Probabilistically Checkable Proofs

Irit Dinur

The Weizmann Institute of Science

September 14, 2010

- Common wisdom: to check a proof you need to read it
- Why bother? instead-
  - Ask for the proof to be supplied in **PCP** format
  - Check randomly by reading only 3 bits.
    Probability of error, i.e. of accepting a bad proof, is at most $1/2$.
    (For error probability 2^{-k}, read $3k$ bits).

"The PCP Theorem"

There is such a format.

caveat: applies only for Mathematical Proofs

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Theorems and Proofs, Problems and Solutions

- What is a mathematical proof?
- Anything that can be verified by a *rigorous* procedure, i.e., an algorithm

- More generally,

a theorem = a problem,
a proof = a solution

- The difference between a theorem and its proof, is *how long* it takes to verify it's correctness

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Linear Equations

Input: A system of linear equations (over a finite field) :

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\begin{aligned}
x_1 + x_2 + x_3 &= 0, \\
x_1 + x_6 - x_2 + x_{90} &= 1 \\
&\vdots
\end{aligned}
$$

Algorithmic goal: Decide if there is a solution to all of the equations

Complexity: easy, by Gaussian elimination

But, "overdetermined" version is hard...

Note:

- Algorithm's efficiency is measured as a function of the input length. Polynomial = good, Exponential = bad.

Computational problems - examples

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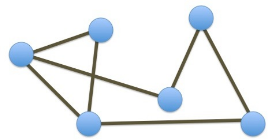
Graph 3 Colorability ($3COL$)

3-Coloring a graph



Input: A graph $G = (V, E)$

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Definition (Computational Problem)

We identify a computational problem with the set of all 'yes' inputs.

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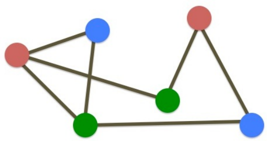

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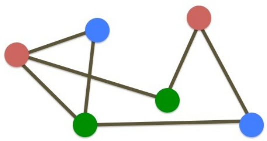

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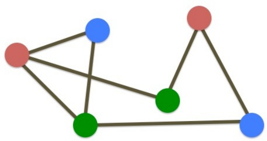

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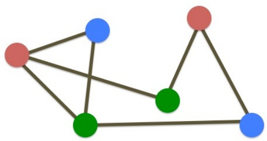

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- P = (polynomial time)
  P is the class of efficiently decidable problems
  e.g. linear equations
- NP = (non-deterministically polynomial time)
  NP is the class of problems with efficiently checkable solutions
  e.g. 3-coloring, max-clique, ...
- $P \neq NP$: $1 Million Question: is discovering a proof as easy as checking it ?
- 3-coloring is "the hardest problem in NP" (aka NP-hard)

Theorem

If 3-coloring is in P then $P = NP$.

To understand NP, enough to study the 3-coloring problem.

P, NP, and all that

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Part II - The PCP Theorem

Arora-Safra,

Arora-Lund-Motwani-Sudan-Szegedy

1991

The PCP Theorem

The NP verifier (Definition)

Every problem $A \in NP$ has an efficient verifier V, that reads

- the input string τ and some randomness
- a constant number of bits from the proof string π

and then accepts or rejects, such that

- **Completeness:** If $\tau \in A$ then there is a proof that V accepts with probability 1.
- **Soundness:** If $\tau \notin A$ then for every π V rejects.

- The "error probability" can be reduced to $(\frac{1}{2})^k$ by k repetitions.
- Striking!

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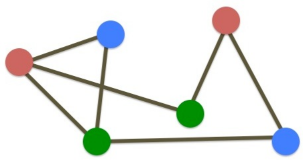
The PCP Theorem for 3-Coloring

The “natural” 3-Coloring verifier reads the coloring

$$c(v_1) = 1, c(v_2) = 2, \ldots$$

and then checks edge-by-edge that endpoints have different colors.



What will the PCP verifier look like?

- Naive attempt: Choose a random edge, read the colors of its endpoints, and accept if true
- Fails! – a non 3-colorable graph may have a 3 coloring with as few as only one monochromatic edge.
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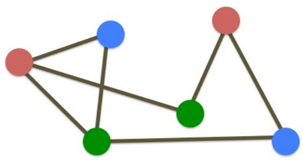

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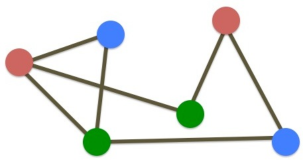

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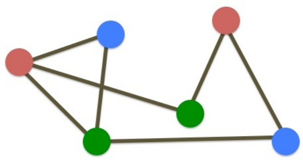

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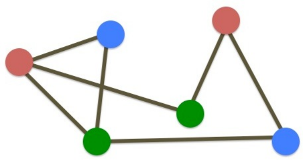

