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Irreducible representations of product of real reductive groups

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Abstract. Let G_1, G_2 be real reductive groups and (π, V) be a smooth admissible representation of $G_1 \times G_2$. We prove that (π, V) is irreducible if and only if it is the completed tensor product of (π_i, V_i), $i = 1, 2$, where (π_i, V_i) is a smooth, irreducible, admissible representation of moderate growth of G_i, $i = 1, 2$. We deduce this from the analogous theorem for Harish-Chandra modules, for which one direction was proven in [AG09, Appendix A] and the other direction we prove here.

As a corollary, we deduce that strong Gelfand property for a pair $H \subset G$ of real reductive groups is equivalent to the usual Gelfand property of the pair $\Delta H \subset G \times H$.
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1. Introduction

Let G_1, G_2 be reductive Lie groups , $\mathfrak{g}_i$ be the Lie algebra of G_i. Fix K_i - a maximal compact subgroup of G_i $(i = 1, 2)$. Let $\mathcal{M}(\mathfrak{g}_i, K_i)$ be the category of Harish-Chandra $(\mathfrak{g}_i, K_i)$-modules and $\mathcal{M}(G_i)$ be the category of smooth admissible Fréchet representations of moderate growth (see [Cas89, Wall92]). We also denote by $Irr(G_i)$ and $Irr(\mathfrak{g}_i, K_i)$ the isomorphism classes or irreducible objects in the above categories.
In this note we prove

Theorem 1.1. Let $M \in Irr(\mathfrak{g}_1 \times \mathfrak{g}_2, K_1 \times K_2)$. Then there exist $M_i \in Irr(\mathfrak{g}_i, K_i)$ such that $M = M_1 \otimes M_2$.

The converse statement, saying that for irreducible $M_i \in \mathcal{M}(\mathfrak{g}_i, K_i)$, $M_1 \otimes M_2$ is irreducible is [AG09, Proposition A.0.6]. By the Casselman-Wallach equivalence of categories $\mathcal{M}(\mathfrak{g}, K) \simeq \mathcal{M}(G)$, these two statements imply

Theorem 1.2. A representation $(\pi, V) \in \mathcal{M}(G_1 \times G_2)$ is irreducible if and only if there exist irreducible $(\pi_i, V_i) \in \mathcal{M}(G_i)$ such that $(\pi, V) \simeq (\pi_1, V_1)\hat{\otimes}(\pi_2, V_2)$.

Finally, we deduce a consequence of this theorem concerning Gelfand pairs. A pair (G, H) of reductive groups is called a *Gelfand pair* if $H \subset G$ is a closed subgroup and the space $(\pi^*)^H$ of H-invariant continuous functionals on any $\pi \in Irr(G)$ has dimension zero or one. It is called a *strong Gelfand pair* or a *multiplicity-free pair* if $\dim \mathrm{Hom}_H(\pi|_H, \tau) \leq 1$ for any $\pi \in Irr(G)$, $\tau \in Irr(H)$.

Corollary 1.3. Let $H \subset G$ be reductive groups and let $\Delta H \subset G \times H$ denote the diagonal. Then (G, H) is a multiplicity-free pair if and only if $(G \times H, \Delta H)$ is a Gelfand pair.

An analog of Corollary 1.3 was proven in [vD09] for generalized Gelfand property of arbitrary Lie groups, with smooth representations replaced by smooth vectors in unitary representations.

An analog of Theorem 1.2 for p-adic groups was proven in [BZ76, §§2.16] and in [Flath79]. For a more detailed exposition see [GH11, §§10.5].

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2. Preliminaries

2.1. Harish-Chandra modules and smooth representations.

In this subsection we fix a real reductive group G and a maximal compact subgroup $K \subset G$. Let $\mathfrak{g}, \mathfrak{k}$ denote the complexified Lie algebras of G, K.

Definition 2.1. A $(\mathfrak{g}, K)$-module is a $\mathfrak{g}$-module π with a locally finite action of K such the two induced actions of $\mathfrak{k}$ coincide and $\pi(ad(k)(X)) = \pi(k)\pi(X)\pi(k^{-1})$ for any $k \in K$ and $X \in \mathfrak{g}$.

A finitely-generated $(\mathfrak{g}, K)$-module is called admissible if any representation of K appears in it with finite (or zero) multiplicity. In this case we also call it a Harish-Chandra module.

Lemma 2.2 ([Wall88], §§4.2)**.** Any Harish-Chandra module π has finite length.

Theorem 2.3 (Casselman-Wallach, see [Wall92], §§§11.6.8)**.** The functor of taking K-finite vectors $HC : \mathcal{M}(G) \to \mathcal{M}(\mathfrak{g}, K)$ is an equivalence of categories.

In fact, Casselman and Wallach construct an inverse functor $\Gamma : \mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(G)$, that is called Casselman-Wallach globalization functor (see [Wall92, Chapter 11] or [Cas89] or, for a different approach, [BK]).

Corollary 2.4.

(i) The category $\mathcal{M}(G)$ is abelian.

(ii) Any morphism in $\mathcal{M}(G)$ has closed image.

