$(O(V \oplus F), O(V))$ IS A GELFAND PAIR FOR ANY QUADRATIC SPACE V OVER A LOCAL FIELD F

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Abstract. Let V be a quadratic space with a form q over an arbitrary local field F of characteristic different from 2. Let $W = V \oplus Fe$ with the form Q extending q with $Q(e) = 1$. Consider the standard embedding $\mathrm{O}(V) \hookrightarrow \mathrm{O}(W)$ and the two-sided action of $\mathrm{O}(V) \times \mathrm{O}(V)$ on $\mathrm{O}(W)$.

In this note we show that any $\mathrm{O}(V)\times\mathrm{O}(V)$-invariant distribution on $\mathrm{O}(W)$ is invariant with respect to transposition. This result was earlier proven in a bit different form in [vD] for $F = \mathbb{R}$, in [AvD] for $F = \mathbb{C}$ and in [BvD] for p-adic fields. Here we give a different proof.

Using results from [AGS], we show that this result on invariant distributions implies that the pair $(O(V), O(W))$ is a Gelfand pair. In the archimedean setting this means that for any irreducible admissible smooth Fréchet representation (π, E) of $\mathrm{O}(W)$ we have $dimHom_{\mathrm{O}(V)}(E, \mathbb{C}) \leq 1$.

A stronger result for p-adic fields is obtained in [AGRS07].

Contents

1. Introduction

Let F be a local field of characteristic different from 2.

Let (W, Q) be a quadratic space defined over F and fix $e \in W$ a unit vector. Consider the quadratic space $V = e^{\perp}$ with $q = Q|_V$. Define the standard imbedding $\mathrm{O}(V) \hookrightarrow \mathrm{O}(W)$ and consider the two-sided action of $\mathrm{O}(V)\times\mathrm{O}(V)$ on $\mathrm{O}(W)$ defined by $(g_1, g_2)h := g_1 h g_2^{-1}$. We also consider the anti-involution τ of O_Q given by $\tau(g) = g^{-1}$. In this paper we prove the following theorem

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Theorem (A)**.** *Any* $\mathrm{O}(V) \times \mathrm{O}(V)$ *invariant distribution on* $\mathrm{O}(W)$ *is invariant under* τ*.*

This theorem has the following corollary in representation theory.

Theorem (B)**.** *Let* (π, E) *be an irreducible admissible representation of* $\mathrm{O}(W)$*. Then*

$$dimHom_{\mathrm{O}(V)}(E, \mathbb{C}) \leq 1$$

Here admissible representation refers to the usual notion in the non-archimedean case and to the notion of admissible smooth Fréchet representation in the archimedean setting.

Our proof for the archimedean and non-archimedean case is uniform, except at one point where the archimedean case requires an extra analysis of a certain normal bundle (see lemma 4.2).

Remark 1.1. *We note that a related result for unitary representations of* $SO(V, Q)$ *is proved in* [BvD] *(for p-adic fields) and in* [vD] *(for the real numbers). In fact, the proof given in those papers implies also theorem* ***A****. Also, an analogous theorem for unitary groups is proven in* [vD2]*.*

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2. From Invariant distributions to Representation theory

In this section we recall a technique due to Gelfand and Kazhdan which allows to deduce theorem **B** from theorem **A**.

Recall the following theorem ([AGS])

Theorem 2.1. *Let* $H \subset G$ *be reductive groups and let* τ *be an involutive anti-automorphism of* G *and assume that* $\tau(H) = H$*. Suppose* $\tau(T) = T$ *for all bi* H*-invariant distributions* [1] *on* G*. Then for any irreducible admissible representation* (π, E) *of* G *we have*

$$dimHom_H(E, \mathbb{C}) \cdot dimHom_H(\widetilde{E}, \mathbb{C}) \leq 1,$$

where $\widetilde{E}$ *denotes the smooth contragredient representation.*

Note that in the non-archimedean case the same result is proven in [Pra].

