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Godbillon–Vey sequence and Françoise algorithm [☆]

Pavao Mardešić [a,*], Dmitry Novikov [b], Laura Ortiz-Bobadilla [c], Jessie Pontigo-Herrera [c]

[a] *Université de Bourgogne, Institut de Mathématiques de Bourgogne – UMR 5584 CNRS, Université de Bourgogne, 9 avenue Alain Savary, BP 47870, 21078 Dijon, France*
[b] *Faculty of Mathematics and Computer Science, Weizmann Institute of Science, Rehovot, 7610001, Israel*
[c] *Instituto de Matemáticas, Universidad Nacional Autónoma de México (UNAM), Área de la Investigación Científica, Circuito exterior, Ciudad Universitaria, 04510, Ciudad de México, Mexico*

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ABSTRACT

We consider foliations given by deformations $dF + \epsilon\omega$ of exact forms dF in $\mathbb{C}^2$ in a neighborhood of a family of cycles $\gamma(t) \subset F^{-1}(t)$.

In 1996 Françoise gave an algorithm for calculating the first nonzero term of the displacement function Δ along γ of such deformations. This algorithm recalls the well-known Godbillon–Vey sequences discovered in 1971 for investigation of integrability of a form ω. In this paper, we establish the correspondence between the two approaches and translate some results by Casale relating types of integrability for finite Godbillon–Vey sequences to the Françoise algorithm settings.

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* Corresponding author.
E-mail addresses: mardesic@u-bourgogne.fr (P. Mardešić), dmitry.novikov@weizmann.ac.il (D. Novikov), laura@matem.unam.mx (L. Ortiz-Bobadilla), pontigo@matem.unam.mx (J. Pontigo-Herrera).

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1. Introduction

Let $\gamma_0 \subset \mathbb{C}^2$ be a regular curve, Σ a transversal to γ_0, F a holomorphic function defined on a tubular neighborhood $U \subset \mathbb{C}^2$ of γ_0, formed by regular curves $\gamma(t) \subset F^{-1}(t)$, $t \in F(\Sigma)$, with $\gamma_0 = \gamma(t_0)$.

Consider the integrable foliation $dF = 0$ and its holomorphic deformation

$$dF + \epsilon\omega = 0 \tag{1.1}$$

in U. We are interested in the displacement function Δ (holonomy along γ minus identity) of (1.1). Here $\Delta(t)$ denotes the holonomy of (1.1) along $\gamma(t)$. It can be developed as

$$\Delta(t) = \sum_{i\geq 1} \epsilon^i M_i(t). \tag{1.2}$$

The functions $M_i(t)$ are called *Melnikov functions*. If $\Delta \equiv 0$, this means that (1.1) has a first integral in a neighborhood of $\gamma(t)$. If not, then there exists a *first non-zero Melnikov function* M_μ.

1.1. Françoise algorithm

Françoise algorithm allows to compute the first nonzero Melnikov function M_μ. Let us first recall the following classical Lemma.

Lemma 1.1. *Given a holomorphic one-form ω and a family of cycles $\gamma(t) \subset \{F^{-1}(t)\}$, the following conditions are equivalent:*

(i) *The form ω verifies*

$$\int_\gamma \omega \equiv 0. \tag{1.3}$$

(ii) *There exists a function r holomorphic in a neighborhood of γ such that*

$$dF \wedge (\omega - dr) \equiv 0. \tag{1.4}$$

(iii) *There exist functions g and r holomorphic in a neighborhood of γ such that*

$$\omega = gdF + dr. \tag{1.5}$$

Note that the functions g and r are univalued in U but in general do not extend to polynomial, nor even univalued functions in $\mathbb{C}^2$.

