POINT COUNTING AND WILKIE'S CONJECTURE FOR NON-ARCHIMEDEAN PFAFFIAN AND NOETHERIAN FUNCTIONS

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Abstract
We consider the problem of counting polynomial curves on analytic or definable subsets over the field $\mathbb{C}((t))$ as a function of the degree r. A result of this type could be expected by analogy with the classical Pila–Wilkie counting theorem in the Archimedean situation.

Some non-Archimedean analogues of this type have been developed in the work of Cluckers, Comte, and Loeser for the field $\mathbb{Q}_p$, but the situation in $\mathbb{C}((t))$ appears to be significantly different. We prove that the set of polynomial curves of a fixed degree r on the transcendental part of a subanalytic set over $\mathbb{C}((t))$ is automatically finite, but we give examples that show their number may grow arbitrarily quickly *even for analytic sets. Thus no analogue of the Pila–Wilkie theorem can be expected to hold for general analytic sets. On the other hand, we show that if one restricts to varieties defined by Pfaffian or Noetherian functions, then the number grows at most* polynomially *in r, thus showing that the analogue of the Wilkie conjecture does hold in this context.*

1. Introduction

1.1. Point counting in Archimedean and non-Archimedean fields

Over the reals, it is known by bounds of Pila and Wilkie [19] that the number of rational points of height at most B on the transcendental part of analytic varieties (or even definable sets in o-minimal structures) is bounded above by cB^{ε} for some $c = c(\varepsilon)$ and with $\varepsilon > 0$. Such bounds also hold over $\mathbb{Q}_p$ by [6], and with uniformity in p by [7]. However, for $\mathbb{C}((t))$, the question about appropriate upper bounds for rational points on analytic (definable) sets is left open in [6]. In this context, it is natural to count points from $\mathbb{C}[t]$ lying in the transcendental part of a definable set as

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a function of the degree r. The question of the finiteness of the set of such polynomials of bounded degree is discussed in [6, Section 5.5], as well as the possibility of Wilkie-type bounds under some extra Pfaffian style conditions, but the methods used there do not allow one to establish any bounds on these sets, not even their finiteness.

1.2. General definable sets: Finiteness and a negative result

In the first part of this paper, we consider the analogue of the Pila–Wilkie counting theorem in the context of the field $\mathbb{C}((t))$. We make two contributions in this direction. First, we show that the set under consideration is indeed finite, so that the counting problem is well-posed. Second, we produce examples of analytic sets where the number of such polynomial curves grows arbitrarily fast as a function of r. In other words, no analogue of the Pila–Wilkie counting theorem can be expected in this context. We also give a variant of the finiteness result over $\mathbb{Q}_p^{\mathrm{unram}}$, the maximal unramified field extension of $\mathbb{Q}_p$. These results are presented in Section 2.

It is easy to explain intuitively why it may be unreasonable to expect a Pila–Wilkie type counting result over the field $\mathbb{C}((t))$. The basic idea behind the Bombieri–Pila method and its subsequent generalization in the work of Pila and Wilkie is that after making a suitable parameterization and cutting the domain into balls of radius $cB^{-\varepsilon}$, the rational points in each ball can be interpolated by a hypersurface of degree $d = d(\varepsilon)$. A similar strategy has been made to work in [6] for the field $\mathbb{Q}_p$. On the other hand, in $\mathbb{C}((t))$ the valuation field is infinite, and it is simply not possible to subdivide a ball of radius 1 into any finite collection of smaller balls. The entire approach, it appears, is doomed to fail—and this is borne out by our counterexamples.

1.3. Wilkie's conjecture

The finiteness result, and the impossibility of Pila–Wilkie type bounds in general, serve as double motivation to search for a framework with additional control where some results in the spirit of the counting theorem can still be obtained. A potent source of intuition in this direction is the Wilkie conjecture. This prominent conjecture due to Wilkie states that for certain natural o-minimal structures (originally $\mathbb{R}_{\exp}$ in Wilkie's formulation) one can sharpen the bound $c(\varepsilon)B^\varepsilon$ to some polynomial in $\log B$. Various special cases of the Wilkie conjecture have been established for sets defined using Pfaffian functions: either in small dimensions (see, e.g., [13], [16], [18]), or for general sets definable in the class of "holomorphic-Pfaffian" functions (see [3]).

Given that in $\mathbb{C}((t))$ we are unable to subdivide the unit ball into any number of smaller balls, the only hope seems to be to show that for some suitable degree d, the rational points can all be approximated *without any subdivision*, that is, using the single unit ball. According to work on the Wilkie conjecture in the Archimedean context, it is known that, using hypersurfaces of degree $d = (\log B)^\alpha$ (rather than

$d = d(\varepsilon)$), it is in fact possible to interpolate all rational points using $(\log B)^{\beta}$ balls (rather than $cB^{-\varepsilon}$). In some cases, even $O(1)$ balls suffice. It therefore appears at least potentially plausible that in the more restrictive context of the Wilkie conjecture, the point counting argument can be salvaged.

Following this intuition, we introduce Pfaffian and Noetherian functions into the non-Archimedean picture. In the second part of the paper with Sections 3–5, we consider an analogue of the Wilkie conjecture, with proof techniques which are independent of Section 2. Namely, we restrict attention to the class of germs of sets defined by *Pfaffian* or *Noetherian* equations over the field of convergent Laurent series $\mathbb{C}(\{t\})$. Many functions of interest in the classical applications of the Pila–Wilkie theorem (e.g., abelian functions, modular functions, period integrals) fall within these classes, and this therefore seems like a natural context in which to pursue non-Archimedean counting theorems. By the general philosophy of the Wilkie conjecture one expects sharper, even polynomial in r, bounds for the counting problem on such sets. Surprisingly, we show that despite the failure of the Pila–Wilkie counting theorem in this context, the analogue of the Wilkie conjecture does in fact hold for arbitrary Pfaffian, or even Noetherian, varieties. Specifically, we show that the subdivision step can be completely avoided in this case, and a single hypersurface of degree polynomial in $\log B$ can indeed be used to interpolate all rational points of height B in the unit ball. We remark that this second part of the paper can be read independently of Section 2.

1.4. Main ideas

We interpret systems of equations over $\mathbb{C}(\{t\})$ as one-parameter deformations of systems over $\mathbb{C}$. A crucial technical difference arises when comparing this local context to the Archimedean one. In the Archimedean context, all known cases of the Wilkie conjecture rely in a crucial way on Khovanskii's Bézout-type bounds for Pfaffian functions over the reals (see [17]), which gives bounds for the number of solutions of systems of Pfaffian equations in terms of their degrees. This theory is *purely real*, and it generally gives bounds only for real solutions of systems of equations. On the other hand, in the local $\mathbb{C}(\{t\})$-context, general bounds are in fact available for *complex*, rather than real, solutions. Indeed, Gabrielov established general bounds for the number of solutions of such complex-analytic deformations of Pfaffian functions in [14]. Moreover, under a small technical restriction, similar results have been established in the class of Noetherian functions by Binyamini and Novikov in [1]. Nothing approaching such general results is known in the Archimedean situation. Using these tools, it is at least plausible to expect that one can treat the case of general Pfaffian or even Noetherian functions by the complex-analytic methods introduced in [3].