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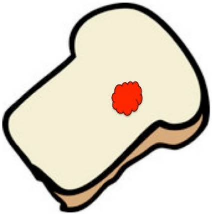

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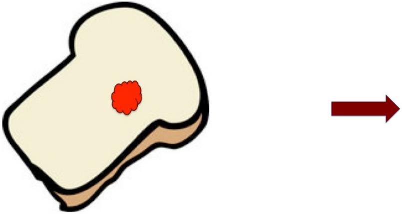

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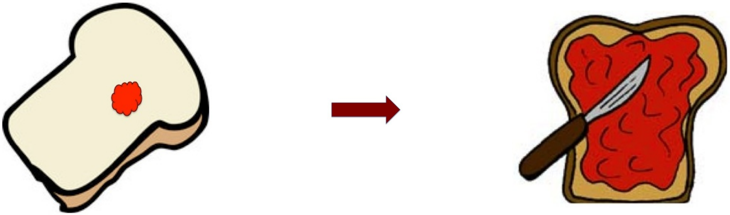

The PCP Theorem & Inapproximability

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- [Feige-Goldwasser-Lovász-Safra-Szegedy, 1990]
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Approximation Problems

- Throughout 70's-80's: many problems discovered to be NP-hard
- Natural to seek approximate solutions. (Almost no known lower bounds)

Optimization

1. Max-LIN: satisfy the largest number of equations.
2. Max-3COL: color the vertices with 3 colors, maximizing number of two-colored edges.

Both these problems are NP-hard (yes, even Max-LIN!)

Approximation

1. satisfy at least $\geq \alpha \cdot OPT$ equations
2. satisfy at least $\geq \alpha \cdot OPT$ edge contraints

Complexity: depends on the problem, and on α

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Hardness of Approximation

Claim:

If there is an efficient algorithm that maps a graph G to a graph G' such that:

Yes: If $OPT(G) = 1$, then $OPT(G') = 1$

No: If $OPT(G) < 1$, then $OPT(G') < 0.99$

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This is a "gap amplifying reduction":

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Claim:

Such a reduction implies the PCP theorem.

PCP & Inapproximability

The PCP Theorem (1) – original formulation

There is an efficient verifier for 3-coloring that reads: the input G, randomness r, and then a constant number of bits from the proof, such that

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1. Read the input G, compute G'.
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(1) $\Longrightarrow$ (2) : exercise.

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(1) $\Longrightarrow$ (2) : exercise.

PCP & Inapproximability

The PCP Theorem (1) – original formulation

There is an efficient verifier for 3-coloring that reads: the input G, randomness r, and then a constant number of bits from the proof, such that

Yes: If G is 3-colorable, then $\Pr_r[V \text{ accepts}] = 1$

No: If G is not 3-colorable, then $\Pr_r[V \text{ accepts}] \leq \frac{1}{2}$.

The PCP Theorem (2) – second formulation

There is an efficient algorithm that maps graphs G to graphs G' such that:

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Tightness of Inapproximability

- For example, Max-3LIN:
  1. NP-hard to 1-approximate, i.e. to solve exactly
  2. Easy to $\frac{1}{2}$-approximate, e.g. by a random assignment
  3. What about α-approximation for $\frac{1}{2} \leq \alpha < 1$?
- Interesting to study boundary between hard and easy ends, possibly pinpoint the point of transition?

Theorem (Håstad '97)

Given a 3LIN instance that is $1 - o(1)$ satisfiable, it is NP-hard to find an assignment satisfying $1/2 + o(1)$ of the clauses.
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Robust Systems

A system is robust (or *stable*) if every approximate solution is close to a perfect solution.

Example (System of Equations)

$$\begin{aligned} x_1^3 x_2 + 7x_3 &= 2, \\ x_1 x_2 x_3 &= 1 \\ &\vdots \end{aligned}$$

1. approximate solution: satisfies $1-\varepsilon$ of the equations.
2. close to: agrees on $1-\delta$ of the coordinates

- Two different measures: (1) equation-based, (2) solution-based.
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Robustness Questions and Inverse Theorems

Robustness is a "desirable" property of systems, and natural to study.