Recall, the classical result of Poincaré and Pontryagin:

$$M_1(t) = -\int_{\gamma(t)} \omega.$$

If $M_1 \equiv 0$, then, by Lemma 1.1,

$$\omega = g_1 dF + dr_1,$$

and in that case, Françoise [3] proves the following theorem (see also [13], [7], [8], [4], [5], [11], [12], [10]):

Theorem 1.2. *Let (1.2) be the displacement function of (1.1). Assume that $M_i(t) \equiv 0$, for $i = 1, \ldots, k$. Then $M_{k+1}(t) = (-1)^{k+1} \int_{\gamma(t)} g_k \omega$, where $g_0 = 1$ and g_i, r_i verify*

$$g_{i-1}\omega = g_i dF + dr_i, \quad i = 1, \ldots, k. \tag{1.6}$$

The existence of the decomposition (1.6), follows by induction from Lemma 1.1.

Definition 1.3. We call any pair (g_i, r_i), verifying (1.6) an i-th Françoise pair associated to the deformation (1.1) and call the sequence (g_i, r_i), $i = 0, 1, \ldots$ a Françoise sequence. We say that the *length of a Françoise sequence* is ℓ, if ℓ is the smallest index such that $g_{\ell+1} = 0$. If there does not exist such an index, we say that the sequence is of infinite length.

1.2. Godbillon–Vey sequence

On the other hand, the classical *Godbillon–Vey sequence* is associated to a foliation defined by a single one form

$$\omega = 0. \tag{1.7}$$

It is a sequence of one-forms $\omega_0 = \omega$, ω_i, $i = 1, \ldots$ such that the formal one-form

$$\Omega = d\epsilon + \omega_0 + \sum_{i=1} \frac{\epsilon^i}{i!}\omega_i \tag{1.8}$$

in $\mathbb{C}^2 \times \mathbb{C}$ verifies the *formal integrability condition*

$$\Omega \wedge \tilde{d}\Omega = 0. \tag{1.9}$$

Here $\tilde{d} = d_\epsilon + d$ denotes the total differential with respect to all variables x, y, ϵ.

Condition (1.9) is equivalent to

$$\begin{array}{rcl} d\omega_0 & = & \omega_0 \wedge \omega_1, \\ d\omega_1 & = & \omega_0 \wedge \omega_2, \\ \dots & & \dots \\ d\omega_n & = & \omega_0 \wedge \omega_{n+1} + \sum_{k=1}^{n} \binom{n}{k} \omega_k \wedge \omega_{n-k+1}. \end{array}$$

We say that the *Godbillon–Vey sequence is of length* n if the forms ω_k vanish for $k \geq n$.

Definition 1.4. Let K be a differential field, G a function and K_G the extension of K by G. We say that the extension K_G is: *Darboux*, *Liouville* or *Riccati*, respectively, if it belongs to a finite sequence of field extensions starting from the field K. The extensions in each step are either algebraic or given respectively by solutions of the equations $dG = \eta_0$, $dG = G\eta_1 + \eta_0$ or $dG = G^2\eta_2 + G\eta_1 + \eta_0$, with η_i one-forms with coefficients in the corresponding field extensions.

In that case, we call the function G Darboux, Liouville or Riccati with respect to K.

In [1], Casale relates the length n of the Godbillon–Vey sequence to the type of first integral of the foliation given by (1.7):

Theorem 1.5.

(i) *There exists a Godbillon–Vey sequence of length* 1 *if and only if* (1.7) *has a Darboux first integral.*
(ii) *There exists a Godbillon–Vey sequence of length* 2 *if and only if* (1.7) *has a Liouvillian first integral.*
(iii) *There exists a Godbillon–Vey sequence of length* 3 *if and only if* (1.7) *has a Riccati first integral.*

Here we develop a version of Godbillon–Vey sequences well-adapted to studying a deformation of an integrable foliation given by (1.1). Recall that on the level $\epsilon = 0$ it is integrable (with first integral F). The Godbillon–Vey sequence gives a condition for verifying if this integrability extends to $\epsilon \neq 0$.