In Sections 4–5, we carry out this program. We introduce a local analogue of the Weierstrass polydisks used in [3], and we apply the results of [1] and [14] to show that these can be constructed with appropriate control over complexity for Pfaffian and Noetherian varieties. It is pleasantly surprising that in this local context we can achieve this in full generality, for both Pfaffian and Noetherian functions, whereas the corresponding results in the Archimedean context are currently far more restricted in scope. Let us finally mention that this seems to be the first study of Pfaffian and Noetherian functions in a non-Archimedean context, as far as we can see. We hope that this will open the way for the study of Pfaffian and Noetherian functions over $\mathbb{Q}_p$ instead of $\mathbb{C}((t))$.

2. Finiteness results

In this section, we state and prove our finiteness result for rational points of bounded height (i.e., with coordinates which are polynomials in $\mathbb{C}[t]$ of bounded degree) on the transcendental part of analytic definable sets (see Theorem 1), and we show that these numbers can grow arbitrarily fast with the degree (see Proposition 1). At the end of Section 2, we adapt the finiteness result to the mixed characteristic case.

Let us make this all very precise. In this section, we write K for $\mathbb{C}((t))$. We recall some of the notions for $K = \mathbb{C}((t))$ from [6], analogous to the corresponding real notions. For a set $X \subset K^n$, let X^{alg} be the union of all semialgebraic sets $C \subset X$ which are of constant local dimension 1. Here semialgebraic means definable with constants from K in the language of valued fields, with symbols $+, -, \cdot, |$, where $x|y$ holds for (x, y) in K^2 if and only if y lies in $x\mathcal{O}_K$ and where $\mathcal{O}_K := \mathbb{C}[[t]]$ is the valuation ring of K. The local dimension of a nonempty semialgebraic set $C \subset K^n$ at $x \in C$ is defined to be the maximal integer $m \geq 0$ such that for all sufficiently small semialgebraic open neighborhoods U of x there is a semialgebraic function $U \cap C \to K^m$ whose range has nonempty interior in K^m. Correspondingly, the transcendental part X^{trans} of X is defined as $X \setminus X^{\mathrm{alg}}$.

As replacement for the o-minimality condition in [19], we will impose a form of analyticity on X as follows. For each integer $n \geq 0$, let $\mathcal{O}_K\langle x_1, \dots, x_n\rangle$ be the t-adic completion of $\mathcal{O}_K[x_1, \dots, x_n]$ inside $\mathcal{O}_K[[x_1, \dots, x_n]]$ for the Gauss norm. Note that $\mathcal{O}_K\langle x_1, \dots, x_n\rangle$ consists of power series $\sum_{i \in \mathbb{N}^n} a_i x^i$, in multi-index notation, with $a_i \in \mathcal{O}_K$ and such that the t-adic norm $|a_i|$ of a_i goes to zero when $i_1 + \cdots + i_n$ goes to $+\infty$. Here the t-adic norm $|x|$ of nonzero $x \in K$ is defined as $e^{-\operatorname{ord} x}$, where $\operatorname{ord} x$ is the t-adic valuation of x, namely, the largest integer n such that x/t^n lies in $\mathcal{O}_K$, the t-adic norm of zero is defined to be zero, and for $x \in K^n$, one defines $|x|$ as the maximum of the $|x_i|$ for $i = 1, \dots, n$. For f in $\mathcal{O}_K\langle x_1, \dots, x_n\rangle$, write $f^{|}$ for the restricted analytic function associated to f, namely, the function $K^n \to K$ sending $z \in \mathcal{O}_K^n$ to the evaluation $f(z)$ of f at z (namely, the limit for the t-adic topology over $s > 0$ of the partial sums $\sum_{i, \max_j i_j < s} a_i z^i$), and sending the remaining z to 0. Let $\mathcal{L}_{\mathrm{an}}^K$ be the language containing the language of valued fields and, for each f in

$\mathcal{O}_K\langle x_1, \ldots, x_n\rangle$ for any $n \geq 0$, a function symbol for the restricted analytic function $f^{\mid}$ associated to f.

Finally, we also follow [6] for the notion of integral points of bounded height on subsets of K^n, which we now recall. For $r \geq 1$, denote by $\mathcal{O}_K(r)$ the subset of $\mathcal{O}_K$ consisting of polynomials $\sum_{i=0}^{r-1} a_i t^i$ with a_i in $\mathbb{C}$ and with degree less than r (in the variable t). For any subset $X \subset K^n$ and any $r \geq 1$, write $X(r)$ for the intersection of X with $(\mathcal{O}_K(r))^n$.

We can now state the first main result of this paper, addressing a question left open in Section 5.5 of [6] (see the partial result, Proposition 5.5.1 of [6]).

THEOREM 1 (Finiteness)
Let $X \subset K^n$ be $\mathcal{L}_{\mathrm{an}}^K$-definable. Then for each $r > 0$,

$$(X^{\mathrm{trans}})(r)$$

is a finite set.

Of course one would like to bound $\# X^{\mathrm{trans}}(r)$ when r grows. However, without extra information about the geometry of X, it is hard to bound the number of points in $X^{\mathrm{trans}}(r)$, as $X^{\mathrm{trans}}(r)$ can grow arbitrarily fast with r by the following result.

PROPOSITION 1
For any sequence of positive integers N_r for $r > 0$, there is an $\mathcal{L}_{\mathrm{an}}^K$-definable set $X \subset K^2$ such that

$$N_r < \#(X^{\mathrm{trans}})(r).$$

Furthermore, for X one can even take the graph of a function $f : \mathcal{O}_K \to \mathcal{O}_K$ given by a power series in $\mathcal{O}_K\langle x\rangle$ in one variable x.

The proof of Theorem 1 uses a quantifier elimination result in a certain expansion $\mathcal{L}_{\mathrm{an}}^{\overline{\mathrm{ac}}}$ of $\mathcal{L}_{\mathrm{an}}^K$ and a reduction to Zariski-constructible conditions on tuples of complex polynomials in t of degree less than r (some similar techniques appear in [2], [3], [6]). We start with developing these ideas, summarized in Proposition 2, which gives that $X(r)$ is a constructible set. Let us mention that languages and formulas are always first order in this paper (as is typical in model theory).

Consider the language $\mathcal{L}_{\mathrm{an}}^{\overline{\mathrm{ac}}}$ with three sorts (resp., valued field, residue field, and value group) containing $\mathcal{L}_{\mathrm{an}}^K$ together with the field inverse $(\cdot)^{-1}$ extended by zero on zero for the valued field sort, the ring language on the residue field, the Presburger language on the value group, an angular component map $\overline{\mathrm{ac}}$ sending nonzero x in K to the coefficient of the leading term in t of x (namely, to $xt^{-\operatorname{ord} x} \bmod (t)$) and sending zero to zero, and the valuation map from $K^\times$ to the value group. Clearly any $\mathcal{L}_{\mathrm{an}}^{K}$-definable set is also $\mathcal{L}_{\mathrm{an}}^{\overline{\mathrm{ac}}}$-definable.

