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Nilpotence of orbits under monodromy and the length of Melnikov functions[☆]

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ABSTRACT

Let $F \in \mathbb{C}[x, y]$ be a polynomial, $\gamma(z) \in \pi_1(F^{-1}(z))$ a non-trivial cycle in a generic fiber of F and let ω be a polynomial 1-form, thus defining a polynomial deformation $dF + \epsilon\omega = 0$ of the integrable foliation given by F.

We study different invariants: the *orbit depth* k, the *nilpotence class* n, the *derivative length* d associated with the couple (F, γ). These invariants bind the *length* ℓ *of the first nonzero Melnikov function* of the deformation $dF+\epsilon\omega$ along γ. We analyze the variation of the aforementioned invariants in a simple but informative example, in which the polynomial F is defined by a product of four lines. We study as well the relation of this behavior with the length of the corresponding Godbillon–Vey sequence. We formulate a conjecture motivated by the study of this example.

1. Introduction, main results and conjectures

This work is motivated by the *16th Hilbert's problem* or rather its *infinitesimal version*. As it is known, the second part of Hilbert's 16-th problem asks for an upper bound in terms of the degree for the number of real *limit cycles*, i.e. isolated periodic orbits of polynomial vector fields in the plane. The problem is far from been solved and the existence of such a number is open even for quadratic vector fields.

Arnold formulated the *infinitesimal Hilbert's problem*, which asks for a bound on the number of (real) limit cycles that can arise under polynomial deformations from an integrable polynomial differential equation in the plane. This bound must be uniform in the sense that it must depend exclusively on the degree of the integrable 1-form defining the corresponding foliation, and the degree of the polynomial that realizes the perturbation.

In one of its forms Arnold's infinitesimal Hilbert problem [1] studies the following situation.

Let $F(x, y) \in \mathbb{R}[x, y]$ be a square-free polynomial, z a regular value of F and $\gamma(z) \subset F^{-1}(z)$ a continuous family of real cycles of $F^{-1}(z)$. We will consider the complexification of the polynomial F which, for the sake of simplicity, we will denote again by F. The polynomial F defines a singular fibration and hence a singular foliation given by the integrable one-form

$$dF = 0\,. \tag{1.1}$$

Consider the polynomial deformations depending on the parameter ϵ

$$dF + \epsilon\omega = 0\,. \tag{1.2}$$

As usual, one defines the displacement function (holonomy map minus identity) Δ_ϵ of the deformation (1.2) along $\gamma(z)$. The isolated zeros of the displacement map Δ_ϵ are in correspondence with the limit cycles of (1.2).

The Taylor series expansion with respect to ϵ at 0 of Δ_ϵ is given by

$$\Delta_\epsilon(z) = \sum_{i=\mu} \epsilon^i M_i(z). \tag{1.3}$$

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The functions M_i are called *Melnikov functions*. We assume $M_\mu \not\equiv 0$ and call it the *first non-zero Melnikov function*. By the implicit function theorem, for regular values of z, it carries the main information on the number of limit cycles that can arise under the deformation. This motivates the study of this function M_μ, the subject of the infinitesimal Hilbert 16th problem.

It is known that M_μ is an iterated integral [2]. Let ℓ denote its *iterated integral length*, or shorter, simply *length*. It measures the complexity of M_μ. Bounding this length, would be a key step towards the bound of the limit cycles arising under perturbation. In [3], Gavrilov and Iliev gave a condition which guarantees that M_μ is an abelian integral (iterated integral of length 1). We showed the necessity of this condition in [4].

Let z be a generic value of F. Consider the fundamental group π_1 of $F^{-1}(z)$, and let L_j be its lower central sequence;

$$L_j = [L_{j-1}, \pi_1], \quad \text{where} \quad L_1 = \pi_1. \tag{1.4}$$

Using the fibration given by F, we define the monodromy group, which is a subgroup of the group of automorphisms of π_1. Then denote by $\mathcal{O} = \mathcal{O}_\gamma$ the normal subgroup of π_1 generated by the orbit of γ under the action by the monodromy group (see [4]).

In [4], we defined the *orbit depth* k of the cycle γ of F by

$$k = \sup\{j \geq 1 | \mathcal{O} \cap L_j \not\subset [\mathcal{O}, \pi_1]\} = \min\{j \geq 1 | \mathcal{O} \cap L_{j+1} \subset [\mathcal{O}, \pi_1]\}, \tag{1.5}$$

and showed that the orbit depth k bounds the length of the iterated integral M_μ, for the displacement map Δ_ϵ of any polynomial deformation (1.2), i.e. that

$$\ell \leq k. \tag{1.6}$$

In [5], we gave an example of a system having unbounded depth and formulated the following conjectures:

Conjecture 1.1.