We define the form

$$\Omega = Rd\epsilon + (dF + \epsilon\omega)G, \tag{1.10}$$

with

$$G = \sum_{i=0} \epsilon^i G_i, \quad R = \sum_{i=0} \epsilon^i R_{i+1} \tag{1.11}$$

unknown functions and $G_0 \equiv 1$. The form (1.10) of Ω comes from the requirement to define the same foliation as (1.1) on each level $\epsilon = const$.

We give a relative version of the definition of different types of first integral for the deformation (1.1).

Definition 1.6. We denote by $K_{F,\omega}$ the field associated to the deformation (1.1). That is, the smallest differential field in a tubular neighborhood U of a cycle γ_0 containing the functions given by coefficients of dF and ω.

Let $F_\epsilon = \sum_{i=0}^{\ell} \epsilon^i F_i$, $\ell < \infty$, be a first integral of (1.1). We say that it is Darboux, Liouville or Riccati, respectively, if all F_i are in the corresponding extension of the field $K_{F,\omega}$.

Theorem 1.7. *[6] There exists a solution (G, R) of the equation*

$$\Omega \wedge \tilde{d}\Omega = 0, \tag{1.12}$$

if and only if the deformation preserves formal integrability along γ i.e. $\Delta \equiv 0$.

Proof. Indeed, if there exists a solution (G, R) of (1.7), then by Frobenius theorem, Ω defines a foliation in a neighborhood of $(\gamma(t), 0)$ in $\mathbb{C}^3$ transversal to $\epsilon = 0$. It follows from the existence of this foliation that the integrability on the level $\epsilon = 0$ is preserved on nearby levels. □

We will also consider the Godbillon–Vey equation up to order k with Ω, G, R given by (1.10) and (1.11):

$$\Omega \wedge \tilde{d}\Omega = 0 \mod \epsilon^{k+1}. \tag{1.13}$$

Definition 1.8. We call any pair (G_i, R_i), verifying (1.13) an i-th *Godbillon–Vey pair associated to the deformation* (1.1), (G_i, R_i), $i = 0, 1, \ldots$ is the *Godbillon–Vey sequence associated to the deformation.* We say that the length of a Godbillon–Vey sequence associated to the deformation is ℓ, if ℓ is the smallest index such that $G_{\ell+1} = 0$. If there does not exist such an index, we say that the sequence is of infinite length.

Remark 1.9. Note that the length is associated to any Françoise sequence or Godbillon–Vey sequence associated to the deformation (1.1).

However, one deformation (1.1) can have Françoise sequences (or Godbillon–Vey sequence) of different lengths. The minimal length is well defined and one can choose a Françoise sequence so that all $g_k = 0$, for $k > \ell$. The same applies for the Godbillon–Vey sequences.

Françoise pairs (g_i, r_i) and Godbillon–Vey pairs (G_i, R_i) exist for all $i = 1, 2, 3, \ldots$ if and only if the deformation preserves integrability along γ.

2. Main theorems

In this section we state our two main results. The first establishes the relationship between the Françoise pairs and the Godbillon–Vey pairs associated to the deformation. In particular it shows that the minimal length of Françoise sequences and Godbillon–Vey sequences coincide:

Theorem 2.1.

(i) *The Melnikov functions M_i, $i = 1, \ldots, k$, are identically equal to zero if and only if one can solve the equation*

$$\Omega \wedge \tilde{d}\Omega = 0 \mod \epsilon^{k+1}. \tag{2.1}$$

(ii) *For each choice of the Françoise sequence (g_i, r_i), $i = 1, \ldots, k$, the Godbillon–Vey sequence (G_i, R_i), $i = 1, \ldots, k$, can be chosen verifying the equations*