1. Additive combinatorics: approximate fields and groups
   If a set is somewhat linear it must be close to a field / group
   [Freiman, Erdös-Szemeredy,. . .]
2. Discrete Fourier analysis & geometry: approximate dictatorships
   If a function $f : \{0,1\}^n \to \mathbb{R}$ is somewhat noise-stable then it must depend on few coordinates [Mossel-O'Donnell-Oleszkiewicz extension of CLT]
3. Extremal set systems: approximate Erdös-Ko-Rado theorems; approximate cliques in certain graphs
   If a clique is somewhat large it must be close to a maximum clique
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5. The PCP Theorem **If a PCP proof is somewhat correct it must be close to perfectly correct proof**

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Part III

(Flavors of)

the Proof of the PCP Theorem

Proving the PCP theorem

- Goal: find an efficient algorithm that maps graphs G to graphs H such that:

  Yes: If $OPT(G) = 1$, then $OPT(H) = 1$
  No: If $OPT(G) < 1$, then $OPT(H) \leq 0.99$

  (let jam$(G) := 1 - OPT(G)$)

- There are two different approaches.
  1. The original "algebraic" proof [Arora-Safra, Arora-Lund-Motwani-Sudan-Szegedy, 1991]
     Technique: H encodes G via low degree polynomials over finite fields
  2. The "combinatorial" proof [Dinur, 2006]
     Technique: gradual gap amplification using graph structure

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Proof Idea

The idea is to proceed in many small steps:

$$G \Longrightarrow G_1 \Longrightarrow G_2 \Longrightarrow \cdots \Longrightarrow G_k =: H$$

such that the "jam" value gets amplified, unless it was zero.
(pictorially, the "jam" is spread little by little)

The basic step

We show a mapping $G \Longrightarrow G'$ for which

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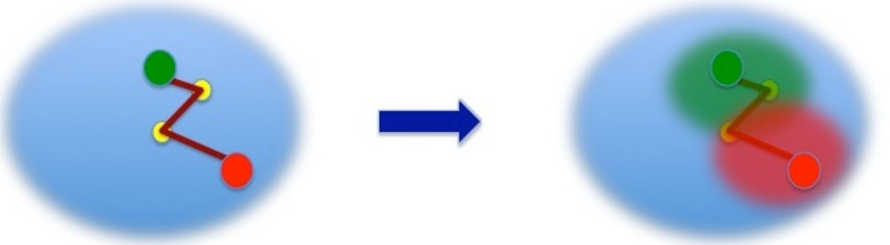

The mapping consists of two sub-steps $G \xrightarrow{1} \hat{G} \xrightarrow{2} G'$:

1. Gather: Each vertex gathers the colors of its neighbors. Encoded by new color of vertex. $\hat{G}$-edges are length-3 paths in G, each tests for inconsistencies.

2. Disperse: This step splits each vertex into several vertices, and replaces the "tests" by regular edges, yielding a 3-coloring instance. Robustly.
   How? by recursion: using a weaker PCP theorem!

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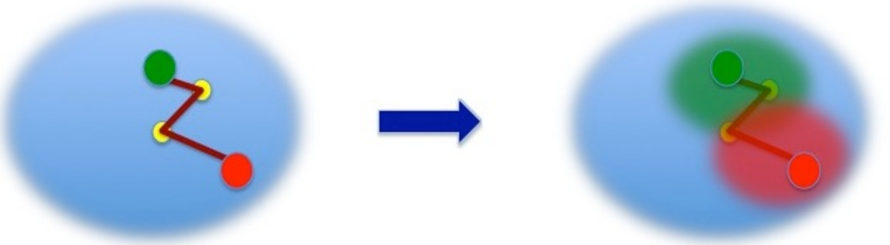

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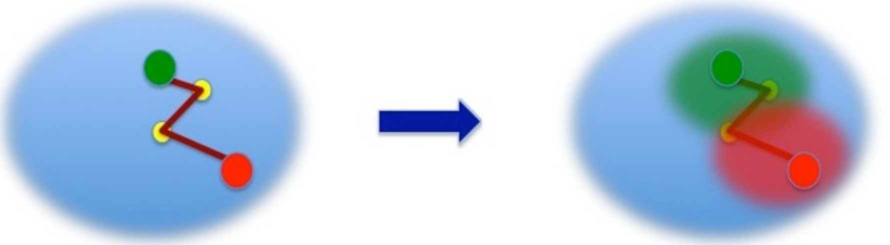

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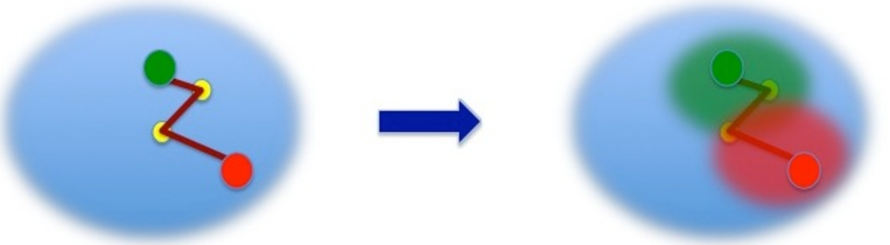

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- A "gathering" step increases the jam value, but looses the 3-coloring structure.
- A "dispersing" step regains the 3-coloring structure using a "gadget" (i.e. local replacement)
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Thank You!