By the quantifier elimination statement of Theorem 4.2 of [11] together with quantifier elimination in the Presburger language and the Chevalley–Tarski theorem, any $\mathcal{L}_{\mathrm{an}}^{\overline{\mathrm{ac}}}$-definable set X is given by a quantifier-free formula in the language $\mathcal{L}_{\mathrm{an}}^{\overline{\mathrm{ac}}}$. (This also follows from Theorem (3.9) of [21], or Theorem 6.3.7 and Example 4.4(1) from [10].)

We can now state and prove the reduction to Zariski-constructible conditions.

PROPOSITION 2
Let $X \subset K^n$ be $\mathcal{L}_{\mathrm{an}}^{\overline{\mathrm{ac}}}$-definable, and let $r > 0$ be an integer. Then $X(r)$ is a Zariski-constructible subset of $\mathbb{C}^{rn}$, where we identify $(\mathcal{O}_K(r))^n$ with $\mathbb{C}^{rn}$ by mapping a complex polynomial $\sum_{i=0}^{r-1} a_i t^i$ in $\mathcal{O}_K(r)$ to the tuple $(a_i)_{i=0}^{r-1}$ in $\mathbb{C}^r$.

Proof
By the mentioned quantifier elimination result for our structure with three sorts K, $\mathbb{C}$, and $\mathbb{Z}$ in the language $\mathcal{L}_{\mathrm{an}}^{\overline{\mathrm{ac}}}$, the set $X \subset K^n$ is given by a quantifier-free formula $\varphi(x)$ in the language $\mathcal{L}_{\mathrm{an}}^{\overline{\mathrm{ac}}}$, with free variables x running over K^n. We proceed by induction on the number of occurrences of the function symbol $(\cdot)^{-1}$ in the quantifier-free formula $\varphi(x)$. By the form of quantifier-free formulas (see Theorem 4.2 of [11]), $\varphi(x)$ is equivalent to a finite Boolean combination of conditions on $x \in K^n$ of the form

(a) $\overline{\mathrm{ac}}(f(x)) = 1$,
(b) $f(x) = 0$,
(c) $\mathrm{ord}(f(x)) \geq 0$,
(d) $\mathrm{ord}(f(x)) \equiv 0 \bmod \lambda$,

for some $\mathcal{L}_{\mathrm{an}}^{\overline{\mathrm{ac}}}$-terms f and some integers $\lambda > 0$. By the definition of $X(r)$, it is enough to prove the proposition when φ itself has one of the above-mentioned forms (a), (b), (c), or (d). Write $x \in (\mathcal{O}_K(r))^n$ as $(\sum_{\ell=0}^{r-1} a_{j\ell} t^\ell)_{j=1}^n$ and write $f(x)$ as $\sum_{s \in \mathbb{Z}} f_s(a) t^s$, with $a = (a_{j\ell})_{j,\ell}$ and functions f_s on $\mathbb{C}^{rn}$.

First, suppose that the term f does not involve field inversion. In this case, we may suppose that f is a finite composition of K-multiples of restricted analytic mappings, that is, $f = \hat{f}_k \circ \hat{f}_{k-1} \circ \cdots \circ \hat{f}_1(x)$, where each $\hat{f}_i$ is a map whose component functions are elements of $\mathcal{O}_K\langle x \rangle \otimes_{\mathcal{O}_K} K$; namely, for each i there are restricted analytic functions $f_{ij}^{|}$ for $j = 1, \ldots, a_i$ and $\lambda_{ij} \in K$ such that $\hat{f}_i(z) = (\lambda_{ij} f_{ij}^{|}(z))_j$, where $j = 1, \ldots, a_i$, $z \in K^{a_{i-1}}$, and where a_i is the arity of $\hat{f}_{i+1}$ if $i < k$ and with $a_k = 1$. (Indeed, since $X(r) \subset \mathcal{O}_K^n$, the global ring operations on K are irrelevant in the presence of all restricted analytic functions.) Write g_i for $\hat{f}_i \circ \hat{f}_{i-1} \circ \cdots \circ \hat{f}_1(x)$ for each $i = 1, \ldots, k$. Let us finish this case by induction on k. In fact, we will prove at the same time an additional statement by induction on k: there is an integer $N > 0$ such that for $x \in \mathcal{O}_K(r)^n$, one either has $f(x) = 0$ or $-N \le \operatorname{ord} f(x) \le N$, and, for any tuple of integers $\hat{\nu} = (\hat{\nu}_{ij})_{i,j}$, there are polynomials $p_{\hat{\nu},s}$ such that $f_s(a) = p_{\hat{\nu},s}(a)$ whenever $x \in \mathcal{O}_K(r)^n$ satisfies $\operatorname{ord} g_{ij}(x) = \hat{\nu}_{ij}$ for all i, j and where a is still such that $x = (\sum_{\ell=0}^{r-1} a_{j\ell} t^{\ell})_{j=1}^n$.

If $k = 1$, then clearly each of the f_s's is a polynomial in the tuple a (no need to specify $\hat{\nu}$), and there is an integer $M > 0$ such that $t^M f$ lies in $\mathcal{O}_K\langle x\rangle$. By the Noetherianity of any polynomial ring in finitely many variables over $\mathbb{C}$, there exists $N > M$ such that $\operatorname{ord} f(x) > N$ implies $f(x) = 0$ for $x \in (\mathcal{O}_K(r))^n$. It is thus sufficient to show the constructibility in $x \in (\mathcal{O}_K(r))^n$ of the following conditions:

(1) $\overline{\mathrm{ac}}(f(x)) = 1 \wedge \operatorname{ord}(f(x)) = \mu$,

(2) $\operatorname{ord}(f(x)) > N$,

(3) $\operatorname{ord}(f(x)) = \mu$,

for integers μ with $-N \le \mu \le N$. Each of these cases is straightforward; for example, condition (1) on $x \in (\mathcal{O}_K(r))^n$ is equivalent to

$$f_\mu(a) = 1 \bigwedge_{s=-N}^{\mu-1} f_s(a) = 0,$$

which is clearly a constructible condition on $a \in \mathbb{C}^{nr}$ (where a corresponds to x as above). Also N is as desired for the additional statement. This finishes the case that $k = 1$.