To finish the deduction of theorem **B** from theorem **A** we will show that

Theorem 2.2. *Let* (π, E) *be an irreducible admissible representation of* $G = O(V)$*. Then* $\widetilde{E} \cong E$ *and in particular*

$$dimHom_H(E, \mathbb{C}) = dimHom_H(\widetilde{E}, \mathbb{C})$$

For the proof we recall proposition I.2 (chapter 4) from [MVW]:

Proposition 2.3. *Let* V *be a quadratic space and let* $g \in O(V)$*. Then* g *is conjugate to* g^{-1}*.*

[1] In fact it is enough to check this only for Schwartz distributions.

Proof of Theorem 2.2. For non-archimedean fields this is a theorem from [MVW] page 91. For archimedean fields we use the Harish-Chandra regularity theorem and the proposition that any element in $g\in O(V)$ is conjugate in $O(V)$ to g^{-1}. Thus, the characters of E and $\widetilde{E}$ are the same and hence $\widetilde{E}\cong E$. □

Remark 2.4. *A related result for the groups $SO(V)$ can be found in* [GP]*, proposition 5.3.*

3. Basic Results on Invariant distributions

In this paper we consider distributions over l-spaces and over smooth manifolds. l-spaces are locally compact totally disconnected topological spaces (see [BZ], section 1).

For X a smooth manifold or an l-space we denote by $\mathcal{D}(X)$ the space of distributions on X. When X is an l-space this means that $\mathcal{D}(X)=S(X)^*$ where $S(X)$ is the space of locally constant functions with compact support on X. For smooth X, we let $\mathcal{D}(X)=C_c^\infty(X)^*$.

The basic tools to study invariant distributions on a G-space X are Bruhat filtration, Frobenius reciprocity ([BZ], [Bar] and [AGS]) and the Bernstein's localization principle ([Ber] and [AG]). Let us remind the statements.

For the simplicity of formulation we provide, for each principle, two versions: for l-spaces and for smooth manifolds.

3.1. **Bruhat Filtration.** Although we will not need the non-archimedean version of this principle, we formulate it for completeness. It is a simple consequence of proposition 1.8 in [BZ].

Theorem 3.1. *Let an l-group G act on an l-space X. Let $X=\bigcup_{i=0}^{l}X_i$ be a G-invariant stratification of X. Let χ be a character of G. Suppose that $\mathcal{D}(X_i)^{G,\chi}=0$. Then $\mathcal{D}(X)^{G,\chi}=0$.*

To formulate the archimedean version we let X be a smooth manifold and $Y\subset X$ a smooth submanifold. We remind the definition of the conormal bundle CN_Y^X. For this denote by T_X the tangent bundle of X and by $N_Y^X:=(T_X|_Y)/T_Y$ the normal bundle to Y in X. The conormal bundle is defined by $CN_Y^X:=(N_Y^X)^*$. Denote by $Sym^k(CN_Y^X)$ the k-th symmetric power of the conormal bundle.

Theorem 3.2. *Let a real reductive group G act on a smooth affine real algebraic variety X. Let $X=\bigcup_{i=0}^{l}X_i$ be a smooth G-invariant stratification of X. Let χ be an algebraic character of G. Suppose that for any $k\in\mathbb{Z}_{\geq 0}$ and any $0\leq i\leq l$ we have $\mathcal{D}(X_i,Sym^k(CN_{X_i}^X))^{G,\chi}=0$. Then $\mathcal{D}(X)^{G,\chi}=0$.*

For proof see [AGS], section B.2.

3.2. **Frobenius reciprocity.** For l-space, the following version of Frobenius reciprocity is proven in [Ber]:

Theorem 3.3 (Frobenius reciprocity)**.** *Let a unimodular l-group G act transitively on an l-space Z. Let $\varphi: X\to Z$ be a G-equivariant continuous map. Let $z\in Z$. Suppose that its stabilizer $\mathrm{Stab}_G(z)$ is unimodular. Let X_z be the fiber of z. Let χ be a character of G. Then $\mathcal{D}(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{D}(X_z)^{\mathrm{Stab}_G(z),\chi}$.*

An archimedean version is considered in [Bar]. Here is a slight generalization (see [AGS]):