(i) *For any polynomial F and any non-trivial cycle γ of F, either the depth is unbounded, or it is* 1, *or* 2.
(ii) *For any F and its cycle γ, either there exist deformations ω, whose first non-zero Melnikov function M_μ is of arbitrary high length, or for any deformation ω, the length of M_ω is* 1 *or* 2.

This conjecture is similar in spirit to the result of Casale [6] , concerning the first integral and the length of the corresponding Godbillon–Vey sequence of a system or the Tits alternative [7].

One of the central components in the study is the *orbit complement abelianization group* or *$\mathcal{O}$-abelianization*

$$\pi_1^{\mathcal{O}-ab} := \frac{\pi_1}{\mathcal{O} \cap L_2}. \tag{1.7}$$

Note that it is associated to the couple (F, γ) and does not depend on the deformation form ω.

In order to give some steps towards the proof of the above conjecture, we consider some other, more classical invariants associated to the group $\pi_1^{\mathcal{O}-ab}$: its *nilpotence class* n and its *derivative length* d, which we relate to the orbit depth k and length ℓ of the first nonzero Melnikov function (see Section 2 for definitions).

Proposition 1.2. *Let $F \in \mathbb{R}[x, y]$ be a polynomial, $\gamma \in \pi_1(F^{-1}(z))$ be a real cycle and ω a polynomial* 1*-form as above, and let $\pi_1^{\mathcal{O}-ab}$ be the group defined in (1.7). Let n be the* nilpotence class *of $\pi_1^{\mathcal{O}-ab}$, d the* derivative length *of $\pi_1^{\mathcal{O}-ab}$, k the* orbit depth *and ℓ the* length of the first nonzero Melnikov *function of the deformation (1.2) along γ. Then the following inequalities hold*

$$\ell \leq k \leq n + 1, \quad d \leq n. \tag{1.8}$$

Remark 1.3. Although $d \leq n$, there is no evident relationship between d and k.

General approach 1.4. The Poincaré (holonomy) map of a deformation (1.2) induces a homomorphism

$$P_\omega : \pi_1 \to \mathbb{C}[\epsilon] \otimes \mathrm{Diff}\,(\mathbb{C}, 0)\,, \quad P_\omega(\delta) = \left\{ z \mapsto z + \epsilon \int_\delta \omega + O(\epsilon^2) \right\}, \tag{1.9}$$

where $P_\omega(\delta)$ is the Poincaré map with respect to the foliation (1.2) along the cycle δ.

Assume that the deformation (1.2) preserves the continuous families of cycles corresponding to elements of $\mathcal{O} \cap L_2$. This means that $P_\omega(\mathcal{O} \cap L_2) = \{Id\}$ and the map (1.9) descends to the homomorphism

$$P_\omega : \pi_1^{\mathcal{O}-ab} \to \mathbb{C}[\epsilon] \otimes \mathrm{Diff}(\mathbb{C}, 0). \tag{1.10}$$

Moreover, if one defines a deformation form ω in such a way that the obtained subgroup of diffeomorphisms is parabolic, then the group $\pi_1^{\mathcal{O}-ab}$ inherits properties of subgroups of parabolic diffeomorphisms. Using these properties, one could conclude that either $\pi_1^{\mathcal{O}-ab}$ is abelian, and so the orbit depth is less than or equal to 2, or it is non-solvable.

However, the difficulty here lies in finding a one-form ω such that, for a cycle $\sigma \in [\pi_1, \pi_1] \setminus \mathcal{O} \cap L_2$, $P_\omega(\sigma) \neq id$, but it is the identity along any commutator in the orbit.

Here, we realize the above approach in the case of a Hamiltonian F given as a product of four real lines

$$\begin{aligned} &F(x, y) = f_1(x, y)f_2(x, y)f_3(x, y)f_4(x, y), \\ &f_i = a_i x + b_i y + c_i, (a_i, b_i) \neq (0, 0), \quad (a_i, b_i) \neq (a_j, b_j), \text{ for } i \neq j. \end{aligned} \tag{1.11}$$

Recall that, as studied in [5], in the above case, if f_i, $i = 1, \ldots, 4$, consist of two pairs of parallel lines, the orbit depth k is infinite. Any Hamiltonian given by a product of four different lines is of one of the following three types:

(1) The four lines are in *generic* position (no parallel lines among f_i, $i = 1, \ldots, 4$, for different indices).
(2) One of the bounded domains bounded by these lines is a quadrilateral with exactly one pair of parallel opposite sides (we call it a *trapezoid*).
(3) One of the bounded domains bounded by these lines is a parallelogram (we call it a *parallelogram*).