The case of general $k > 1$ goes as follows. By induction on k, we have constructibility and the additional statement for g_{k-1} for some $N > 0$ and all $\hat{\nu}$. Let us fix a tuple $\hat{\nu}$ with $-N \le \hat{\nu}_{ij} \le N$ for all i and j and a maximal consistent set W of conditions on $x \in (\mathcal{O}_K(r))^n$ of the form $\operatorname{ord} g_{ij}(x) = \hat{\nu}_{ij}$ or of the form $g_{ij}(x) = 0$, where $i < k$. Now the case that $x \in (\mathcal{O}_K(r))^n$ satisfies the conjunction of $\varphi(x)$ with the conditions from W can be finished as the argument for $k = 1$ (including the additional statement) by using again the Noetherianity of polynomial rings. This finishes the case that the term f does not involve the function symbol for field inversion.

Next, suppose that f contains a subterm g which is of the form g_0^{-1} for some term g_0 which does not involve field inversion. Choose $N > 0$ for g_0 as given by the base case of our induction and its additional statement, as shown above. For each integer ν with $-N \le \nu \le N$, consider the condition $B_\nu(x, \xi)$ on $(x, \xi) \in K^{n+1}$ being the conjunction of $\operatorname{ord}(\xi) = 0$ with

$$\operatorname{ord} g_0(x) = \nu \wedge \overline{\mathrm{ac}}\big(\xi g_0(x)\big) = 1$$

and with $\varphi(x)$, where we recall that φ is the condition (a), (b), (c), or (d).

Note that there is an $\mathcal{L}_{\mathrm{an}}^{\overline{\mathrm{ac}}}$-term $h(x, \xi)$, with (x, ξ) running over K^{n+1}, which does not involve field inversion and such that for $x \in \mathcal{O}_K^n$ and $\xi \in \mathbb{C}^\times \subset \mathcal{O}_K^\times$ with

$\operatorname{ord} g_0(x) = \nu \wedge \overline{\mathrm{ac}}(\xi g_0(x)) = 1$, we can write

$$g(x) = \frac{1}{g_0(x)} = \xi t^{-\nu} \frac{1}{1-(1-\xi t^{-\nu} g_0(x))} = h(x,\xi).$$

Indeed, h comes from developing $1/(1-m)$ as a power series in m running over the maximal ideal $t\mathcal{O}_K$ of $\mathcal{O}_K$ and evaluating at $m = 1-\xi t^{-\nu} g_0(x)$. Let us now, inside f, replace the subterm $g(x)$ by $h(x,\xi)$. Then we see by our ongoing induction on the number of occurrences of the function symbol $(\cdot)^{-1}$ that condition $B_\nu(x,\xi)$ on $(x,\xi) \in \mathcal{O}_K(r)^{n+1}$ yields a constructible subset A_ν of $\mathbb{C}^{r(n+1)}$. Hence, also the set A'_ν consisting of (x,ξ) in A_ν with moreover $\xi \in \mathbb{C}^\times \times \{0\}^{r-1} \subset \mathbb{C}^r$ is constructible. By the Chevalley–Tarski theorem, the image A''_ν of A'_ν under the coordinate projection to $\mathbb{C}^{nr}$ is also constructible, and A''_ν equals the set of $x \in \mathcal{O}_K(r)^n$ satisfying $\varphi(x) \wedge \operatorname{ord} g_0(x) = \nu$, where φ is still (a), (b), (c), or (d).

Finally, consider the condition $B^0(x)$ which is the conjunction of $g_0(x) = 0$ and the condition $\varphi'(x)$ obtained from $\varphi(x)$ by replacing the subterm $g(x)$ by zero in the term $f(x)$. By induction on the number of occurrences of the function symbol $(\cdot)^{-1}$, the condition $B^0(x)$ on $x \in \mathcal{O}_K^n(r)$ is Zariski-constructible and the extra statement also holds.

By construction, for $x \in \mathcal{O}_K^n(r)$, x lies in $X(r)$ if and only if $B^0(x)$ holds or $x \in A''_\nu$ for some ν with $-N \le \nu \le N$. This finishes the proof of the proposition. $\square$

Remark 3

In fact, we will prove Theorem 1 more generally for any $\mathcal{L}_{\mathrm{an}}^{\overline{\mathrm{ac}}}$-definable set $X \subset K^n$.

Proof of Theorem 1

We will prove the theorem for a subset $X \subset K^n$ which is $\mathcal{L}_{\mathrm{an}}^{\overline{\mathrm{ac}}}$-definable. Let $r > 0$ be an integer. By Proposition 2, $X(r)$ is a constructible subset of $\mathbb{C}^{rn}$. We prove the finiteness of $(X^{\mathrm{trans}})(r)$ by induction on the dimension ℓ of $X(r)$. If $\ell = 0$, then there is nothing left to prove since $X(r)$ is finite in this case and since $(X^{\mathrm{trans}})(r) \subset X(r)$. If $\ell > 0$, then write A for $X(r)$ and choose an algebraic family of algebraic (locally closed) curves $C_v \subset \mathbb{A}_{\mathbb{C}}^{rn}$ for $v \in V \subset \mathbb{A}_{\mathbb{C}}^s$ for some $s \ge 0$ such that the union of the sets $C_v(\mathbb{C})$ over $v \in V(\mathbb{C})$ equals $A(\mathbb{C}) \setminus F$, where F is a finite set. Such a family clearly exists. For each $v \in V(\mathcal{O}_K)$, let S_v be the image of $C_v(\mathcal{O}_K)$ under the map $p : \mathcal{O}_K^{rn} \to \mathcal{O}_K^n$ sending (x_{jk}) to $(\sum_{k=0}^{r-1} x_{jk} t^k)_{j=1}^n$. Clearly S_v is semialgebraic and of dimension 1 for each $v \in V(\mathcal{O}_K)$, since S_v is infinite and equal to the image under a semialgebraic function of an algebraic curve. (See, e.g., Section 3 of [20], or the dimension theory of [12] and Theorem 6.3.7 from [10].) For each $v \in V(\mathcal{O}_K)$, let S'_v be the subset of $S_v \cap X$ consisting of x such that $S_v \cap X$ is locally of dimension 1 at x. Let X' be $X \setminus (\bigcup_{v \in V(\mathcal{O}_K)} S'_v)$. Then clearly X' is $\mathcal{L}_{\mathrm{an}}^{\overline{\mathrm{ac}}}$-definable. Moreover, by construction, we have that $X'(r)$ is of dimension less than ℓ (as a constructible subset of $\mathbb{C}^{rn}$). Indeed, $C_v(\mathbb{C}) \cap X'(r)$ is finite for any $v \in V(\mathbb{C})$, since $(S_v \cap X) \setminus S'_v$ is finite for each $v \in V(\mathcal{O}_K)$. (Here we have used that the subset of points x in a semialgebraic set S of local dimension 0 at x is finite; see again [20] or [12].) Since clearly $X^{\text{trans}}(r)$ is contained in $X'^{\text{trans}}(r)$, we can replace X by X' and thus we are done by induction on ℓ. □

Proof of Proposition 1

Let N_k be a strictly increasing sequence, and consider the analytic function $\mathcal{O}_K \to \mathcal{O}_K$ given by the following converging power series f in $K[[x]]$:

$$f(x) = \sum_{i>0} t^i P_{N_i}, \quad \text{where } P_k = x^k \prod_{j=1}^{k} (x - j). \tag{1}$$