Proof. (i) $\mathcal{M}(\mathfrak{g}, K)$ is clearly abelian and by the theorem is equivalent to $\mathcal{M}(G)$.

(ii) Let $\phi : \pi \to \tau$ be a morphism in $\mathcal{M}(G)$. Let $\tau' = \overline{Im\phi}$, $\pi' = \pi/\ker \phi$ and $\phi' : \pi' \to \tau'$ be the natural morphism. Clearly ϕ' is monomorphic and epimorphic in the category

$\mathcal{M}(G)$. Thus by (i) it is an isomorphism. On the other hand, $Im\phi' = Im\phi \subset \overline{Im\phi} = \tau'$. Thus $Im\phi = \overline{Im\phi}$. ∎

We will also use the embedding theorem of Casselman.

Theorem 2.5. Any irreducible $(\mathfrak{g}, K)$-module can be imbedded into a $(\mathfrak{g}, K)$-module of principal series.

Lemma 2.2, Theorems 2.3 and 2.5 and Corollary 2.4 have the following corollary.

Corollary 2.6. The underlying topological vector space of any admissible smooth Fréchet representation of moderate growth is a nuclear Fréchet space.

Definition 2.7. Let G_1 and G_2 be real reductive groups. Let $(\pi_i, V_i) \in \mathcal{M}(G_i)$ be admissible smooth Fréchet representations of moderate growth of G_i. We define $\pi_1 \otimes \pi_2$ to be the natural representation of $G_1 \times G_2$ on the space $V_1 \widehat{\otimes} V_2$.

Proposition 2.8 ([AG09], Proposition A.0.6)**.** Let G_1 and G_2 be real reductive groups. Let $\pi_i \in Irr(\mathfrak{g}_i, K_i)$ be irreducible Harish-Chandra modules of G_i. Then $\pi_1 \otimes \pi_2 \in Irr(\mathfrak{g}_1 \times \mathfrak{g}_2, K_1 \times K_2)$.

We will use the classical statement on irreducible representations of compact groups.

Lemma 2.9. Let K_1, K_2 be compact groups. A representation τ of $K_1 \times K_2$ is irreducible if and only if there exist irreducible representations τ_i of K_i such that $\tau \simeq \tau_1 \otimes \tau_2$. Note that τ_i are finite-dimensional, and $\otimes$ is the usual tensor product.

Corollary 2.10. Let G_1 and G_2 be real reductive groups and $(\pi_i, V_i) \in \mathcal{M}(G_i)$. Then we have a natural isomorphism $(\pi_1 \otimes \pi_2)^{HC} \simeq \pi_1^{HC} \otimes \pi_2^{HC}$.

3. Proof of Theorem 1.1

Throughout the section ρ_i always denote irreducible representations of K_1, σ_j always denote irreducible representations of K_2. For a representation V of K_1 (or of K_2) we will denote by V^ρ (resp. by V^σ) the corresponding isotypic component.
Let $K := K_1 \times K_2$ and $\mathfrak{g} := \mathfrak{g}_1 \times \mathfrak{g}_2$.

Let (π, V) be an irreducible admissible $(\mathfrak{g}, K)$ - module. We show that there exist non-zero irreducible and admissible $(\mathfrak{g}_1, K_1)$-module V_1 and $(\mathfrak{g}_2, K_2)$-module V_2 and a non-zero morphism $V_1 \bigotimes V_2 \to V$. From the irreducibility of V and $V_1 \bigotimes V_2$, we obtain that $V \simeq V_1 \bigotimes V_2$.

Let's first find the module V_1. Choose $\tau \in Irr(K)$ such that the isotypic component V^τ is non-zero. By Lemma 2.9 $\tau \simeq \rho \otimes \sigma$ for some $\rho \in Irr(K_1)$, $\sigma \in Irr(K_2)$. Let W be the $(\mathfrak{g}_1, K_1)$-module generated by V^τ. Note that since the actions of $(\mathfrak{g}_1, K_1)$ and $(\mathfrak{g}_2, K_2)$ commute, W is also a K_2-module and $W = W^\sigma$. We claim that W is an admissible $(\mathfrak{g}_1, K_1)$-module. Indeed, let ρ_1 be an irreducible representation of K_1. Then $W^{\rho_1} \subseteq V^{\rho_1 \otimes \sigma}$ and as a corollary

$$\dim(W^{\rho_1}) \leq \dim\left(V^{\rho_1 \otimes \sigma}\right) < \infty,$$

since V is an admissible $(\mathfrak{g}, K)$-module.

Now by Lemma 2.2 W has finite length and thus there is an irreducible admissible $(\mathfrak{g}_1, K_1)$-submodule $V_1 \subseteq W$. Thus, we finished the first stage of the proof.
Let

$$W_2' := \mathrm{Hom}_{(\mathfrak{g}_1, K_1)}(V_1, V).$$

Clearly, $W_2' \neq 0$. Since actions of $(\mathfrak{g}_1, K_1)$ and $(\mathfrak{g}_2, K_2)$ on V commute, W_2' has a natural structure of $(\mathfrak{g}_2, K_2)$-module. Take any non-zero morphism $L \in W_2'$ and let $W_2 \subset W_2'$ be the $(\mathfrak{g}_2, K_2)$-module generated by L.