Theorem 1.5. *Let F be the product of four real lines (1.11), and $\gamma \in \pi_1(F^{-1}(z))$ a continuous family of real cycles of F.*

(1) *In the case that the four lines are in generic position, then $\pi_1^{\mathcal{O}-ab}$ is abelian, so its nilpotence class n and derived length d are equal to 1. Hence the orbit depth $k \leq 2$, and the length of the first non-zero Melnikov function is $\ell \leq 2$.*
(2) *In the trapezoid case $\pi_1^{\mathcal{O}-ab}$ is non-solvable $d = n = \infty$, but the orbit depth is $k \leq 2$. Hence, the length of the first non-zero Melnikov function of any deformation is bounded by 2, $\ell \leq 2$.*
(3) *In the parallelogram case $\pi_1^{\mathcal{O}-ab}$ is non-solvable $d = n = \infty$ and the orbit depth is infinite $k = \infty$.*

Remark 1.6. We stress that the realization of the length is required in order to prove the conjecture in [4] that states that the orbit depth is an optimal bound for the length of Melnikov functions. In [5] we provide a deformation of the parallelogram that has a first non-zero Melnikov function of length 3, which already distinguishes this case from the trapezoid, in terms of the length, where the length is less than or equal to 2.

The case of four lines in generic position has been studied in [4,8]. We recall it here for completeness. In this work, our aim is to focus on more degenerated situations, the trapezoid and parallelogram cases.

Since the nilpotence class n of $\pi_1^{\mathcal{O}-ab}$ provides an upper bound for the orbit depth k, we also want to understand conditions under which $\pi_1^{\mathcal{O}-ab}$ is non-nilpotent (or non-solvable). From the definition of the homomorphism in (1.10) we see that the nilpotence class of $\pi_1^{\mathcal{O}-ab}$ is greater than the nilpotence class of its image under P_ω. In this sense, the class of nilpotence of $\pi_1^{\mathcal{O}-ab}$ is related with the type of deformations preserving centers in $\mathcal{O}\cap L_2$, and therefore, with the type of integrability of these deformations.

Theorem 1.7.

(1) *In the parallelogram and the trapezoid cases, there exist polynomial* 1*-forms ω such that the deformations* (1.2) *preserve the pairs of parallel lines and have a first integral of Riccati type.*
(2) *Moreover, there are deformations, preserving the pairs of parallel lines, having Godbillon–Vey sequences of any finite length.*

Remark 1.8. In [9] the authors define the length of a foliation as the minimal length among all Godbillon–Vey sequences for the foliation. They mention that they do not know any example of finite length greater than 4. Deformations in the second part of Theorem 1.7 could provide such examples if one can prove that its Godbillon–Vey sequence has optimal length.

Conjecture 1.9.

(1) *The non-solvability of the group $\pi_1^{\mathcal{O}-ab}$ is characterized by the presence of a pair of parallel curves in the Hamiltonian foliation.*
(2) *The non-bounded orbit depth is characterized by the presence of two pairs of parallel curves in the Hamiltonian foliation.*
(3) *The type of singularity given by a pair of parallel curves at the line at infinity characterizes the non-solvability of $\pi_1^{\mathcal{O}-ab}$.*

Here by parallel curves, we mean two level curves $f^{-1}(c_1)$ and $f^{-1}(c_2)$, of the same function f, for $c_1\neq c_2$. Parallel curves intersect only at the line at infinity.

2. Nilpotence class and derivative length

Definition 2.1.

(i) Given a group G, let G_i be its *lower central sequence*:

$$G=G_1\supset G_2\supset\cdots,\quad G_{j+1}=[G_j,G].$$

If there exists $j\in\mathbb{N}$ such that $G_j=\{e\}$, we say that the group is *nilpotent* and define its *nilpotence class* $n=n(G)$ as

$$n=\min\{j\geq 1|G_{j+1}=\{e\}\},\tag{2.1}$$

where e is the identity element in G.

(ii) Similarly, the *upper central sequence* G^j is

$$G=G^0\supset G^1\supset\cdots,\quad G^{j+1}=[G^j,G^j].$$

If there exists j such that $G^j=\{e\}$, we say that the group is *solvable* and define its *derived length* d as

$$d=\min\{j\geq 1|G^j=\{e\}\}.\tag{2.2}$$

Note that

$$G_{j+1}\supset G^j.\tag{2.3}$$

This gives $d\leq n$ and in particular any nilpotent group is solvable.

We apply these two notions to the orbit complement abelianization group $\pi_1^{\mathcal{O}-ab}$ given in (1.7).

Proof of Proposition 1.2. In order to prove that $k\leq n+1$, it suffices to prove that $(\pi_1^{\mathcal{O}-ab})_j=\{\bar{e}\}$ implies $\mathcal{O}\cap L_{j+1}\subset[\mathcal{O},\pi_1]$, for any j.