Let $X \subset \mathcal{O}_K^2$ denote the analytic set $y = f(x)$ for $x \in \mathcal{O}_K$. For each $i > 0$ and $j = 1, \dots, N_i$, note that $f(j)$ is a polynomial in t of degree less than i and therefore

$$\#X(i) \geq N_i. \tag{2}$$

On the other hand,

$$\operatorname{supp}_x f(x) \subset \bigcup_i [N_i, 2N_i], \quad \text{where } \operatorname{supp}_x\Big(\sum c_j x^j\Big) = \{j : c_j \neq 0\}. \tag{3}$$

If we choose N_i to grow sufficiently quickly (e.g., if for every $d \in \mathbb{N}$, eventually $N_i > 2dN_{i-1}$), then (3) implies that $f(x)$ is transcendental. Indeed, if f is algebraic of degree d, then by the Bézout theorem its order of contact with its $2N_{i-1}$-Taylor approximation should not exceed $2dN_{i-1}$. By choosing N_i in this manner, we obtain from (2) the lower bound $\#X^{\text{trans}}(i) = \#X(i) \geq N_i$, which finishes the proof. □

A mixed characteristic variant

In this subsection, let L be $\mathbb{Q}_p^{\text{unram}}$, the maximal unramified field extension of $\mathbb{Q}_p$ for some prime p. Let $\tau : \mathbb{F}_p^{\text{alg}} \to \mathcal{O}_L$ be the Teichmüller lifting, namely, the unique multiplicative section of the natural projection map $\mathcal{O}_L \to \mathbb{F}_p^{\text{alg}}$ with $\mathcal{O}_L$ the valuation ring of L, and where $\mathbb{F}_p^{\text{alg}}$ is an algebraic closure of $\mathbb{F}_p$.

We use analogous notation and definitions for L as above for K, with the following natural adaption to define $\mathcal{O}_L(r)$. For any integer $r \geq 1$, denote by $\mathcal{O}_L(r)$ the subset of $\mathcal{O}_L$ consisting of elements of the form $\sum_{i=0}^{r-1} \tau(a_i) p^i$. These elements can informally be seen as polynomials in p with coefficients in $\tau(\mathbb{F}_p^{\text{alg}})$ and degree less than r. We identify $\mathcal{O}_L(r)$ with $(\mathbb{F}_p^{\text{alg}})^r$ by sending $\sum_{i=0}^{r-1} \tau(a_i) p^i$ to the tuple $(a_i)_i$.

THEOREM 2 (Finiteness in mixed characteristic)
Let $X \subset L^n$ be $\mathcal{L}_{\text{an}}^L$-definable. Then for each $r > 0$, $X(r)$ is a constructible subset of

$(\mathbb{F}_p^{\mathrm{alg}})^{rn}$ and

$$(X^{\mathrm{trans}})(r)$$

is a finite set.

Proof
The proof is, as for Proposition 2 and Theorem 1, using the usual operations on Witt-vectors, with natural adaptations, in particular putting p in the role of t. For the first part, one adapts the proof of Proposition 2 by using also higher-order angular component maps and identifying $\mathcal{O}_L/p^k\mathcal{O}_L$ with $(\mathbb{F}_p^{\mathrm{alg}})^k$ in order to have quantifier elimination. A higher-order angular component map is a map $L \to \mathcal{O}_L/p^k\mathcal{O}_L$ for some integer $k > 0$ sending nonzero x to $p^{-\operatorname{ord} x}x \bmod p^k\mathcal{O}_L$ and zero to zero. For the second part, one repeats the proof of Theorem 1 where one takes a lift of the family of curves $C_v \subset \mathbb{A}^{rn}_{\mathbb{F}_p^{\mathrm{alg}}}$ to a family of curves C_v in $\mathbb{A}^{rn}_L$ by applying τ to the coefficients of the defining polynomials. $\square$

3. Non-Archimedean Wilkie-type conjecture for Pfaffian and Noetherian varieties

Our goal is to prove an analogue of the Wilkie conjecture over the field of convergent Laurent series $\mathbb{C}(\{t\})$ for varieties defined using Pfaffian and Noetherian equations. We begin by recalling these notions. Denote by $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ a system of coordinates on $\mathbb{C}^n$.

Definition 4 (Pfaffian and Noetherian functions)
A *Pfaffian chain* of order ℓ and degree α at $(\mathbb{C}^n, 0)$ is a sequence of holomorphic functions $\phi_1, \ldots, \phi_\ell : (\mathbb{C}^n, 0) \to \mathbb{C}$ satisfying a triangular system of differential equations

$$\mathrm{d}\phi_j = \sum_{i=1}^{n} P_{i,j}\big(\mathbf{x}, \phi_1(\mathbf{x}), \ldots, \phi_j(\mathbf{x})\big)\,\mathrm{d}\mathbf{x}_i, \quad j = 1, \ldots, \ell, \tag{4}$$

where $P_{i,j}$ are polynomials of degrees not exceeding α.

A *Noetherian chain* is defined similarly by dropping the triangularity condition, that is, replacing (4) by

$$\mathrm{d}\phi_j = \sum_{i=1}^{n} P_{i,j}\big(\mathbf{x}, \phi_1(\mathbf{x}), \ldots, \phi_\ell(\mathbf{x})\big)\,\mathrm{d}\mathbf{x}_i, \quad j = 1, \ldots, \ell. \tag{5}$$

Given such a Pfaffian (resp., Noetherian) chain, a germ $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ of the form $f(\mathbf{x}) = P(\mathbf{x}, \phi_1(\mathbf{x}), \ldots, \phi_\ell(\mathbf{x}))$, where P is a polynomial of degree not exceeding β with coefficients in $\mathbb{C}$, is called a *Pfaffian* (resp., *Noetherian*) *function* of order ℓ and degree (α, β).

Let $K = \mathbb{C}(\{t\})$ or $K = \mathbb{C}(t)$. Let $\Omega = (\mathbb{C}^n_{\mathbf{x}} \times \mathbb{C}_t, 0)$ denote the germ of $\mathbb{C}^n_{\mathbf{x}} \times \mathbb{C}_t$ at the origin. If P is a polynomial over the field K, then we will say that $f : \Omega \to \mathbb{C}$ of the form $f(\mathbf{x}, t) = P(\mathbf{x}, \phi_1(\mathbf{x}), \ldots, \phi_\ell(\mathbf{x}))$ as above is Pfaffian (resp., Noetherian) over K. If $K = \mathbb{C}(t)$, then we also define the t-degree of f by considering the total degree of P as a polynomial over $\mathbb{C}$ in $n+1$ variables, where we treat t as an additional variable and clearing the denominators.

Remark 5
In contrast to Section 2, note that we now restrict to K given by *convergent* rings. We also restrict, in Definition 4, to systems of differential equations with coefficients independent of t.