Theorem 3.4 (Frobenius reciprocity)**.** *Let a unimodular Lie group G act transitively on a smooth manifold Z. Let $\varphi : X \to Z$ be a G-equivariant smooth map. Let $z \in Z$. Suppose that its stabilizer* $\mathrm{Stab}_G(z)$ *is unimodular. Let X_z be the fiber of z. Let χ be a character of G. Then $\mathcal{D}(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{D}(X_z)^{\mathrm{Stab}_G(z),\chi}$. Moreover, for any G-equivariant bundle E on X, $\mathcal{D}(X,E)^{G,\chi}$ is canonically isomorphic to $\mathcal{D}(X_z, E|_{X_z})^{\mathrm{Stab}_G(z),\chi}$.*

3.3. **Bernstein's Localization principle.** For l-spaces it is taken from [Ber]:

Theorem 3.5 (Localization principle)**.** *Let X and T be l-spaces and $\phi : X \to T$ be a continuous map. Let an l-group G act on X preserving the fibers of ϕ. Let χ be a character of G. Suppose that for any $t \in T$, $\mathcal{D}(\phi^{-1}(t))^{G,\chi} = 0$. Then $\mathcal{D}(X)^{G,\chi} = 0$.*

For real smooth algebraic varieties, the following theorem is proven in [AG], Corollary A.0.3:

Theorem 3.6 (Localization principle)**.** *Let a real reductive group G act on a smooth affine real algebraic variety X. Let Y be a smooth real algebraic variety and $\phi : X \to Y$ be an algebraic G-invariant submersion. Suppose that for any $y \in Y$ we have $\mathcal{D}(\phi^{-1}(y))^{G,\chi} = 0$. Then $\mathcal{D}(X)^{G,\chi} = 0$.*

4. Proof of Theorem **A**

Recall the setting. (W, Q) is a quadratic space over F, $e \in W$ with $Q(e) = 1$. Also (V, q) is defined by $V = e^{\perp}$ and $q = Q|_V$.

We need some further notations.

- $O_q = O(V, q)$ is the group of isometries of the quadratic space (V, q).
- $G_q = O(V, q) \times O(V, q)$.
- $\Delta : O_q \to G_q$ the diagonal. $H_q = \Delta(O_q) \subset G_q$.
- $\tau(g_1, g_2) = (g_2, g_1)$.
- $\widetilde{G_q} = G_q \rtimes \{1, \tau\}$, same for $\widetilde{H_q}$
- $\chi : \widetilde{G_q} \to \{+1, -1\}$ the non trivial character with $\chi(G_q) = 1$.
- $\widetilde{G_Q}$ acts on O_Q by $(g_1, g_2)x = g_1 x g_2^{-1}$ and $\tau(x) = x^{-1}$.

Clearly Theorem **A** follows from the following theorem:

Theorem 4.1. $\mathcal{D}(O_Q)^{\widetilde{G_q},\chi} = 0$

4.1. **Proof of theorem 4.1.** We denote by $\Gamma = \{w \in W : Q(w) = 1\}$. Note that by Witt's theorem Γ is an O_Q transitive set and therefore $\Gamma \times \Gamma$ is a transitive $\widetilde{G_Q}$ set where the action of G_Q is the standard action on $W \oplus W$ and τ acts by flip.

Applying Frobenuis reciprocity (3.3, 3.4) to projections of $O_Q \times \Gamma \times \Gamma$ first on $\Gamma \times \Gamma$ and then on O_Q we have

$$\mathcal{D}(O_Q)^{\widetilde{G_q},\chi} = \mathcal{D}(O_Q \times \Gamma \times \Gamma)^{\widetilde{G_Q},\chi}$$

and also that

$$\mathcal{D}(O_Q \times \Gamma \times \Gamma)^{\widetilde{G_Q},\chi} = \mathcal{D}(\Gamma \times \Gamma)^{\widetilde{H_Q},\chi}$$

In what follows we will abuse notation and write $Q(u, v)$ for the bilinear form defined by Q. Define a map $D : \Gamma \times \Gamma \to Z$ where $Z = \{(v, u) \in W \oplus W : Q(v, u) = 0, Q(v + u) = 4\}$ by

$$D(x, y) = (x + y, x - y).$$

D defines an $\widetilde{G}_Q$-equivariant homeomorphism and thus we need to show that

$$\mathcal{D}(Z)^{\widetilde{H_Q},\chi} = 0$$

Here, the action of G_Q on $Z \subset W \oplus W$ is the restriction of its action on $W \oplus W$ while the action of τ is given by $\tau(v,u) = (v,-u)$.