Let us show that W_2 is admissible. Choose $\sigma_2 \in Irr(K_2)$. Let $\rho_2 \in Irr(K_1)$ such that $V_1^{\rho_2} \neq 0$. Then $V_1^{\rho_2}$ generates V_1 and thus for any $L', L'' \in W_2^{\sigma_2}$ if L' agrees with L'' on $V_1^{\rho_2}$ then $L' = L''$. This gives a linear embedding from $W_2^{\sigma_2}$ into the finite-dimensional space $\mathrm{Hom}_{\mathbb{C}}(V_1^{\rho_2}, V^{\rho_2 \otimes \sigma_2})$. Thus W_2 is an admissible $(\mathfrak{g}_2, K_2)$-module.

Thus W_2 has finite length and therefore there is an irreducible admissible submodule $V_2 \subseteq W_2$. Define a linear map $\phi : V_1 \bigotimes V_2 \to V$ by the formula

$$\phi(v \otimes l) := l(v)$$

on the pure tensors. Clearly, this is a non-zero $(\mathfrak{g}, K)$-map.

The result $V_1 \bigotimes V_2 \simeq V$ follows now from the irreducibility of V and of $V_1 \bigotimes V_2$ (Proposition 2.8).

Remark 3.1. An alternative way to prove this theorem is to remark that the category $\mathcal{M}(\mathfrak{g}, K)$ is equivalent to the category of admissible modules over the idempotented algebra $\mathcal{H}(\mathfrak{g}, K)$ of K-finite distributions on G supported in K (see [Flath79]), then show that this algebra is the tensor product of $\mathcal{H}(\mathfrak{g}_i, K_i)$ and thus the proofs from [BZ76, Flath79] extend to this case. We estimate that such proof would be of similar length, but slightly less elementary.

4. Proof of Theorem 1.2 and Corollary 1.3

Proof. [Proof of Theorem 1.2] First take $\pi_i \in Irr(G_i)$, for $i = 1, 2$. Then $\pi_i^{HC} \in Irr(\mathfrak{g}_i, K_i)$ and by Proposition 2.8 $\pi_1^{HC} \otimes \pi_2^{HC} \in Irr(\mathfrak{g}_1 \times \mathfrak{g}_2, K_1 \times K_2)$. By Corollary 2.10 $(\pi_1 \otimes \pi_2)^{HC} \simeq \pi_1^{HC} \otimes \pi_2^{HC} \in Irr(\mathfrak{g}_1 \times \mathfrak{g}_2, K_1 \times K_2)$. This implies $\pi_1 \otimes \pi_2 \in Irr(G_1 \times G_2)$.

Now take $\pi \in Irr(G_1 \times G_2)$. Then $\pi^{HC} \in Irr(\mathfrak{g}_1 \times \mathfrak{g}_2, K_1 \times K_2)$ and by Theorem 1.1 there exist $(M_i) \in Irr(\mathfrak{g}_i, K_i)$ such that $\pi^{HC} \simeq M_1 \otimes M_2$. By Theorem 2.3 there exist $\pi_i \in Irr(G_i)$ such that $\pi_i^{HC} \simeq M_i$. Then $\pi^{HC} \simeq \pi_1^{HC} \otimes \pi_2^{HC} \simeq (\pi_1 \otimes \pi_2)^{HC}$ and by Theorem 2.3 this implies $\pi \simeq \pi_1 \otimes \pi_2$. ∎

Corollary 1.3 follows from Theorem 1.2 and the following lemma.

Lemma 4.1. Let $H \subset G$ be real reductive groups. Let (π, E) and (τ, W) be admissible smooth Fréchet representations of moderate growth of G and H respectively. Then $Hom_H(\pi, \tau)$ is canonically isomorphic to $Hom_{\Delta H}(\pi \otimes \widetilde{\tau}, \mathbb{C})$, where $\widetilde{\tau}$ denotes the contragredient representation.

Proof. For a nuclear Fréchet space V we denote by V' its dual space equipped with the strong topology. Let $\widetilde{W} \subset W'$ denote the underlying space of $\widetilde{\tau}$. By the theory of nuclear Fréchet spaces ([T67, Chapter 50], we know $Hom_{\mathbb{C}}(E,W) \cong E'\widehat{\otimes}W$ and $Hom_{\mathbb{C}}(E\widehat{\otimes}\widetilde{W},\mathbb{C}) \cong E'\widehat{\otimes}\widetilde{W}'$. Thus we have canonical embeddings

$$Hom_H(\pi,\tau) \hookrightarrow Hom_{\Delta H}(\pi \otimes \widetilde{\tau},\mathbb{C}) \hookrightarrow Hom_H(\pi,\widetilde{\tau}')$$

Since the image of any H-equivariant map from π to $\widetilde{\tau}'$ lies in the space of smooth vectors $\widetilde{\widetilde{\tau}}$, which is canonically isomorphic to τ, the lemma follows. ∎

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