The former restriction is a technical convenience meant to give a situation more closely analogous to the analytic deformations studied in [14] and [1]. We believe it is likely one can prove analogous results in the Pfaffian case also for formal deformations. The latter restriction is more serious: in both the Pfaffian and the Noetherian contexts, the existing results produce bounds when one deforms a system of Pfaffian or Noetherian equations with a *fixed* chain, but not when one deforms the chain itself.

Denote by π_x (resp., π_t) the projection from Ω to the $\mathbb{C}^n$ (resp., $\mathbb{C}$) factor. We will say that an analytic germ $X \subset \Omega$ is *flat* if it is flat with respect to the projection π_t, that is, if X has no components contained in the fiber $\pi_t^{-1}(0)$. We will identify a K-variety in $(\mathbb{A}^n_K, 0)$ with the flat holomorphic variety in Ω defined by identifying t with the second factor in $\mathbb{C}^n \times \mathbb{C}$ and removing any nonflat components. This will allow us to apply results from the usual theory of Pfaffian varieties over $\mathbb{C}$ to the study of Pfaffian varieties over K.

Definition 6 (Pfaffian and Noetherian varieties)
If $f_1, \ldots, f_k : \Omega \to \mathbb{C}$ are Pfaffian (resp., Noetherian) over K with degree (α, β) and a common Pfaffian chain of order ℓ, then we define $V(f_1, \ldots, f_k) \subset \Omega$ to be the analytic germ obtained from $\{f_1 = \cdots = f_k = 0\}$ by removing any components contained in the fiber $t = 0$. We call a germ obtained in this manner a *Pfaffian* (resp., *Noetherian*) *variety* over K. Note that since we remove components over $t = 0$ there will be no harm in assuming from the start that the coefficients of the polynomials defining $f_1, \ldots, f_k$ are in fact holomorphic at $t = 0$, and the corresponding functions are in fact holomorphic germs.

Example 7

Let $n = 2$, and consider the Noetherian chain whose elements are given by x, y. The two equations $f_1 = x$ and $f_2 = x - ty$ define the isolated point $x = y = 0$ over generic t, but over $t = 0$ the entire y-axis appears as an extra component. In our notation $V(f_1, f_2)$ will be the analytic germ defined by $x = y = 0$.

Below we will suppose that a Pfaffian chain has been fixed, and in our asymptotic notation we will allow the constants to depend on α, n, ℓ (we note that all constants can be explicitly computed). We will refer to β in Definition 6 as the complexity of the Pfaffian (resp., Noetherian) variety. If $K = \mathbb{C}(t)$, then we define the notion of t-complexity by replacing degrees with t-degrees.

Let $X \subset \Omega$ be a flat analytic germ. If $p : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ is an analytic germ, then we denote by $\tilde{p} : (\mathbb{C}, 0) \to \Omega$ the map $t \to (p(t), t)$. If every coordinate of p is a polynomial, of degree less than r for some $r > 0$, then we write $\deg p < r$ (otherwise we set $\deg p = \infty$).

Definition 8

We denote

$$X(r) := \{p : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0) : \operatorname{Im} \tilde{p} \subset X, \deg p < r\}. \tag{6}$$

More generally, if $\mathbf{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$ is holomorphic, then we denote

$$X(\mathbf{g}, r) := \{\operatorname{Im} \tilde{p} : \operatorname{Im} \tilde{p} \subset X, \deg \mathbf{g}(p) < r\}. \tag{7}$$

Our first main result is the following analogue of the Wilkie conjecture for Pfaffian varieties over $\mathbb{C}(\{t\})$.

THEOREM 3

Let $X \subset \Omega$ be a Pfaffian variety over $\mathbb{C}(\{t\})$ of complexity β, and let $r \in \mathbb{N}$. Then there exists a collection $\{W_\eta\}_\eta$ of irreducible algebraic varieties over $\mathbb{C}(t)$ such that $W_\eta \subset X$ as germ at the origin and $X(r) \subset \bigcup_\eta W_\eta(r)$. Moreover,

$$\#\{W_\eta\} = \operatorname{poly}(\beta, r), \qquad \deg W_\eta = \operatorname{poly}(\beta, r). \tag{8}$$

Theorem 3 is analogous to the Wilkie conjecture combined with Pila's "blocks" formalism, where each algebraic W_η can be thought of as a block. In particular, any positive-dimensional W_η lies by definition in X^{alg}, and

$$X^{\mathrm{trans}}(r) = \bigcup_{\eta : \dim W_\eta = 0} W_\eta(r). \tag{9}$$

By irreducibility, $\#W_\eta(r) \leq 1$ for the 0-dimensional W_η, and one concludes that $\#X^{\text{trans}}(r) = \text{poly}(r, \beta)$.

Our second main result is the following analogue of the Wilkie conjecture for Noetherian varieties over $\mathbb{C}(t)$. We are unable to prove the same result over $\mathbb{C}(\{t\})$ in the Noetherian category due to certain limitations in the available complexity estimates for the Noetherian category (see Fact 17 and the following discussion). On the other hand, we remark that many of the classical transcendental functions involved in applications of the Pila–Wilkie theorem do lie in the Noetherian (and not in the Pfaffian) category, and the limitation of algebraic dependence on the variable t does not seem to be overly restrictive.

THEOREM 4
Let $X \subset \Omega$ be a Noetherian variety over $\mathbb{C}(t)$ of t-complexity β, and let $r \in \mathbb{N}$. Then there exists a collection $\{W_\eta\}_\eta$ of irreducible algebraic varieties over $\mathbb{C}(t)$ such that $W_\eta \subset X$ as germ at the origin and $X(r) \subset \bigcup_\eta W_\eta(r)$. Moreover,

$$\#\{W_\eta\} = \text{poly}(\beta, r), \qquad \deg W_\eta = \text{poly}(\beta, r). \tag{10}$$

The proofs of Theorems 3 and 4 are given in Section 5, after developing some preliminary material in Section 4.

4. Weierstrass polydisks and interpolation

4.1. Analytic germs and Weierstrass coordinates

Definition 9
Let $X \subset \Omega$ be a flat analytic germ of pure dimension $m+1$. Let $\mathbf{x} := \mathbf{z} \times \mathbf{w} : \mathbb{C}^n \to \mathbb{C}^p_{\mathbf{z}} \times \mathbb{C}^{n-p}_{\mathbf{w}}$ be a unitary linear map. We will say that $\mathbf{z} \times \mathbf{w}$ are *Weierstrass coordinates* for X if $p = m$ and the projection $\pi_z \times \pi_t : X \to (\mathbb{C}^m \times \mathbb{C}, 0)$ is finite. We denote by $e(X, \mathbf{x})$ the degree of this projection, that is, the number of points (counted with multiplicities) in the fiber of any point $p \in (\mathbb{C}^m \times \mathbb{C}, 0)$.

LEMMA 10
Weierstrass coordinates exist for every flat analytic germ $X \subset \Omega$ of pure dimension $m+1$.