We claim first that the assumption $(\pi_1^{\mathcal{O}-ab})_j=\{\bar{e}\}$, implies

$$L_j\subset\mathcal{O}.\tag{2.4}$$

Indeed, let $\sigma=[[\cdots[\gamma_1,\gamma_2],\ldots,\gamma_{j-1}],\gamma_j]$ be a generator of L_j and let $\hat{\sigma}\in\pi_1^{\mathcal{O}-ab}$ be the class of this σ_i in $(\pi_1^{\mathcal{O}-ab})$. Then $\hat{\sigma}\in\left(\pi_1^{\mathcal{O}-ab}\right)_j=\{e\}$, i.e. $\sigma\in\mathcal{O}\cap L_2$. We thus have (2.4).

Hence, $L_{j+1}=[L_j,\pi_1]\subset[\mathcal{O},\pi_1]$, showing that $\mathcal{O}\cap L_{j+1}\subset[\mathcal{O},\pi_1]$.

The relation $d\leq n$ comes from expression (2.3). On the other hand, it is known from [4] that the length ℓ of the first non-zero Melnikov function is bounded by the orbit depth k. Therefore, $\ell\leq k\leq n+1$. □

3. Germs of diffeomorphisms

Given a germ of diffeomorphism $f\in\mathrm{Diff}(\mathbb{C},0)$, we say that it is *parabolic* if it is of the form $f(z)=z+o(z)$. If f is not the identity, then $f=z+az^{p+1}+o(z^{p+1})$, with $a\neq 0$. We call p the *level* of f. Let $\mathrm{Diff}_1(\mathbb{C},0)\subset\mathrm{Diff}(\mathbb{C},0)$ denote the subgroup of parabolic germs.

The general approach given in 1.4 is based on the following well-known facts about the solvability of the group of parabolic germs $\mathrm{Diff}_1(\mathbb{C},0)$.

Lemma 3.1 (*Proposition 6.11, [10]*)**.** *Let $f=z+az^{p+1}+\cdots$ and $g=z+bz^{q+1}+\cdots$. Then $[f,g](z)=z+ab(p-q)z^{p+q+1}+o(z^{p+q+1})$.*

Proposition 3.2 (*Lemma 6.13 [10]*)**.** *Let G be a finitely generated subgroup of parabolic germs $\mathrm{Diff}_1(\mathbb{C},0)$. Then (i) or (ii) holds*

(i) *G is abelian, i.e. of nilpotence class $n(G)\leq 1$.*
(ii) *G is not solvable (hence not nilpotent).*

Lemma 3.3 (*[10]*)**.** *Let G be a finitely generated subgroup of parabolic germs $\mathrm{Diff}_1(\mathbb{C},0)$. Then G is solvable if and only if it is abelian. Moreover, one of the following statements holds:*

(i) *G is not abelian and there exist two diffeomorphisms f and g in G of different levels.*
(ii) *G is abelian and all diffeomorphisms in G are of the same level.*

Proof. Suppose $G\neq\{id\}$. We now consider two cases: either there exist two germs $f=z+az^{p+1}+o(z^{p+1})$ and $g=z+bz^{q+1}+o(z^{q+1})$ in G of different level $q\neq p$, or all germs are of the same level.

In the first case, by Lemma 3.1, $h=[f,g](z)=z+ab(p-q)z^{p+q+1}+o(z^{p+q+1})\neq z$. Therefore, $G_1=[G,G]\neq\{id\}$. Applying inductively the same reasoning on h and g, one can show that $G_\ell\neq\{id\}$ and $G^\ell\neq\{id\}$, for all ℓ. Thus, G is neither nilpotent, nor solvable.

Now, suppose that all the elements of G have the same level p. Then, by Lemma 3.1, given $f=z+az^{p+1}+o(z^{p+1})$ and $g=z+bz^{p+1}+o(z^{p+1})$, the element $[f,g]$ is either of a level strictly greater than p or the identity. The first option is impossible by the assumption, so it follows that $G_2=G^1=\{id\}$. Thus G is abelian. □

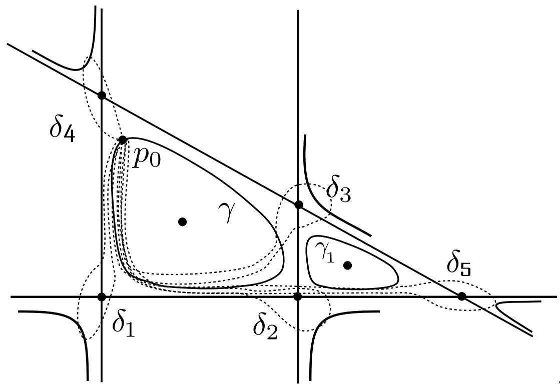

Fig. 1. The real loops $\gamma(z)$ and $\gamma_1(z)$ and the complex vanishing loops $\delta_i(z)$ as elements of $\pi_1\left(F^{-1}(z), p_0\right)$.