$$G_i = (-1)^i g_i, \quad R_i = (-1)^{i+1} i r_i. \tag{2.2}$$

(iii) *If Ω verifies (2.1) then*

a) *there exists a function $N = 1 + \sum_{i=1}^k \epsilon^i n_i$ such that*

$$\Omega = N\tilde{d}F_\epsilon \mod \epsilon^{k+1}.$$

Then the function F_ϵ is of the form

$$F_\epsilon = F + \sum_{i=1}^{k} (-1)^{i+1} \epsilon^i r_i. \tag{2.3}$$

and

$$\tilde{d}F_\epsilon = \tilde{R}d\epsilon + \tilde{G}\left(dF + \epsilon\omega\right). \tag{2.4}$$

b) *Let $\tilde{G}$ and $\tilde{R}$ be given in (2.4) and (G_i, R_i), $i = 1, \ldots, k$, be its coefficients as in (1.11). Then the functions (g_i, r_i), $i = 1, \ldots, k$, given by (2.2) are Françoise pairs.*

Our second result gives the type of local first integral F_ϵ of the deformation (1.1) if the length of its Françoise sequence is finite. The first result is that the first integral is in a finite sequence of extensions of Darboux type. The second shows that it is in a single extension of Liouvillian type.

Theorem 2.2. *Let $\eta_\epsilon = dF + \epsilon\omega$ as in (1.1) be such that there exists a Françoise sequence of finite length ℓ.*

(i) *Then* (1.1) *admits a univalued first integral which is Darboux with respect to the field* $K_{F,\omega}$ *of the deformation* (1.1)*.*

(ii) *Then there exists a meromorphic form* $\tilde{\eta}_\epsilon$ *verifying the Godbillon–Vey sequence of length* 2*:*

$$d\eta_\epsilon = \eta_\epsilon \wedge \tilde{\theta}_\epsilon$$
$$d\tilde{\theta}_\epsilon = 0,$$

such that there exists a (possibly multivalued) first integral $\tilde{F}_\epsilon$ *of* (1.1) *verifying*

$$d\tilde{F}_\epsilon = f\eta_\epsilon,$$

where

$$df = f\tilde{\theta}_\epsilon$$

is a (possibly multivalued) function in a tubular neighborhood U *of the cycle* γ_0*. In particular, the function* f *belongs to a Liouville extension of* $K_{F,\omega}$ *and* $\tilde{F}_\epsilon$ *belongs to a Darboux extension of this Liouville extension.*

Remark 2.3. Note that we are restricting our study to a tubular neighborhood U of a cycle γ_0. A first integral F_ϵ which is Darboux in U can be more complicated (Liouville, Riccati, ...) when studied globally.

Remark 2.4. In Theorem 2.2 (ii) we prove in particular that if the deformation (1.1) has a finite Françoise sequence, then it has a Liouvillian first integral. The converse is an interesting question.

Remark 2.5. In Theorem 2.2 we suppose that (1.1) has a Françoise sequence of finite order. What happens in the case of $\ell = \infty$? In particular, is it possible to give a condition assuring that a deformation (1.1) has a Liouville or a Riccati first integral in these terms?

3. Proof of Theorem 2.1

Proof. We first prove the direct implication of the statement (i), the converse will follow from (iii)(b). If the functions M_i identically vanish for $i = 1, \dots, k$, then one can build a first integral F_ϵ of $dF + \epsilon\omega \mod \epsilon^{k+1}$ in the following way: extend F to transversal Σ to $\gamma \times \{0\}$ in $\mathbb{C}^3 = \mathbb{C}^2_{x,y} \times \mathbb{C}_\epsilon$ as $F(x, y, \epsilon) = F(x, y)$, and extend it to a neighborhood U of $\gamma \times \{0\}$ in $\mathbb{C}^3$ by the flow. The extension F_ϵ is a multivalued function, but different branches of F_ϵ agree $\mod \epsilon^{k+1}$ on Σ by assumption, and therefore everywhere in U. In other words, in the decomposition $F_\epsilon = F + \sum_{i\geq 1} \epsilon^i F_i$, $F_i = F_i(x, y)$, the functions F_i are univalued for $i = 1, \dots, k$.