Proof
Since X is flat, the fiber X_0 over $t = 0$ has pure dimension m. Let $\mathbf{x} = \mathbf{z} \times \mathbf{w}$ be unitary coordinates, and let $\Delta_z \times \Delta_w$ be a Weierstrass polydisk (in the sense of [3]) for X_0, that is, $X_0 \cap (\bar{\Delta}_z \times \partial\Delta_w) = \emptyset$. Since X is closed, for a sufficiently small disk

$D_t \subset (\mathbb{C},0)$ we also have $X \cap (\bar{\Delta}_z \times \partial\Delta_w \times \bar{D}_t) = \emptyset$. Then $(\Delta_z \times D_t) \times (\Delta_w)$ is a Weierstrass polydisk for X, and in particular the projection $\pi_z \times \pi_t : X \to (\mathbb{C}^m \times \mathbb{C}, 0)$ is finite. ☐

PROPOSITION 11

Let $X \subset \Omega$ be a flat analytic germ of pure dimension $m+1$ and $\mathbf{x}$ Weierstrass coordinates for X, and set $\nu := e(X,\mathbf{x})$. Then for any $f \in \mathcal{O}(\Omega)$ there exists a function

$$P \in \mathcal{O}_0(\mathbb{C}^m \times \mathbb{C})[w], \qquad \deg_{\mathbf{w}_i} P \le \nu - 1, \quad i = 1,\dots,n-m \tag{11}$$

such that $f|_X \equiv P|_X$, and where $\mathcal{O}_0$ stands for the holomorphic germs at zero.

Proof

The claim follows from [3, Proposition 7] applied to a Weierstrass polydisk in the $\mathbf{x}, t$ coordinates as constructed in the proof of Lemma 10. ☐

4.2. Interpolation determinants

In this section, we give a non-Archimedean analogue of the interpolation determinant method of Bombieri and Pila [4].

Let $X \subset \Omega$ be a flat analytic germ of pure dimension $m+1$ and $\mathbf{x}$ Weierstrass coordinates for X, and set $\nu := e(X,\mathbf{x})$. Let $\mathbf{p} = (p_1,\dots,p_\mu)$ with $p_j : (\mathbb{C},0) \to \mathbb{C}^n$, and let $\mathbf{f} = (f_1,\dots,f_\mu)$ with $f_j \in \mathcal{O}(\Omega)$. We define the *interpolation determinant* of $\mathbf{f}$ and $\mathbf{p}$ to be $\Delta(\mathbf{f},\mathbf{p}) := \det(f_i(\tilde{p}_j))$.

PROPOSITION 12

Suppose that $\operatorname{Im} \tilde{p}_j \subset X$ *for all j. Then*

$$\operatorname{ord}_t \Delta(\mathbf{f},\mathbf{p}) \ge C_n \nu^{-(n-m)/m} \mu^{1+1/m}, \tag{12}$$

where the constant C_n depends only on n.

Proof

Expand each f_j using Proposition 11, and then expand the determinant $\Delta(\mathbf{f},\mathbf{p})$ by linearity in each column over $\mathbb{C}\{\!\{t\}\!\}$. Each term of order k in the $\mathbf{x}$ variables, when evaluated at $\tilde{p}_j$, has order at least k in t. Moreover, in the decomposition of each f_i there are fewer than $\nu^{n-m}k^{m-1}$ linearly independent terms of each order k. It follows that

$$\operatorname{ord}_t \Delta(\mathbf{f},\mathbf{p}) \ge \sum_{k=0}^{b} (\nu^{n-m}k^{m-1}) \cdot k, \tag{13}$$

where

$$\mu \geq \sum_{k=0}^{b} \nu^{n-m} k^{m-1}. \tag{14}$$

Then we have $b \sim (\mu/\nu^{n-m})^{1/m}$, and plugging into (13) gives (12). □

4.3. Polynomial interpolation determinants

Fix $d \in N$, and let μ denote the dimension of the space of polynomials of degree at most d in $m+1$ variables. Note that $\mu \sim d^{m+1}$.

For $\mathbf{g}$ an $(m+1)$-tuple of functions and $\mathbf{p}$ a μ-tuple of points, we define the *polynomial interpolation determinant* $\Delta^d(\mathbf{g},\mathbf{p}) := \Delta(\mathbf{f},\mathbf{p})$, where $\mathbf{f}$ is the tuple of all monomials of degree at most d in the coordinates of $\mathbf{g}$.

PROPOSITION 13

Let $\mathbf{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}^{m+1}, 0)$*, and suppose that*

$$p_i \in X(\mathbf{g}, r+1), \quad j = 1, \ldots, \mu. \tag{15}$$

Then

$$\deg \Delta^d(\mathbf{g},\mathbf{p}) \leq E_n d^{m+2} r, \tag{16}$$

where E_n *is some constant depending only on* n*.*

Proof

There are $\mu \sim d^{m+1}$ columns in the matrix defining $\deg \Delta^d(\mathbf{g},\mathbf{p})$ and each of them has degree at most dr in t. □

COROLLARY 14

Let $\mathbf{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}^{m+1}, 0)$*. Then there exists a constant* A_n *depending only on* n *such that if*

$$d > A_n \nu^{n-m} r^m, \tag{17}$$

then $X(\mathbf{g}, r+1)$ *is contained in the zero locus of a polynomial* $P \in \mathbb{C}(\{t\})[\mathbf{g}]$ *of degree at most* d*.*

Proof

By elementary linear algebra, if $\Delta^d(\mathbf{g},\mathbf{p})$ vanishes for each $\mathbf{p}_i \in X(\mathbf{g}, r+1)$, then the conclusion of the corollary follows. To see that this indeed happens, we use Propositions 12 and 13, and we note that the order at zero of a polynomial cannot exceed its degree. Therefore, unless $\Delta^d(\mathbf{g},\mathbf{p})$ vanishes we have

$$C_n \nu^{-(n-m)/m} \mu^{1+1/m} \le \operatorname{ord}_t \Delta(\mathbf{g},\mathbf{p}) \le \deg \Delta^d(\mathbf{g},\mathbf{p}) \le E_n d^{m+2} r. \tag{18}$$

Since $\mu \sim d^{m+1}$ this gives

$$C_n d^{1/m} \le E_n \nu^{(n-m)/m} r, \tag{19}$$

and for d as in (17) we indeed obtain a contradiction. □

5. Proofs of the main theorems

5.1. The Pfaffian case

We will use the following result of Gabrielov.

FACT 15 ([14, Theorem 2.1])

Let $f_1,\dots,f_n : \Omega \to \mathbb{C}$ be Pfaffian over $\mathbb{C}(\{t\})$ of degree (α,β) over a common Pfaffian chain of order ℓ, and let $X = V(f_1,\dots,f_n)$. Then the number of isolated points (counted with multiplicities) in the fiber X_t converging to the origin as $t \to 0$ is bounded by $O(\beta^{n+\ell})$, where the constants depend only on α, n, ℓ.