4. Proof of Theorem 1.5

Proof. In the case (1), of product of lines in generic position, it has been proved in [8] that $[\pi_1, \pi_1] \subseteq \mathcal{O}$. Therefore, $\pi_1^{\mathcal{O}-ab}$ is abelian, and $k \leq 2$. Hence, the orbit depth of the real cycle, as well as the length of the first non-zero Melnikov function for any deformation, is bounded by 2, by [4].

To prove that $\pi_1^{\mathcal{O}-ab}$ is non-solvable for cases (2) and (3) we follow the same strategy. In both cases, by an affine change of coordinates we can assume that the Hamiltonian is given by $F = (x-1)(x+1)f_3f_4$, where f_3 and f_4 are linear factors (non-parallel in case (2) and parallel in case (3)).

Now, consider the foliation $\mathcal{F}_\omega = \{dF + \epsilon\omega = 0\}$, where $\omega = F^2(\frac{dx}{x-1} + F\frac{dx}{x+1})$, and the homeomorphims defined by the holonomy with respect to $\mathcal{F}_\omega$:

$$P_\omega : \pi_1^{\mathcal{O}-ab} \to \mathbb{C}[\epsilon] \otimes Diff^{\,1}(\mathbb{C}, 0).$$

We define the subgroup $G := P_\omega(\pi_1^{\mathcal{O}-ab})$ of the group of parabolic diffeomorphism with coefficients depending on ϵ. Note that the morphism P_ω is well defined. Namely, since $\mathcal{F}_\omega$ has a reflexion symmetry with respect to y-axis, then it preserves the center at the origin. Moreover, since $P_\omega[\delta_1, \delta_2] = z + \epsilon^2 W(z^2, z^3)\int_{[\delta_1,\delta_2]} \frac{dx}{x-1}\frac{dx}{x+1} + O(\epsilon^3) \neq z$, where W is the Wronskian, then G is not Abelian, and therefore non-solvable.

In [5] it is proved that the parallelogram has unbounded orbit depth.

It remains to prove that for the trapezoid the orbit depth is $k \leq 2$. By an affine change of coordinates we may assume that F is defined by

$$F = (x-1)(x+1)f_3f_4\,, \tag{4.1}$$

and has the configuration given in Fig. 1. The zero level $F^{-1}(0)$ consists of four lines, enclosing a quadrilateral and a triangle. Let γ be the real cycle in $F^{-1}(z')$, $z' > 0$, close to zero and enclosed by the quadrilateral, and let γ_1 be the real cycle in $F^{-1}(z'')$, $z'' < 0$, close to zero and enclosed by the triangle, both with positive orientation. Let p_i be the vertices of the quadrilateral oriented positively with p_2 and p_3 the common vertices with the triangle and let p_5 be the last vertex of the triangle. Let δ_i, be the vanishing cycles at the vertices p_i, $i = 1, \ldots, 5$ taken so that the intersection numbers verify $(\gamma, \delta_i) = 1$, $i = 1, \ldots, 4$ and $(\gamma_1, \delta_5) = 1$.

Using the Gauss–Manin connection, we continue analytically $\gamma, \gamma_1, \delta_i$ to a non-singular curve $F^{-1}(z)$, $0 < |z| \ll 1$, along segments $[z', z]$ and $[z'', z]$. Recall that, by [11], the intersection numbers verify:

$$(\gamma, \gamma_1) = 1 \tag{4.2}$$

and

$$(\gamma_1, \delta_2) = (\gamma_1, \delta_3) = 1. \tag{4.3}$$

Remark 4.1. Theorem 1.5 is also true in the complex case if conditions (4.2) and (4.3) are fulfilled.

In [4], we obtained the orbit of the real cycle for a first integral of triangle type, that is

$$\mathcal{O}_{\gamma_1} =< \gamma_1, \delta_2\delta_3\delta_5, [\delta_2, \delta_3] > \quad \mod [\mathcal{O}_{\gamma_1}, \pi_1],$$

and in [5], for the quadrilateral type. The quadrilateral case in a neighborhood of the quadrilateral is the same as in the parallelogram case. Now, due to (4.2), in our case (4.1), the orbit of γ is generated by the union of the orbits in these two cases. This gives:

$$\begin{aligned}&\frac{\mathcal{O}_\gamma}{[\mathcal{O}_\gamma, \pi_1]} = \\ &=< \{\gamma, \delta_1\delta_2\delta_3\delta_4, [\delta_1\delta_2, \delta_2\delta_3], [\delta_1\delta_2, [\delta_2, \delta_2\delta_3]], \ldots, \\ &\delta_1\delta_2, [\delta_2, \ldots [\delta_2, \delta_2\delta_3]], \ldots\} \cup \mathcal{O}_{\gamma_1} > .\end{aligned}$$