This implies that in the decomposition

$$\tilde{d}F_\epsilon = (F_\epsilon)'_\epsilon\, d\epsilon + (dF + \epsilon\omega)\, G,$$

the coefficients $(F_\epsilon)'_\epsilon, G$ are univalued modulo terms of order $\geq k+1$, and we take $\Omega := j_\epsilon^{k-1}(F_\epsilon)'_\epsilon\, d\epsilon + (dF + \epsilon\omega)\, j_\epsilon^k G$, where j_ϵ^k denotes the k-th jet with respect to ϵ. Then Ω verifies (2.1).

Using (1.6), the proof of (ii) follows from the computation:

$$\begin{aligned}(dF + \epsilon\omega)\left(1 + \sum_{i=1}^{k}(-1)^i \epsilon^i g_i\right) =& dF + \sum_{i=1}^{k} \epsilon^i \left((-1)^i g_i dF + (-1)^{i-1} g_{i-1}\omega\right)\\ =& dF + \sum_{i=1}^{k} \epsilon^i (-1)^{i-1} dr_i \mod \epsilon^{k+1},\end{aligned}$$

where $g_0 \equiv 1$. Therefore, by (2.2) and (1.11),

$$\begin{aligned}\Omega = Rd\epsilon + G(dF + \epsilon\omega) =& \left(\sum_{i=1}^{k}(-1)^{i-1} i\epsilon^{i-1} r_i\right) d\epsilon + dF + \sum_{i=1}^{k} \epsilon^i (-1)^{i-1} dr_i\\ =& \tilde{d}\left(F + \sum_{i=1}^{k} \epsilon^i (-1)^{i-1} r_i\right) \mod \epsilon^{k+1}\end{aligned}$$

is closed up to order ϵ^{k+1} and therefore satisfies (2.1).

We prove statement (iii)(a) and (iii)(b) simultaneously by induction. We define weights of monomials by posing $w(x) = w(y) = w(dx) = w(dy) = 0$ and $w(\epsilon) = w(d\epsilon) = 1$, so d and $\tilde{d}$ preserve weights. We will denote by o_k any collection of terms of weight $> k$. In these notations, (2.1) is equivalent to

$$\Omega \wedge \tilde{d}\Omega = o_{k+1}. \tag{3.1}$$

Let $N_j = 1 + \sum_{i=1}^{j} \epsilon^i n_i$. We construct the function $N = N_k$ by induction.

Consider first $k = 0$. A simple computation shows that

$$\Omega \wedge \tilde{d}\Omega = dF \wedge d\epsilon \wedge (\omega - dR_1) + o_1,$$

so, simplifying by $\wedge d\epsilon$, (3.1) for $k = 0$ is equivalent to

$$dF \wedge (\omega - dR_1) = 0.$$

By Lemma 1.1, this equation can be solved if and only if $\int_\gamma \omega \equiv 0$, i.e. if and only if the first Françoise condition $M_1 \equiv 0$ is satisfied. Therefore, the existence of Ω satisfying

(2.1) for $k = 0$ is equivalent to the first Françoise condition, and we can choose r_1 in (1.6) to be equal to R_1,

$$\omega = dr_1 + g_1 dF.$$

Hence,

$$\Omega = r_1 d\epsilon + (dF + \epsilon\omega)(1 + \epsilon\beta_1) + o_1 \tag{3.2}$$

for some function β_1. Therefore,

$$\Omega = [r_1 d\epsilon + (dF + \epsilon\omega)(1 - \epsilon g_1)]\,(1 + \epsilon(\beta_1 + g_1)) + o_1 = N_1 \tilde{d}F_{\epsilon,1} + o_1,$$

where $N_1 = 1 + \epsilon(\beta_1 + g_1)$ and $F_{\epsilon,1} = F + \epsilon r_0$.

