough to show that $a^{-1}(\mathcal{O}'_c) \neq B$.

$$\begin{aligned} a^{-1}(\mathcal{O}'_c) &= \{b \in B \mid \sum \psi(E_\alpha) E_{-\alpha} \in \mathfrak{g}_{bx} + \mathfrak{b}\} = \{b \in B \mid \sum \psi(E_\alpha) ad(b^{-1}) E_{-\alpha} \in \mathfrak{g}_x + \mathfrak{b}\} = \\ &= \{tu \in B \mid \sum \psi(E_\alpha) ad(t^{-1}) E_{-\alpha} \in \mathfrak{g}_x + \mathfrak{b}\} = \{tu \in B \mid \sum \psi(E_\alpha) \alpha(t) E_{-\alpha} \in \mathfrak{g}_x + \mathfrak{b}\} \end{aligned}$$

By Lemma 3.5.3 we can choose $\alpha \in \Delta$ such that $\mathfrak{g}_{-\alpha} x \not\subset \mathfrak{b}x$. For any $\varepsilon > 0$ there exists $t \in T$ such that $\alpha(t) = 1$ and $\forall \beta \neq \alpha \in \Delta$ we have $|\beta(t)| < \varepsilon$. It is easy to see that for ε small enough, $t \notin a^{-1}(\mathcal{O}'_c)$. □

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