We know that $[\delta_2, \delta_3]$ belongs to the orbit of γ_1 and hence to $\mathcal{O}_\gamma$. Hence, $[\delta_2, \delta_2\delta_3] = \delta_2[\delta_2, \delta_3]\delta_2^{-1}$ belongs to the orbit as well. This gives that in the above expression for the orbit $\mathcal{O}_\gamma$ the commutator $[\delta_1\delta_2, [\delta_2, \delta_2\delta_3]]$ belongs to $[\mathcal{O}, \pi_1]$. The same is therefore true for all terms which contain it (all the terms following it in the above braces). We know [4] that the depth of $\mathcal{O}_{\gamma_1}$ is two. All the terms following the term $[\delta_1\delta_2, \delta_2\delta_3]$ are in $[\mathcal{O}, \pi_1]$, showing that the depth k verifies $k \leq 2$ and hence by [4] the length of the first non-zero function M_μ of any deformation ω is $\ell \leq 2$. $\square$

5. Types of integrability of the deformations

We will study the Godbillon–Vey sequence for the foliation

$$dF + \epsilon\omega = 0, \tag{5.1}$$

with $F = f_1f_2f_3f_4$ and $\omega = p_1(F)\frac{df_1}{f_1} + p_2(F)\frac{df_2}{f_2}$, where $f_1 = x-1$, $f_2 = x+1$, p_1, p_2 are polynomials in F, and f_3, f_4 are linear factors different from f_1 and f_2. For the parallelogram, f_3 and f_4 define parallel lines, while for the trapezoid they do not. By explicit computation of Godbillon–Vey sequences we will show that the foliation $dF + \epsilon\left(\frac{df_1}{f_1} + F\frac{df_2}{f_2}\right) = 0$ is Liouville integrable, while $dF + \epsilon\left(F\frac{df_1}{f_1} + F^2\frac{df_2}{f_2}\right) = 0$ is Riccati integrable. Moreover, for $\omega = p_1(F)\frac{df_1}{f_1} + p_2(F)\frac{df_2}{f_2}$, with $n = \deg\{\deg p_1, \deg p_2\}$, foliation (5.1) admits a Godbillon–Vey sequence of length $\leq n$, which increases complexity of the first integral for those cases. Therefore, for $n \geq 3$ the foliation (5.1) could have a first integral which is not of Riccati type, with infinite dimensional pseudo-group of holonomy.

We recall that a Godbillon–Vey sequence for a meromophic 1-form η_0 is a sequence $\{\eta_0, \eta_1, \ldots\}$, such that

$$\begin{aligned}
d\eta_0 &= \eta_0 \wedge \eta_1\\
d\eta_1 &= \eta_0 \wedge \eta_2\\
d\eta_2 &= \eta_0 \wedge \eta_3 + \eta_1 \wedge \eta_2\\
d\eta_3 &= \eta_0 \wedge \eta_4 + 2\eta_1 \wedge \eta_3\\
&\ \ \vdots\\
d\eta_n &= \eta_0 \wedge \eta_{n+1} + \sum_{k=1}^{n}\binom{n}{k}\eta_k \wedge \eta_{n-k+1}
\end{aligned} \tag{5.2}$$

It is of length ℓ if $\eta_j = 0$ for all $j \geq \ell$.

It is known that η_0 has a first integral of Liouville type (respectively of Riccati type), if and only if, it admits a Godbillon–Vey sequence of length 2 (respectively of length 3).

5.1. Degree on F equals 1

Let us start by analyzing the case $\omega = F\frac{df_1}{f_1}$. We denote $\eta_0 := dF + \epsilon F\frac{df_1}{f_1}$. Then

$$d\eta_0 = \epsilon dF \wedge \frac{df_1}{f_1}.$$

Thus, we can take $\eta_1 := \epsilon\frac{df_1}{f_1} = \frac{df_1^\epsilon}{f_1^\epsilon}$. Then, $d\eta_1 = 0$, which means $\eta_2 = 0$. Hence, η_0 admits a Godbillon–Vey sequence of length 2, and therefore (5.1) is Liouville integrable. Moreover, Casale's article [6] provides a way to compute from this sequence the first integral. This is given by $dH = G\eta_0$, where $\eta_1 = \frac{dG}{G}$. In this case, we have $G = f_1^\epsilon$. Therefore the first integral H is given by

$$f_1^\epsilon\eta_0 = f_1^\epsilon(dF + F\frac{df_1^\epsilon}{f_1^\epsilon}) = f_1^\epsilon dF + Fdf_1^\epsilon = d(f_1^\epsilon F);$$

thus, $H = f_1^\epsilon F$, which lies in a Liouvillian extension.