Now, let $k > 0$ and assume (3.1). In particular, it means that $\Omega \wedge \tilde{d}\Omega = o_k$. By induction, we have

$$\Omega = N_{k-1}\tilde{d}F_{\epsilon,k-1} + o_{k-1}, \quad \text{where} \quad F_{\epsilon,k-1} = F + \epsilon r_0 - \ldots + (-1)^{k-2}\epsilon^{k-1} r_{k-1}.$$

Define

$$\Theta = N_{k-1}^{-1}\Omega = \tilde{d}F_{\epsilon,k-1} + \theta_k + o_k,$$

where θ_k is homogeneous of weight k. We have $\Theta \wedge \tilde{d}\Theta = \Omega \wedge \tilde{d}\Omega = o_k$.

But $\tilde{d}\Theta = \tilde{d}\theta_k$ has weight k. Therefore

$$\Theta \wedge \tilde{d}\Theta = dF \wedge \tilde{d}\theta_k + o_k.$$

Note that Θ has form (1.10), with G_i, R_i as in (2.2) for $i \leq k-1$. Separating terms of weight k, we get

$$\theta_k = \epsilon^{k-1} R_k d\epsilon + \epsilon^k G_k dF + (-1)^{k-1}\epsilon^k g_{k-1}\omega.$$

Therefore

$$0 = dF \wedge \tilde{d}\theta_k = \epsilon^{k-1} dF \wedge d\epsilon \wedge \left((-1)^{k-1} k g_{k-1}\omega - dR_k\right). \tag{3.3}$$

As Θ is a solution of (2.1), this equation is solvable, which, by Lemma 1.1, means that $\int_\gamma g_{k-1}\omega \equiv 0$, i.e. that the k-th Melnikov function vanishes identically. Moreover, (3.3) implies

$$k g_{k-1}\omega = (-1)^{k-1} dR_k + k g_k dF,$$

i.e. Françoise decomposition (1.6) of $g_{k-1}\omega$ with k-th Francoise pair (g_k, r_k), such that $R_k = (-1)^{k-1} k r_k$.

Therefore

$$\begin{aligned}\Theta &= \left(r_0 + \ldots + \epsilon^{k-1}(-1)^{k-1} k r_k\right) d\epsilon \\ &+ (dF + \epsilon\omega)\left(1 + \ldots + (-1)^{k-1}\epsilon^{k-1} g_{k-1} + \epsilon^k G_k\right) + o_k = \\ &= \left(1 + \epsilon^k (G_k + (-1)^{k-1} g_k)\right) \tilde{d}F_{\epsilon,k} + o_k,\end{aligned}$$

where $F_{\epsilon,k} = F + \epsilon r_0 - \ldots + (-1)^{k-1}\epsilon^k r_k$, and

$$\Omega = N_k \tilde{d}F_{\epsilon,k} + o_k, \quad N_k = N_{k-1}\left(1 + \epsilon^k (G_k + (-1)^{k-1} g_k)\right),$$

as required. □

4. Proof of Theorem 2.2

Proof. Proof of (i): Let (g_i, r_i), $i = 0, 1, 2, \ldots$ be a Françoise sequence and assume that $g_i = 0$, for $i \geq \ell + 1$ (see Remark 1.9). Let

$$\eta_\epsilon = dF + \epsilon\omega, \quad G = \sum_{i=0}^{\ell} (-1)^i \epsilon^i g_i, \quad F_\epsilon = F + \sum_{i=1}^{\ell} (-1)^{i+1} \epsilon^i r_i. \tag{4.1}$$

It follows from the definition of Françoise pairs (1.6) that

$$G\eta_\epsilon = dF_\epsilon. \tag{4.2}$$

Differentiating (1.6) and dividing by dF (that is, applying the Gelfand–Leray derivative), one obtains

$$dg_i = \frac{dg_{i-1} \wedge \omega}{dF} + g_{i-1}\frac{d\omega}{dF} =: \eta_{i-1}.$$

By induction, from the Definition 1.4, g_i is Darboux, for $i = 0, \ldots, \ell$. It now follows from (1.6) that r_i, $i = 1, \ldots, \ell$, is Darboux as well and by Definition 1.6, the first integral F_ϵ is Darboux with respect to the field $K_{F,\omega}$.