We now deduce a general result on Weierstrass coordinates for Pfaffian varieties over $\mathbb{C}(\{t\})$. For an analytic germ $X \subset \Omega$, we denote by $X^{\le m}$ the union of the irreducible components of X having dimension m or less (and similarly for X^m).

THEOREM 5

Let $X \subset \Omega$ be a Pfaffian variety over $\mathbb{C}(\{t\})$ of complexity β, and let $1 \le m \le n$. Then there exists an analytic germ $Z \subset \Omega$ of pure dimension m satisfying $X^{\le m} \subset Z$ and Weierstrass coordinates $\mathbf{x}$ for Z such that $e(Z,\mathbf{x}) = O(\beta^{n+\ell})$.

Proof

Let $\tilde f_1,\dots,\tilde f_{n+1-m}$ be $n+1-m$ generic linear combinations of the Pfaffian functions defining X, and set $\tilde X := V(\tilde f_1,\dots,\tilde f_{n+1-m})$ and $Z = \tilde X^m$. It is easy to see that for a sufficiently generic choice, every component of $X^{\le m}$ is contained in a component of $\tilde X$ of dimension m and hence $X^{\le m} \subset Z$.

Let $\mathbf{x}$ be a set of Weierstrass coordinates for Z, and set $\pi^Z := \pi_z \times \pi_t|_Z$. Denote by Z_b the set of components of $\tilde X$ of dimension greater than m. Then $Z \cap Z_b$ has dimension strictly smaller than m, and $Y := \pi^Z(Z \cap Z_b)$ is a (strict) analytic germ in $(\mathbb{C}^m \times \mathbb{C}, 0)$. It follows that for a generic choice of the vector $v \in \mathbb{C}^m$, the line $\mathbb{C}\cdot(v,1)$ meets Y only at the origin. Thus, a fiber of the map π^Z over any point $(v\cdot t, t)$ with $t \in (\mathbb{C},0)$ and $t \ne 0$ consists of $\nu = e(Z,\mathbf{x})$ isolated points in $Z \setminus Z_b$ which converge to the origin as $t \to 0$. Each such isolated point is an isolated solution of the system

$$\{\tilde{f}_1 = \cdots = \tilde{f}_{n+1-m} = 0, \mathbf{z}_1 = v_1 \cdot t, \ldots, \mathbf{z}_m = v_m \cdot t\}, \tag{20}$$

and the bound on ν thus follows from Fact 15. □

The following proposition gives the basic induction step for the proof of the Wilkie conjecture over $\mathbb{C}(\{t\})$.

PROPOSITION 16

Let $X \subset \Omega$ be a Pfaffian variety over $\mathbb{C}(\{t\})$. Let $W \subset \Omega$ be an irreducible algebraic variety over $\mathbb{C}(\{t\})$ of dimension k over $\mathbb{C}(\{t\})$, and suppose that $X \subset W$ and $X \neq W$.

Then for any $r > 0$, there exists an algebraic hypersurface $H \not\supset W$ over $\mathbb{C}(\{t\})$ of degree $O(\beta^{(n+\ell)(n-k+1)} r^{k-1})$ such that $X(r) \subset (X \cap H)(r)$.

Proof

Let $\mathbf{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$ be a linear projection which is dominant on W. Note that necessarily $\dim_{\mathbb{C}} X < \dim_{\mathbb{C}} W = k + 1$. By Theorem 5, we choose an analytic germ $Z \supset X$ of pure dimension k (over $\mathbb{C}$) and Weierstrass coordinates $\mathbf{x}$ for Z such that $\nu := e(Z, \mathbf{x}) = O(\beta^{n+\ell})$. Then, by Corollary 14, we may choose a polynomial $P \in \mathbb{C}(\{t\})[\mathbf{g}]$ of degree at most $O(\nu^{n+1-k} r^{k-1})$ such that

$$X(r) \subset Z(\mathbf{g}, r) \subset H, \quad H = \{P = 0\} \tag{21}$$

and $W \not\subset H$ since $\mathbf{g}$ was assumed to be dominant on W. □

Proof of Theorem 3

First, set $X' = X, W' = \mathbb{A}^n_{\mathbb{C}(\{t\})}$. If $X' = W'$, then we can finish with $\{W_\eta\} = \{W'\}$. Otherwise, we may apply Proposition 16 with $W = W'$ to find a collection of $\operatorname{poly}(\beta, r)$ hypersurfaces $H_j \subset \mathbb{A}^n_{\mathbb{C}(\{t\})}$ such that

$$X'(r) \subset \bigcup_j (X' \cap H_j)(r). \tag{22}$$

Denote by $\{H'_k\}_k$ the collection of irreducible components of the hypersurfaces H_j. It will be enough to prove the claim for each pair $(X' \cap H'_k, W' \cap H'_k)$ and take the union of all the resulting collections $\{W_\eta\}$. For this, we repeat the same argument as above with (X', W') replaced by this pair.

Proceeding in this manner for n steps, we end up with W' of dimension 0 over $\mathbb{C}(\{t\})$ and $X' \subset W'$. If $X' = W'$, then we can take $\{W_\eta\} = \{W'\}$, and otherwise the intersection $X' \cap W'$ is empty and we take $\{W_\eta\} = \emptyset$. □

5.2. *The Noetherian case*

The proof in the Noetherian case is entirely analogous to the proof in the Pfaffian case, and we leave the detailed derivation for the reader. The only significant difference is that in this case Fact 15 should be replaced by the following.

FACT 17 ([1, Theorem 1])
Let $f_1,\dots,f_n:\Omega\to\mathbb{C}$ *be Noetherian over* $\mathbb{C}(t)$ *of* t*-degree* (α,β) *over a common Pfaffian chain of order* ℓ*, and let* $X=V(f_1,\dots,f_n)$*. Then the number of isolated points (counted with multiplicities) in the fiber* X_t *converging to the origin as* $t\to 0$ *is bounded by* $\operatorname{poly}(\beta)$*, where the constants depend only on* α,n,ℓ*.*

Note that, comparing with Fact 15, in Fact 17 the bound depends on the t-degree of the equations with respect to t. We must therefore restrict to varieties over $\mathbb{C}(t)$. It has been conjectured by Gabrielov and Khovanskii [15] that a similar result should hold without dependence on the t-degree, and this conjecture would indeed imply an analogue of Theorem 3 without the restriction to $\mathbb{C}(t)$.

5.3. *A final remark*

In new work in [5] it is shown that the finiteness result from Theorem 1 no longer holds if one weakens the condition of $\mathcal{L}_{\mathrm{an}}^{K}$-definability to just definability in a Hensel minimal structure. Hensel minimality is a non-Archimedean analogue of o-minimality introduced in [8] and [9]. Other bounds based on dimension (instead of on counting) are put forward and are shown in the Hensel minimal curve case in [5]. The higher-dimensional case remains open and will represent the non-Archimedean analogue of the bounds in o-minimal structures from [19].

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