Now we analyze the following case $\eta_0 := dF + \epsilon\left(\frac{df_1}{f_1} + F\frac{df_2}{f_2}\right)$, which will give us a way to proceed for higher degree. By substituting $f_1 = x-1, f_2 = x+1$, we obtain $\eta_0 = dF + \epsilon\left(\frac{1}{x-1} + \frac{F}{x+1}\right)dx$. Denote $\varphi(x,F) = \frac{1}{x-1} + \frac{F}{x+1}$. Then $\eta_0 = dF + \epsilon\varphi dx$, and

$$d\eta_0 = \epsilon d\varphi \wedge dx. \tag{5.3}$$

Note that $d\varphi = \varphi_F dF + \varphi_x dx$, where $\varphi_F = \frac{\partial\varphi}{\partial F}(x,F) = \frac{1}{x+1}$, and $\varphi_x = \frac{\partial\varphi}{\partial x}(x,F) = \frac{-1}{(x-1)^2} - \frac{F}{(x+1)^2}$. Then, (5.3) is equal to

$$d\eta_0 = \epsilon\varphi_F dF \wedge dx. \tag{5.4}$$

Define $\eta_1 := \epsilon\varphi_F dx$. It satisfies the equation $d\eta_0 = \eta_0 \wedge \eta_1 = (dF + \epsilon\varphi dx) \wedge \epsilon\varphi_F dx$. On the other hand, $d\eta_1 = \epsilon d\varphi_F \wedge dx$, where $d\varphi_F = \varphi_{FF}dF + \varphi_{Fx}dx$. That is, $d\eta_1 = \epsilon\varphi_{FF}dF \wedge dx$, which is zero because $\varphi_{FF} = 0$. Then, we can take $\eta_2 = 0$. Thus, η_0 admits a Godbillon–Vey sequence of length 2, and therefore is Liouville integrable. To compute the first integral H, we have to solve the equations $dH = G\eta_0$, and $\eta_1 = \frac{dG}{G}$. For this purpose, we note that $\eta_1 = \epsilon\frac{dx}{x+1} = \frac{df_2^\epsilon}{f_2^\epsilon}$. Then, $dH = f_2^\epsilon\eta_0 = f_2^\epsilon\left(dF + \epsilon\left(\frac{df_1}{f_1} + F\frac{df_2}{f_2}\right)\right)$, which corresponds to

$$f_2^\epsilon\left(dF + \frac{df_1^\epsilon}{f_1^\epsilon} + F\frac{df_2^\epsilon}{f_2^\epsilon}\right) = f_2^\epsilon dF + f_2^\epsilon\frac{df_1^\epsilon}{f_1^\epsilon} + Fdf_2^\epsilon = d\left(f_2^\epsilon F + \int f_2^\epsilon\frac{df_1^\epsilon}{f_1^\epsilon}\right).$$

Therefore, $H = f_2^\epsilon F + \int f_2^\epsilon\frac{df_1^\epsilon}{f_1^\epsilon}$. By substituting the expressions of f_1, f_2, we get $\int f_2^\epsilon\frac{df_1^\epsilon}{f_1^\epsilon} = \int \epsilon(x+1)^\epsilon\frac{dx}{x-1}$.

We stress that the construction above is analogous for f_1 and f_2 defining two arbitrary parallel lines.

6. Proof of Theorem 1.7

Proof.

We consider the deformation $dF + \epsilon(F\frac{df_1}{f_1} + F^2\frac{df_2}{f_2})$, where $f_1 = x-1, f_2 = x+1$, and define

$$\eta_0 := dF + \epsilon\left(\frac{F}{x-1} + \frac{F^2}{x+1}\right)dx. \tag{6.1}$$

Denote $\varphi(x,F) = \frac{F}{x-1} + \frac{F^2}{x+1}$. Then $\eta_0 = dF + \epsilon\varphi dx$, and since $d\varphi = \varphi_F dF + \varphi_x dx$, where φ_F is the partial derivative of φ with respect to F: $\varphi_F = \frac{1}{x-1} + \frac{2F}{x+1}$, we have

$$d\eta_0 = \epsilon\varphi_F dF \wedge dx. \tag{6.2}$$

Thus, we define $\eta_1 := \epsilon\varphi_F dx$. It satisfies the equation $d\eta_0 = \eta_0 \wedge \eta_1$. Now we consider $d\eta_1 = \epsilon d\varphi_F \wedge dx$. Then, again, writing $d\varphi_F = \varphi_{FF}dF + \varphi_{Fx}dx$, we have $d\eta_1 = \epsilon\varphi_{FF}dF \wedge dx$. So, we can take $\eta_2 := \epsilon\varphi_{FF}dx$. It satisfies the equation $d\eta_1 = \eta_0 \wedge \eta_2$. Continuing with the Godbillon–Vey sequence we consider $d\eta_2 = \epsilon d\varphi_{FF} \wedge dx$. But, since the degree of φ in F is 2, φ_{FF} depends only on x, therefore $d\varphi_{FF} \wedge dx = 0$. Now, for the equation $d\eta_2 = \eta_0 \wedge \eta_3 + \eta_1 \wedge \eta_2$, since $\eta_1 \wedge \eta_2 = 0$, we can take $\eta_3 = 0$. Hence, η_0 admits a Godbillon–Vey sequence of length 3, and therefore it has a first integral of Riccati type.