Proof of (ii): Let η_ϵ, G and F_ϵ be as in (4.1). Now from (4.2) it follows that

$$d\eta_\epsilon = \frac{dG^{-1}}{G^{-1}} \wedge G^{-1} dF_\epsilon = \theta_\epsilon \wedge \eta_\epsilon, \quad \text{for } \theta_\epsilon = \frac{dG^{-1}}{G^{-1}}.$$

Hence, $d\theta_\epsilon = 0$. The two equations together give a Godbillon–Vey sequence of length 2 in a Liouville extension of the space of forms with coefficients in $K_{F,\omega}$ in $(x, y) \in U$ and holomorphic with respect to the parameter ϵ.

Now Singer's theorem [9] (see also [2]) gives that there exists a form $\tilde{\theta}_\epsilon$ with coefficients in $K_{F,\omega}$ (holomorphic with respect to ϵ) verifying the same Godbillon–Vey equations:

$$d\eta_\epsilon = \eta_\epsilon \wedge \tilde{\theta}_\epsilon,$$
$$d\tilde{\theta}_\epsilon = 0.$$

That is $\tilde{\theta}_\epsilon$ is closed. Hence, there exists a (possibly multivalued) function f defined in U such that

$$df = f\tilde{\theta}_\epsilon.$$

One verifies that the form $f\tilde{\theta}_\epsilon$ is closed. This means that there exists a (possibly multivalued) function $\tilde{F}_\epsilon$ verifying

$$d\tilde{F}_\epsilon = f\eta_\epsilon. \quad \square$$

5. Classical Godbillon–Vey sequences and examples

Let Ω be the form given by (1.10). We apply the classical Godbillon–Vey condition (1.9) to the form

$$\frac{\Omega}{R} = d\epsilon + \eta_0 + \epsilon\eta_1 + \ldots + \tfrac{\epsilon^i}{i!}\eta_i + \ldots$$

Comparing to the closed form (1.10), we conclude that the forms $\eta_1, \ldots$ defined by

$$\eta_\epsilon = \sum \frac{\epsilon^i}{i!}\eta_i = \frac{dF_\epsilon}{d_\epsilon F_\epsilon} = \frac{dF + \sum_{i=1}^{\infty} \epsilon^i (-1)^{i-1} dr_i}{\sum_{i=1}^{\infty} (-1)^{i-1} i\epsilon^{i-1} r_i} \tag{5.1}$$

from a Godbillon–Vey sequence of $\frac{dF}{R_1}$:

$$\eta_0 = R_1^{-1} dF, \quad \eta_1 = R_1^{-1}(2R_2 dF + dR_1), \quad \ldots,$$

and the forms

$$\tilde{\eta}_1 = 2R_1^{-1}(R_2 dF + dR_1), \ldots, \tilde{\eta}_i = R_1^{i-1}\eta_i,$$

form a Godbillon–Vey sequence of dF. This sequence could be infinite.

In classical setting one starts from a given foliation $\omega = 0$ for $\epsilon = 0$, and looks for a simplest perturbation $\omega_\epsilon = 0$ such that the form $d\epsilon + \omega_\epsilon$ is integrable. Results of [2] say that if the foliation $\omega = 0$ is Darboux integrable, Liouville integrable or Riccati integrable, then one can find perturbations such that ω_ϵ either does not depend on ϵ or is polynomial in ϵ of degree 1 or 2, respectively, i.e. that the Godbillon–Vey sequence has finite length.