Now we cover $Z = U_1 \cup U_2$ where

$$U_1 = \{(v,u) \in Z : Q(v) \neq 0\}$$

and

$$U_2 = \{(v,u) \in Z : Q(u) \neq 0\}$$

We will show $\mathcal{D}(U_1)^{\widetilde{H_Q},\chi} = 0$, and the proof for U_2 is analogous. This will finish the proof.

Lemma 4.2. $\mathcal{D}(U_1)^{\widetilde{H_Q},\chi} = 0$

Proof for non-archimedean F. Consider $\ell_1 : U_1 \to F - \{0\}$ defined as $\ell_1(v,u) = Q(v)$. By the localization principle, it is enough to show $\mathcal{D}(U_1^\alpha)^{\widetilde{H_Q},\chi} = 0$ where $U_1^\alpha = \ell_1^{-1}(\alpha)$, for any $\alpha \in F - \{0\}$. But

$$U_1^\alpha = \{(v,u)|Q(v) = \alpha, Q(u) = 4 - \alpha, Q(v,u) = 0\}$$

Let $W^\alpha = \{w \in W|Q(w) = \alpha\}$ and let $p_1 : U_1^\alpha \to W^\alpha$ be given by $p_1(v,u) = v$.

On W^α our group acts transitively. Fix a vector $v_0 \in W^\alpha$.

Denote $H(v_0) := H_{(Q|_{v_0^\perp})}$ and $\widetilde{H}(v_0) := \widetilde{H}_{(Q|_{v_0^\perp})}$.

The stabilizer in $\widetilde{H}_Q$ of v_0 is $\widetilde{H}(v_0)$. The fiber $p_1^{-1}(v_0) = \{a \in v_0^\perp|Q(a) = 4-\alpha\}$.

Frobenius reciprocity implies that

$$\mathcal{D}(U_1^\alpha)^{\widetilde{H_Q},\chi} = \mathcal{D}(p_1^{-1}(v_0))^{\widetilde{H}(v_0),\chi}$$

But clearly $\mathcal{D}(p_1^{-1}(v_0))^{\widetilde{H}(v_0),\chi} = 0$ as $-Id \in H(v_0)$. □

Proof for archimedean F. Now let us consider the archimedean case. Define $U := \{(v,u) \in U_1|u \neq 0\}$. Note that the map $\ell_1|_U$ is a submersion, so the same argument as in the non-archimedean case shows that $\mathcal{D}(U)^{\widetilde{H_Q},\chi} = 0$. Let $Y := \{(v \in W|Q(v) = 4\} \times \{0\}$ be the complement to U in U_1. By theorem 3.2, it is enough to prove $\mathcal{D}(Y, Sym^k(CN_Y^{U_1}))^{\widetilde{H_Q},\chi} = 0$.

Note that the action of $\widetilde{H}_Q$ on Y is transitive, and fix a point $(v,0) \in Y$. The stabilizer in $\widetilde{H}_Q$ of $(v,0)$ is $\widetilde{H}(v)$, and the normal space to Y at $(v,0)$ is $v^\perp$. So Frobenius reciprocity (theorem 3.4) implies that

$$\mathcal{D}(Y, Sym^k(CN_Y^{U_1}))^{\widetilde{H_Q},\chi} = Sym^k(v^\perp)^{\widetilde{H}(v),\chi}$$

But clearly $Sym^k(v^\perp)^{\widetilde{H}(v),\chi} = 0$ as $-Id \in H(v)$. □

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