According to Casale's method [6] we can compute the first integral by solving the following equations:

$$\begin{aligned} dH &= G_1\eta_0 \\ dG_1 &= G_1\left(\eta_1 + \frac{2}{G_2}\eta_0\right) \\ dG_2 &= -\frac{G_2^2}{2}\eta_2 - G_2\eta_1 - \eta_0, \end{aligned} \tag{6.3}$$

with $\eta_0 = dF + \epsilon\left(\frac{F}{x-1} + \frac{F^2}{x+1}\right)dx$, $\eta_1 = \epsilon\left(\frac{1}{x-1} + \frac{2F}{x+1}\right)dx$, and $\eta_2 = \epsilon\left(\frac{2}{x+1}\right)dx$.

One can check that $H = \frac{-1}{Ff_1^\epsilon} + \int\frac{df_2^\epsilon}{f_1^\epsilon f_2^\epsilon}$, $G_1 = \frac{1}{F^2f_1^\epsilon}$ and $G_2 = -F$ satisfy Eqs. (6.3).

Now, we analyze what happens when we increase the degree in F for the foliation (5.1). Let

$$\eta_0 := dF + \epsilon\left(F^2\frac{df_1}{f_1} + F^3\frac{df_2}{f_2}\right). \tag{6.4}$$

Substituting $f_1 = x-1$, $f_2 = x+1$, we obtain $\eta_0 = dF + \epsilon\left(\frac{F^2}{x-1} + \frac{F^3}{x+1}\right)dx$. Following the same process as before, we write $\eta_0 = dF + \epsilon\varphi dx$, where $\varphi(x,F) = \frac{F^2}{x-1} + \frac{F^3}{x+1}$. Since $d\varphi = \varphi_F dF + \varphi_x dx$, we have

$$d\eta_0 = \epsilon\varphi_F dF \wedge dx, \tag{6.5}$$

where $\varphi_F = \frac{2F}{x-1} + \frac{3F^2}{x+1}$.

Thus, we define $\eta_1 := \epsilon\varphi_F dx$, note that $\deg_F\varphi_F = 2$. It satisfies the equation $d\eta_0 = \eta_0 \wedge \eta_1$. Now we consider $d\eta_1 = \epsilon d\varphi_F \wedge dx$. Writing $d\eta_1 = \varphi_{FF}dF + \varphi_{Fx}dx$, we have $d\eta_1 = \epsilon\varphi_{FF}dF \wedge dx$. So, we can take again $\eta_2 := \epsilon\varphi_{FF}dx$, where now $\deg_F\varphi_{FF} = 1$. It satisfies the equation $d\eta_1 = \eta_0 \wedge \eta_2$. Then $d\eta_2 = \epsilon d\varphi_{FF} \wedge dx = \epsilon\varphi_{FFF}dF \wedge dx$. We want η_3 to be such that $d\eta_2 = \eta_0 \wedge \eta_3 + \eta_1 \wedge \eta_2$. Since $\eta_1 \wedge \eta_2 = 0$, we can take $\eta_3 = \epsilon\varphi_{FFF}dx$. Consider $d\eta_3 = \epsilon d\varphi_{FFF} \wedge dx$. This is zero, since φ_{FFF} depends only on x. We want now η_4 verifying the equation $d\eta_3 = \eta_0 \wedge \eta_4 + 2\eta_2 \wedge \eta_3$. But, since $\eta_2 \wedge \eta_3 = 0$, we can take $\eta_4 = 0$. Moreover, from this construction $\eta_k \wedge \eta_\ell = 0$ for all $k, \ell \geq 1$, hence this Godbillon–Vey sequence has length 4. Therefore, η_0 could have a first integral which is not of Riccati type. With the same procedure we can obtain a deformation with a Godbillon–Vey sequence of any finite length n, by taking the degree in F of the deformation equal to $n-1$.

We stress that the above computations do not depend on the factors f_3 and f_4, therefore they are valid for the parallelogram, as well as for the trapezoid. □

Remark 6.1. From these cases one can observe that the length of the Godbillon–Vey sequence for the foliation (5.1) increases with the degree in F of the function $\varphi(x,F)$. So, increasing the degree in F for the deformation, we can have a first integral of more complicated type.

In particular, for $n = \deg_F\varphi > 4$ one should obtain a Godbillon–Vey sequence of finite length n higher than 4. In the paper [9] (p. 25) the authors say they do not know any example of finite length greater than 4. However, the length is defined as the minimal length among all the Godbillon–Vey sequences that the foliation can admit. So, in order to prove that these examples for $n > 4$ have Godbillon–Vey length higher than 4, one should prove that the constructed Godbillon–Vey is optimal. That is that there does not exist some other Godbillon–Vey sequence of smaller length.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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