In this paper, given a perturbation (1.1), and we construct a one-form Ω such that its restriction to the planes $\{\epsilon = \text{const}\}$ defines the same foliation as the initial one. In other words, unlike the classical settings, here the perturbation of the foliation is almost uniquely prescribed, the only freedom being the coefficients G, R in (1.10). Thus the length of the corresponding Godbillon–Vey sequence can be infinite even if ω_ϵ is Liouville integrable for all ϵ.

Example 5.1. Let $F = x^2 + y^2$ and $\omega = y^2dx$. For symmetry reasons, the perturbation (1.1) is integrable. Computation shows that

$$g_n = \frac{(-1)^n}{n!}x^n, \quad r_n = \frac{(-1)^n}{n!}\left(\frac{2}{n+2}x^{n+2} - x^n y^2\right). \tag{5.2}$$

Then a first integral is given by

$$\begin{aligned} F_\epsilon =& x^2 + y^2 + \sum_{n=1}^{\infty}(-1)^{n-1}\epsilon^n r_n = e^{\epsilon x}\left(y^2 + 2\,\frac{x}{\epsilon} - 2\,\epsilon^{-2}\right), \\ \tilde{d}F_\epsilon =& e^{\epsilon x}(\epsilon y^2 + 2x)dx + 2e^{\epsilon x}ydy, \end{aligned}$$

and therefore the Godbillon–Vey forms ω_i are Taylor coefficients in ϵ of

$$\left(\frac{\partial}{\partial\epsilon}F_\epsilon\right)^{-1} dF_\epsilon = dF + \sum_{n=1}^{\infty}\epsilon^i\omega_i. \tag{5.3}$$

One can see that this series is not polynomial in ϵ, though the first integral F_ϵ is of Liouville type. Some authors call this type of functions generalized Darboux.

Example 5.2. For a trivially integrable perturbation $\omega = gdF$, the Françoise pairs are given by $g_i = g^i$, $i = 1, \ldots$, and $r_1 \equiv 1$, $r_i = 0$ for $i = 2, \ldots$. Therefore the first integral is

$$F_\epsilon = F + \epsilon, \qquad R \equiv 1, \quad G = 1 + \sum_{i=1}^{\infty}(-1)^i\epsilon^i g^i = (1 - \epsilon g)^{-1},$$

and

$$\Omega = d\epsilon + (dF + \epsilon g dF)(1 - \epsilon g)^{-1} = dF_\epsilon.$$

Example 5.3. For a Darboux integrable perturbation (1.1) with $\omega = F\frac{dr}{r}$ we have

$$g_i = (-1)^i\frac{(\log r)^i}{i!}, \qquad r_i = -Fg_i,$$

so

$$F_\epsilon = F + \sum_{i=1}^{\infty}(-1)^{i-1}\epsilon^i r_i = F + F\sum_{i=1}^{\infty}\epsilon^i \frac{(\log r)^i}{i!} = Fe^{\epsilon \log r} = Fr^\epsilon.$$

Then

$$\tilde{d}F_\epsilon = Fr^\epsilon \log r d\epsilon + \left(r^\epsilon dF + \epsilon F r^{\epsilon-1} dr\right) = Fr^\epsilon \log r \left[d\epsilon + \left(\frac{dF}{F\log r} + \epsilon \frac{dr}{r\log r}\right)\right].$$

Therefore the forms

$$\omega_1 = \frac{dr}{r\log r}, \quad \omega_i = 0, \qquad i = 2, \ldots$$

form a Godbillon–Vey sequence for $\frac{dF}{F\log r}$, and hence

$$\tilde{\omega}_1 = \frac{dr}{r\log r} + \frac{dF}{F} + \frac{dr}{r}, \quad \tilde{\omega}_i = 0, \qquad i = 2, \ldots$$

form a Godbillon–Vey sequence for dF, so this Godbillon–Vey sequence for $dF + \epsilon\omega$ has length 1.

Conflict of interest statement

No conflict of interest.

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